ROUGH SOLUTIONS OF THE EINSTEIN CONSTRAINT EQUATIONS WITH NONCONSTANT MEAN CURVATURE

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ABSTRACT. We consider the conformal decomposition of Einstein's constraint equations introduced by Lichnerowicz and York, on a compact manifold with boundary. We first develop some technical results for the momentum constraint operator under weak assumptions on the problem data, including generalized Korn inequalities on manifolds with boundary not currently in the literature. We then consider the Hamiltonian constraint, and using order relations on appropriate Banach spaces we derive weak solution generalizations of known sub- and super-solutions (barriers). We also establish some related *a priori* L^∞-bounds on any $W^{1,2}$-solution. The barriers are combined with variational methods to establish existence of solutions to the Hamiltonian constraint in $L^\infty \cap W^{1,2}$. The result is established under weak assumptions on the problem data, and for scalar curvature R having any sign; non-negative R requires additional positivity assumptions either on the matter energy density or on the trace-free divergence-free part of the extrinsic curvature. Although the formulation is different, the result can be viewed as extending the regularity of the recent result of Maxwell on "rough" CMC solutions in $W^{k,2}$ for $k > 3/2$ down to $L^\infty \cap W^{1,2}$. The results for the individual constraints are then combined to establish existence of non-CMC solutions in $W^{1,p}$, $p > 3$ for the three-metric and in L^q, $q = 6p/(3+p)$ for the extrinsic curvature. The result is obtained using fixed-point iteration and compactness arguments directly, rather than by building a contraction map. The non-CMC result can be viewed as a type of extension of the regularity of the 1996 non-CMC result of Isenberg and Moncrief down to $W^{1,p}$ for $p > 3$, and extending their result to R having any sign. Similarly, the result can also be viewed as type of extension of the recent work of Maxwell on rough solutions from the CMC case to the non-CMC case. Although our presentation is for 3-manifolds, the results also hold in higher dimensions with minor adjustments. The results should also extend to other cases such as closed and (fully or partially) open manifolds without substantial difficulty.

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Contents

1. Introduction

In this article, we give an analysis of the coupled Hamiltonian and momentum constraints in the Einstein equations on 3-dimensional compact manifolds with boundary. We consider the equations with matter sources satisfying an energy condition implied by the dominant energy condition in the 4-dimensional spacetime; the unknowns are a Riemannian three-metric and a two-index symmetric tensor. The equations form an under-determined system; therefore, we focus entirely on a standard reformulation used in both mathematical and numerical general relativity, called the conformal method, introduced by Lichnerowicz and York [37, 52, 53]. The conformal method assumes that the unknown metric is known up to a scalar field called a conformal factor, and also assumes that the trace and a term proportional to the trace-free divergence-free part of the two-index symmetric tensor is known, leaving as unknown a term proportional to the traceless symmetrized derivative of a vector. Therefore, the new unknowns are a scalar and a vector field, transforming the original under-determined system for a metric and a symmetric tensor into a (potentially) well-posed elliptic system for a scalar and a vector field. See [6] for a recent review article. We point out just some of the quite substantial number of previous related works, including: the original work on the Lichnerowicz equation [37]; the development of the conformal method [52, 53, 54, 55]; the initial solution theory for the Hamiltonian constraint [41, 42, 43]; the thin sandwich alternative to the conformal method [5, 40]; the complete classification of CMC initial data [29] and the few known non-CMC results [30, 31, 13]; various technical results on transverse-traceless tensors and the conformal Killing operator [7, 9]; the more recent development of the conformal thin sandwich formulation [56]; initial data for black holes [8, 10]; initial data for Kerr-like black holes [16, 17]; and the gluing approach to generating initial data [15].

The conformal method gives rise to a coupled nonlinear elliptic system for the unknown scalar and vector fields; the trace of the symmetric tensor plays an important role: in the case that the trace is constant (referred to as the *constant mean curvature or CMC case*), the two equations decouple, giving rise to the term "semi-decoupling decomposition" which is sometimes used to describe the conformal method [6]. In this case, a linear equation for the unknown vector can be solved first, and then a semi-linear equation for the scalar field can be solved, where a coefficient in the nonlinearity depends quadratically on derivatives of the vector unknown. Almost all of the previous work on developing a solution theory for the constraints has focussed on the conformal decomposition in the CMC case, primarily in the case of compact manifolds without boundary [6]. A notable exception is the non-CMC existence and uniqueness result in Hölder-classes for a particular physical scenario, which was established in [30].

In this article, we extend the solution theory for the individual and coupled Hamiltonian and momentum constraints on compact manifolds with boundary in three ways:

(i) Some technical results, including generalized Korn inequalities, are established for the conformal Killing operator on compact Riemannian manifolds with boundary, under several different boundary condition assumptions. The results, which are not currently in the literature, allow us to establish well-posedness of the momentum constraint equation in $W^{1,2}(\mathcal{M})$ on a manifold $\mathcal{M}$ with boundary, using either variational or Riesz-Schauder methods. The assumptions we make on the data using either method are weak enough that standard techniques to establish additional regularity are not available.

(ii) Existence (and in some cases, uniqueness) results are established for weak (or rough) CMC solutions to Hamiltonian constraint, for weaker solution spaces than appeared previously in [30, 38]. In particular, using variational methods we establish existence of weak solutions to the Hamiltonian constraint in $L^\infty(\mathcal{M}) \cap W^{1,2}(\mathcal{M})$ on compact manifolds with boundary, under assumptions on the data that do not allow for the use of standard techniques to establish additional regularity. The variational methods we employ make use of (generalized) barriers for the Hamiltonian constraint equation; in §4.3 we summarize the barriers we use for different values of the Ricci scalar of the background metric. We also establish some related *a priori* L^∞-bounds on any $W^{1,2}$-solution to the Hamiltonian constraint in §4.4. Although such results are standard for semi-linear scalar problems with monotone nonlinearities (see for example [32]), our results hold for a class of non-monotone nonlinearities that includes the Hamiltonian constraint nonlinearity and appear to be new.

(iii) Existence results are established for non-CMC solutions to the coupled system of constraints on compact manifolds with boundary, in the setting of weaker (rougher) solutions spaces and for more general physical scenarios than appeared previously in [30]. In particular, we establish existence of solutions to the coupled Hamiltonian and momentum constraints, in $W^{1,p}(\mathcal{M})$ for $p > 3$ for the conformal factor and in $W^{1,q}(\mathcal{M})$ for $q = 6p/(3+p)$ for the momentum vector, on compact manifolds with boundary, with no restrictions on the sign of the scalar curvature R. For the case of non-negative R, either the matter

energy density or the trace-free divergence-free part of the extrinsic curvature must be globally positive. The technical condition on the trace of the extrinsic curvature (called the "near-CMC" condition in [6]) used to produce the coupled system result in [30] is still present here, although it now involves weaker norms (see §5). In addition, this condition is only used here to construct a global super-solution to the Hamiltonian constraint, and is not used a second distinct time as part of the fixed-point argument as was needed in [30].

The results above imply that the weakest differentiable solutions of the Einstein constraint equations we have found correspond to CMC hypersurfaces with physical spatial metric h_{ab} and extrinsic curvature k_{ab} satisfying

$$h_{ab} \in L^\infty(\mathcal{M}) \cap W^{1,2}(\mathcal{M}), \qquad k_{ab} \in L^2(\mathcal{M}). \tag{1.1}$$

The curvature of such data can be computed in a distributional sense, following [24].

There are at least four distinct, but related, motivations for establishing the extensions outlined above. First, as outlined in [6], new results for the non-CMC case, beyond the case analyzed in [30], are of great interest in both mathematical and numerical relativity. Second, there is currently substantial research activity in rough solutions to the Einstein evolution equations, which rest on rough/weak solution results for the initial data [34]. Third, the role of boundary conditions and bounded domains in the solution and approximation theory is of importance particularly in numerical relativity; most existing results are for closed (compact without boundary), open, or only partially bounded domains. Finally, the approximation theory for Petrov-Galerkin-type methods (including finite element, wavelet, spectral, and other methods) for the constraints and similar systems previously developed in [28] establishes convergence of numerical solutions in very general physical situations, but rests on assumptions about the solution theory; the results in the present paper help to complete this approximation theory framework.

An outline of the paper describing the results is as follows.

In §1.1, we give a brief outline of the notation used throughout the paper. In §2, we quickly overview the conformal decomposition, describe the classical strong formulation of the resulting coupled elliptic system, and then define weak formulations of the constraint equations that will allow us to develop solution theories for the constraints in the spaces with the weakest possible regularity. Our formulation allows for a mix of Dirichlet and Robin boundary conditions for modeling e.g. black hole and other physically important scenarios.

In §3, we develop some basic technical results for the momentum constraint equation on compact manifolds with boundary that we will need later for analysis of the Hamiltonian constraint and for analysis of the coupled system. We first develop the weak formulation of the momentum constraint for a given scalar conformal factor in §3.1, and then develop some preliminary results related to the Korn inequality in §3.2. In particular, we establish generalized Korn inequalities for the conformal Killing operator on compact manifolds with boundary under several boundary condition scenarios; these results do not appear to be in the literature. In §3.3, we use the preliminary results from §3.2, together with a variational argument, to establish existence and uniqueness of weak solutions to the momentum constraint in $W^{1,2}(\mathcal{M})$ when the Dirichlet part of the boundary is non-empty, with assumptions on the data that do not allow for additional regularity in the sense described earlier. In §3.4 we give a second (non-variational) argument for existence and uniqueness again in $W^{1,2}(\mathcal{M})$ using a Riesz-Schauder (Fredholm alternative) argument following. This second argument is more general than the variational argument in the sense that the Dirichlet part of the boundary might be empty. While the techniques we use in the analysis of the momentum constraint are standard, this collection of results of the momentum constraint operator on compact domains with boundary apparently are not in the literature. Regularity of solutions to the momentum constraint is discussed briefly in §3.5.

In §4, we give a corresponding analysis of the Hamiltonian constraint on compact manifolds with boundary. We first develop the weak formulation of the Hamiltonian constraint for a given momentum vector variable in §4.1, and establish some preliminary results on local and global barriers (constant sub- and super-solutions) for weak solutions in §4.2. The term local means that the barrier depends on the given momentum vector variable solution of the momentum constraint equation, while global means that the barrier is not local. These barriers are non-trivial extensions of those in [29, 30] to nonlinearities with coefficients in $W^{-1,2}(\mathcal{M})$. We also establish some related *a priori* L^∞-bounds on any $W^{1,2}$-solution to the Hamiltonian constraint; although such results are standard for semi-linear problems with monotone nonlinearities, our results hold for a class of non-monotone nonlinearities that includes the Hamiltonian constraint nonlinearity and appear to be new. The weak solution barriers are critical to extending the solution theory for the Hamiltonian constraint to the weakest possible setting of $W^{1,2}(\mathcal{M})$ in §4.5, and are also key to extending the solution theory for the coupled system to weaker spaces and to new physical scenarios in §5. In §4.5, we use the barriers from §4.2, together with a variational argument, to establish existence and uniqueness of solutions to the Hamiltonian constraint in the weakest possible setting of $L^\infty(\mathcal{M}) \cap W^{1,2}(\mathcal{M})$. Due to the lack of Gâteaux-differentiability of the nonlinearity in $W^{1,2}(\mathcal{M})$, the connection between the energy used for the variational argument and the Hamiltonian constraint as its Euler condition is non-trivial, and is established through several Lemmas. We note that our arguments allow for scalar curvature to have any sign, and the assumptions on the data are such that additional regularity is not possible in the sense described earlier. The results for non-negative R require an assumption of global positivity either on the trace-free and divergence-free part of the extrinsic curvature or on the matter energy density. Although our problem formulation is somewhat different, the result can be viewed as extending the regularity of the recent result of Maxwell [38] on "rough" CMC solutions in $W^{k,2}(\mathcal{M})$ for $k > 3/2$ down to $L^\infty(\mathcal{M}) \cap W^{1,2}(\mathcal{M})$. In §4.6 we give a second (non-variational) argument for existence and uniqueness, using the barriers (sub-/super-solution) approach as in most of the earlier work [29, 30, 38]. Unlike the case of the momentum constraint in §3.4, where the non-variational technique allows for the development of a solution theory in same weak setting of $W^{1,2}(\mathcal{M})$ as does the variational method, the Hamiltonian constraint barriers approach requires additional regularity beyond what the variational approach in §4.5 requires. Regularity of solutions is discussed briefly in §4.7.

Finally, in §5 we use the results for the individual constraints derived earlier to establish a new non-CMC result for the coupled system. In §5.1, we establish existence of non-CMC solutions to the coupled constraints through fixed-point iteration and compactness arguments directly, rather than by using the Contraction Mapping Theorem as was done in the original work of Isenberg and Moncrief in [30]. The "near-CMC" assumption on the trace of the extrinsic curvature required in [30] is still present here, although it now involves weaker norms (see §5). In addition, the condition is only used here to construct a global super-solution to the Hamiltonian constraint, and is not used a second distinct time as part of the fixed-point argument as was necessary in [30]. If a global super-solution can be constructed without the near-CMC assumption, then this new coupled system result would still hold for "far-from-CMC" scenarios. The result requires more regularity than that needed for the result established in §3 and §4 for the individual constraints, with solutions in $W^{1,p}(\mathcal{M})$ for $p > 3$ for the conformal factor and in $W^{1,q}(\mathcal{M})$ for $q = 6p/(3+p)$ for the momentum vector, but still extends the existing theory for the system in two distinct ways. First, although our problem formulation is somewhat different (bounded domains with matter), the result can be viewed as a type of extension of the 1996 non-CMC result of Isenberg and Moncrief in [30] from the scalar curvature $R = -1$ case to R having any sign (with non-negative R requiring the assumption that either the matter energy density or the trace-free divergence-free part of the extrinsic curvature be globally positive), and to weaker solution spaces. Second, again although the problem formulation is different, the result could be viewed as a type of extension of the recent rough CMC solution work of Maxwell in [38] to the non-CMC case. Although our presentation is for 3-manifolds, the results hold in higher spatial dimensions with minor adjustments, and the techniques we employ should extend to other cases such as closed and (fully or partially) open manifolds through the use of tools such as weighted Sobolev spaces.

We summarize our results in §6.

1.1. **Notation and conventions.** Let $(\mathcal{M}, h_{ab})$ be a a Riemannian manifold, where $\mathcal{M}$ is a 3-dimensional, smooth, compact manifold with non-empty boundary $\partial\mathcal{M}$, and h_{ab} is a C^2 metric on $\overline{\mathcal{M}}$, that is, a symmetric, positive definite, covariant, two-index tensor on $\mathcal{M}$ with all components in a smooth coordinate system having two continuous derivatives. Latin indices denote abstract indices as they are defined in [50], §2.4. The metric defines an inner product on $T_x\mathcal{M}$, the vector space tangent to $\mathcal{M}$ at the point $x \in \mathcal{M}$. Denote by h^{ab} the inverse of the metric tensor h_{ab}, that is, $h_{ac}h^{bc} = \delta_a{}^b$, where $\delta_a{}^b : T_x\mathcal{M} \to T_x\mathcal{M}$ is the identity map. We use the convention that repeated indices, one upper-index and one sub-index, denote contraction. Let ∇_a be the Levi-Civita connection associated with the metric h_{ab}, that is, the unique torsion-free connection satisfying $\nabla_a h_{bc} = 0$. Let $R_{abc}{}^d$ be the Riemann tensor of the connection ∇_a, where the sign convention used in this article is $(\nabla_a\nabla_b - \nabla_b\nabla_a)v_c := R_{abc}{}^d v_d$. Denote by $R_{ab} := R_{acb}{}^c$ the Ricci tensor and by $R := R_{ab}h^{ab}$ the Ricci curvature scalar of this connection. (Only in §2.1 we modify the notation for the connection ∇_a.)

Indices on tensors will be raised and lowered with h^{ab} and h_{ab}, respectively. For example, given the tensor $s^{ab}{}_c$ we denote $s_{abc} = h_{aa_1}h_{bb_1}\, s^{a_1b_1}{}_c$, and $s^{abc} = h^{cc_1}\, s^{ab}{}_{c_1}$; notice that the order of the indices is important in the case that the tensor s_{abc} or s^{abc} is not symmetric. We say that a tensor is an n-**index** tensor iff it can be transformed into a tensor $s_{a_1\cdots a_n}$ by lowering appropriate indices. We denote by $C^\infty(\mathcal{M}, n)$ the set of all smooth n-index tensor fields on $\mathcal{M}$. Given an arbitrary tensor $s^{a_1\cdots a_n}{}_{b_1\cdots b_m}$, which is an $n+m$-index tensor, we define its magnitude at any point $x \in \mathcal{M}$ as the real-valued function given by

$$|s| := (s^{a_1\cdots b_m} s_{a_1\cdots b_m})^{1/2}. \tag{1.2}$$

Integration on $\mathcal{M}$ is performed with the volume element dx associated to the metric h_{ab}. A norm of an arbitrary smooth tensor field $s^{a_1\cdots a_n}{}_{b_1\cdots b_m}$ on $\mathcal{M}$ can be defined for any $1 \leqslant p < \infty$ and for $p = \infty$ respectively using (1.2) as follows,

$$\|s\|_p := \Big[\int_{\mathcal{M}} |s|^p \, dx\Big]^{1/p}, \qquad \|s\|_\infty := \text{ess} \sup_{x\in\mathcal{M}} |s|. \tag{1.3}$$

We introduce the **Lebesgue spaces** $L^p(\mathcal{M}, n)$, for $1 \leqslant p \leqslant \infty$, of n-index tensor valued fields as the completion of $C^\infty(\mathcal{M}, n)$ under the norm in Eq. (1.3), and this norm is called the L^p**-norm**. The Lebesgue spaces $L^p(\mathcal{M}, n)$ are Banach spaces; they are separable when $1 \leqslant p < \infty$ and reflexive when $1 < p < \infty$. For the case $p = 2$ the spaces $L^2(\mathcal{M}, n)$ form a Hilbert space with the inner product and norm given by

$$(s, r) := \int_{\mathcal{M}} s_{a_1\cdots a_n} r^{a_1\cdots a_n} \, dx, \qquad \|s\| := \sqrt{(s, s)} = \|s\|_2. \tag{1.4}$$

Covariant derivatives of tensor fields are denoted as

$$\nabla^m s := \nabla_{b_1,\cdots,b_m} s^{a_1\cdots a_n} := \nabla_{b_1} \cdots \nabla_{b_m} s^{a_1\cdots a_n},$$

where the super-script m indicates the total number of derivatives, which plays the role of the number $|\alpha|$ for a multi-index α, in the multi-index notation used in the PDE literature. Using again Eq. (1.2) introduce the real-valued function

$$|\nabla^m s| := \big[(\nabla_{b_1} \cdots \nabla_{b_m} s^{a_1\cdots a_n})(\nabla^{b_1} \cdots \nabla^{b_m} s_{a_1\cdots a_n})\big]^{1/2},$$

as the starting point for defining L^p-type norms involving derivatives. One such norm in the vector space $C^\infty(\mathcal{M}, n)$ is given for any non-negative integer k and a real number p with $1 \leqslant p < \infty$, and separately for $p = \infty$, as follows

$$\|s\|_{k,p} := \Big[\sum_{m=0}^{k} \|\nabla^m s\|_p^p\Big]^{1/p}, \qquad \|s\|_{k,\infty} := \max_{0\leqslant m\leqslant k} \|\nabla^m s\|_\infty. \tag{1.5}$$

We introduce the **Sobolev spaces** $W^{k,p}(\mathcal{M}, n)$ of n-index tensor valued fields as the completion of $C^\infty(\mathcal{M}, n)$ under the norm in Eq. (1.5), and this norm is called the $W^{k,p}$**-norm**. The Sobolev spaces $W^{k,p}(\mathcal{M}, n)$ are Banach spaces; being based on $L^p(\mathcal{M}, n)$, they are separable when $1 \leqslant p < \infty$ and reflexive when $1 < p < \infty$. For the case $p = 2$ the spaces $W^{k,2}(\mathcal{M}, n)$ form a Hilbert space with the inner product and norm given by

$$(s, r)_k := \sum_{m=0}^{k} (\nabla^m s, \nabla^m r), \qquad \|s\|_{k,2} = \sqrt{(s, s)_k}, \tag{1.6}$$

where we have introduced the notation

$$(\nabla^m s, \nabla^m r) := \int_{\mathcal{M}} (\nabla_{b_1} \cdots \nabla_{b_m} s_{a_1\cdots a_n})(\nabla^{b_1} \cdots \nabla^{b_m} r^{a_1\cdots a_n}) \, dx.$$

Therefore we have that $L^p(\mathcal{M}, n) = W^{0,p}(\mathcal{M}, n)$ and $\|s\|_p = \|s\|_{0,p}$. These definitions follow [27] for the case of scalar fields and [45] for the case of arbitrary tensor fields. These definitions can also be extended, using appropriate partitions of the unity and Fourier transforms, from non-negative integers k to real numbers s. For example see [49].

In this article we are mainly concerned with spaces of scalar-valued fields and vector-valued fields on $\mathcal{M}$, so we introduce the following notation for these special cases,

$$C^\infty := C^\infty(\mathcal{M},0), \qquad \boldsymbol{C}^\infty := C^\infty(\mathcal{M},1),$$
$$L^p := L^p(\mathcal{M},0), \qquad \boldsymbol{L}^p := L^p(\mathcal{M},1),$$
$$W^{k,p} := W^{k,p}(\mathcal{M},0), \qquad \boldsymbol{W}^{k,p} := W^{k,p}(\mathcal{M},1).$$

However, we will not suppress the manifold from the notation of these spaces when this information is important in a given situation. In a similar way, we use both notations $\boldsymbol{w}$ and w^a to denote a vector field. We consider in this article that the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ can be divided in the following two, possible different, ways as follows,

$$\partial\mathcal{M} = \partial\mathcal{M}_D \cup \partial\mathcal{M}_N, \quad \overline{\partial\mathcal{M}}_D \cap \overline{\partial\mathcal{M}}_N = \emptyset, \tag{1.7}$$
$$\partial\mathcal{M} = \partial\mathcal{M}_{I\!\!D} \cup \partial\mathcal{M}_{I\!\!N}, \quad \overline{\partial\mathcal{M}}_{I\!\!D} \cap \overline{\partial\mathcal{M}}_{I\!\!N} = \emptyset. \tag{1.8}$$

Introduce the trace operators

$$\mathsf{tr}_D : W^{s,p} \to W^{s-\frac{1}{p},p}(\partial\mathcal{M}_D,0), \qquad \mathsf{tr}_{I\!\!D} : \boldsymbol{W}^{s,p} \to W^{s-\frac{1}{p},p}(\partial\mathcal{M}_{I\!\!D},1),$$

for $s > 1/p$, which are the continuous extensions to Sobolev spaces of the operators defined on smooth fields given by $\mathsf{tr}_D\phi := \phi|_{\partial\mathcal{M}_D}$, and $\mathsf{tr}_{I\!\!D}\boldsymbol{w} := \boldsymbol{w}|_{\partial\mathcal{M}_{I\!\!D}}$, see for example [47]. Both spaces $W^{s-\frac{1}{p},p}(\partial\mathcal{M}_D,0)$ and $W^{s-\frac{1}{p},p}(\partial\mathcal{M}_{I\!\!D},1)$ are Banach spaces and we denote their norms as $\|\phi\|_{s-\frac{1}{p},p,D}$ and $\|\boldsymbol{w}\|_{s-\frac{1}{p},p,I\!\!D}$, respectively. In the particular case $p=2$ these spaces become Hilbert spaces and we denote their inner product as $(\phi,\underline{\phi})_{s-\frac{1}{2},D}$ and $(\boldsymbol{w},\underline{\boldsymbol{w}})_{s-\frac{1}{2},I\!\!D}$. We will be mainly concerned with the case $s=1$, $p=2$, and in this case we denote their inner product and norms as follows

$$(\hat{\phi},\underline{\hat{\phi}})_D, \quad \|\hat{\phi}\|_D = (\hat{\phi},\hat{\phi})_D^{1/2}, \quad (\hat{\boldsymbol{w}},\underline{\hat{\boldsymbol{w}}})_{I\!\!D}, \quad \|\hat{\boldsymbol{w}}\|_{I\!\!D} = (\hat{\boldsymbol{w}},\hat{\boldsymbol{w}})_{I\!\!D}^{1/2},$$

for all $\hat{\phi}$, $\underline{\hat{\phi}} \in W^{\frac{1}{2},2}(\partial\mathcal{M}_D,0)$ and all $\hat{\boldsymbol{w}}$, $\underline{\hat{\boldsymbol{w}}} \in W^{\frac{1}{2},2}(\partial\mathcal{M}_{I\!\!D},1)$.

In an analogous way, introduce the trace operators tr_N and $\mathsf{tr}_{I\!\!N}$ and the spaces $W^{s-\frac{1}{p},p}(\mathcal{M}_N,0)$ and $W^{s-\frac{1}{p},p}(\mathcal{M}_{I\!\!N},1)$. We use the notation

$$W_0^{1,p} := \{\phi \in W^{1,p} : \mathsf{tr}_D\phi = 0,\ \mathsf{tr}_N\phi = 0\}, \qquad W_D^{1,p} := \{\phi \in W^{1,p} : \mathsf{tr}_D\phi = 0\},$$
$$\boldsymbol{W}_0^{1,p} := \{\boldsymbol{w} \in W^{1,p} : \mathsf{tr}_{I\!\!D}\boldsymbol{w} = 0,\ \mathsf{tr}_{I\!\!N}\boldsymbol{w} = 0\}, \quad \boldsymbol{W}_{I\!\!D}^{1,p} := \{\boldsymbol{w} \in W^{1,p} : \mathsf{tr}_{I\!\!D}\boldsymbol{w} = 0\},$$

for the function spaces. The dual spaces of some Sobolev spaces will be denoted as follows:

$$W^{-k,p} := \big[W^{k,p'}\big]^*, \quad W_0^{-k,p} := \big[W_0^{k,p'}\big]^*, \quad W_D^{-k,p} := \big[W_D^{k,p'}\big]^*,$$
$$\boldsymbol{W}^{-k,p} := \big[\,\boldsymbol{W}^{k,p'}\big]^*, \quad \boldsymbol{W}_0^{-k,p} := \big[\,\boldsymbol{W}_0^{k,p'}\big]^*, \quad \boldsymbol{W}_{I\!\!D}^{-k,p} := \big[\,\boldsymbol{W}_{I\!\!D}^{k,p'}\big]^*,$$

where we denote by p' the conjugate of p in the sense $\frac{1}{p}+\frac{1}{p'}=1$. These are Banach spaces with the norm

$$\|s^*\|_{-k,p} := \sup_{0\neq r\in W^{k,p'}(\mathcal{M},n)} \frac{|s^*(r)|}{\|r\|_{k,p'}},$$

where $s^* \in W^{-k,p}(\mathcal{M},n)$, and the asterisk is introduced to emphasize that s^* is a linear and bounded map $s^* : W^{k,p'}(\mathcal{M},n) \to \mathbb{R}$. The product of an element in $u \in L^\infty(\mathcal{M},0)$ by an element in $s^* \in W^{-1,p}(\mathcal{M},n)$, with $1<p<\infty$ will be denoted by $(su)^*$. Such element is a well-defined functional in $W^{-1,p}(\mathcal{M},n)$. The proof of this statement is given in the Appendix using appropriate Gelfand triple structures.

We will need order structure in some Sobolev spaces. See the Appendix for a review on ordered Banach spaces, where we explain the particular notation from this field we use in this article, and where we define the main order cones needed in this article: L^∞_+, L^p_+, and $W^{k,p}_+$. We use both notations $(u-v)\in X_+$ and $u \geqslant v$ to state that the element u in a Banach space X is bigger than or equal to another element v in that space. The former notation specifies which Banach space the elements belong to and also which order cone is used in that particular Banach space; while both pieces of information are not explicitly displayed in the latter notation. We use the notation $u \geqslant v$ when there is no ambiguity, otherwise we use the notation $(u-v)\in X_+$. We also write $-(u-v)\in X_+$ to denote $u \leqslant v$.

Given Banach spaces X, Y and an operator $A : D_A \subset X \to R_A \subset Y$, we denote by D_A and R_A the domain and range of A, respectively, while N_A denotes the null space of A.

We recall the **generalized Hölder inequality** (see [22], page 904), which is used in several places in this article, and says that given n-index tensor fields $s_i \in L^{p_i}(\mathcal{M},n)$ with $i=1,\cdots,k$ for $k\in\mathbb{N}$, the pointwise product is well-defined a.e. in $\mathcal{M}$, the tensor field $s_1\cdots s_k \in L^p(\mathcal{M},nk)$, where $p=\sum_{i=1}^k 1/p_i$, and the following estimate holds,

$$\|s_1\cdots s_k\|_p \leqslant \|s_1\|_{p_1}\cdots\|s_k\|_{p_k}.$$

One last comment on the notation: Given a Banach space X, the elements x, $\underline{x}\in X$ denote different elements. This notation usually appears in equations written in weak form, where the element x denotes the trial function and the element $\underline{x}$ denotes the test function.

2. The constraint equations

We give a quick overview of the conformal decomposition, describe the classical strong formulation of the resulting coupled elliptic system, and then define weak formulations of the constraint equations that will allow us to develop solution theories for the constraints in the spaces with the weakest possible regularity. Our formulation allows for a mix of Dirichlet and Robin boundary conditions for modeling e.g. black hole and other physically important scenarios.

2.1. **The conformal decomposition method.** Let $(M, g_{\mu\nu})$ be a smooth 4-dimensional spacetime, that is, M is a 4-dimensional, smooth manifold, and $g_{\mu\nu}$ is a smooth, Lorentzian metric on M with signature $(-,+,+,+)$. Let ∇_μ be the Levi-Civita connection associated with the metric $g_{\mu\nu}$, that is, the unique torsion-free connection satisfying $\nabla_\sigma g_{\mu\nu}=0$. The Einstein equation is

$$G_{\mu\nu}=\kappa T_{\mu\nu},$$

where $G_{\mu\nu}=R_{\mu\nu}-R\,g_{\mu\nu}/2$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor, and $\kappa=8\pi G/c^4$, with G the gravitation constant and c the speed of light. The Ricci tensor is $R_{\mu\nu}=R_{\mu\sigma\nu}{}^{\sigma}$ and $R=R_{\mu\nu}g^{\mu\nu}$ is the Ricci scalar, where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, that is $g_{\mu\sigma}g^{\sigma\nu}=\delta_\mu{}^\nu$. The Ricci tensor is defined as a contraction of the Riemann tensor $R_{\mu\nu\sigma}{}^{\rho}w_\rho=\big(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu\big)w_\sigma$, where w_μ is any 1-form on M. The stress-energy tensor $T_{\mu\nu}$ is assumed to be symmetric, to satisfy the integrability condition $\nabla_\mu T^{\mu\nu} = 0$, and the **dominant energy condition**, that is, the vector $-T^{\mu\nu}v_\nu$ is timelike and future-directed, where v^μ is any timelike and future-directed vector field (see [50], page 219). In this section Greek indices μ, ν, σ, ρ denote abstract spacetime indices, that is, tensorial character on the 4-dimensional manifold M. They are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively. Later on Latin indices a, b, c, d will denote tensorial character on a 3-dimensional manifold.

The map $t : M \to \mathbb{R}$ is a **time function** iff the function t is differentiable and the vector field $-\nabla^\mu t$ is a timelike, future-directed vector field on M. Introduce the hypersurface $\mathcal{M} := \{x \in M : t(x) = 0\}$, and denote by n_μ the unit 1-form orthogonal to vector fields tangent to $\mathcal{M}$. By definition of $\mathcal{M}$ the 1-form n_μ has the form $n_\mu = -\alpha \nabla_\mu t$, where α is a positive function such that $n_\mu n_\nu g^{\mu\nu} = -1$, which is called the **lapse function**. Since the lapse function is positive, the vector field n^μ is future-directed. Let $\hat{h}_{\mu\nu}$ and $\hat{k}_{\mu\nu}$ be the first and second fundamental forms of the hypersurface $\mathcal{M}$, that is,

$$\hat{h}_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu, \qquad \hat{k}_{\mu\nu} := -\hat{h}_\mu{}^\sigma \nabla_\sigma n_\nu.$$

The Einstein constraint equations on $\mathcal{M}$ are given by

$$\big(G_{\mu\nu} - \kappa T_{\mu\nu}\big)\, n^\nu = 0.$$

It is a straightforward albeit long computation to express these equations involving tensors on M as an equation involving tensors on $\mathcal{M}$. The result is the following equations,

$$^3\hat{R} + \hat{k}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \tag{2.1}$$

$$\hat{D}^a\hat{k} - \hat{D}_b\hat{k}^{ab} + \kappa\hat{\jmath}^a = 0, \tag{2.2}$$

where tensors $\hat{h}_{ab}$, $\hat{k}_{ab}$, $\hat{\jmath}^a$ and $\hat{\rho}$ on a 3-dimensional manifold are the pull-back on $\mathcal{M}$ of the tensors $\hat{h}_{\mu\nu}$, $\hat{k}_{\mu\nu}$, $\hat{\jmath}^\mu$ and $\hat{\rho}$ on the 4-dimensional manifold M. We have introduced the energy density $\hat{\rho} := n_\mu n_\mu T^{\mu\nu}$ and the momentum current density $\hat{\jmath}^\mu := -\hat{h}^\mu{}_\nu n_\sigma T^{\nu\sigma}$. We have denoted by $\hat{D}_a$ the Levi-Civita connection associated to $\hat{h}_{ab}$, so $(\mathcal{M}, \hat{h}_{ab})$ is a 3-dimensional Riemannian manifold, with $\hat{h}_{ab}$ having signature $(+,+,+)$, and we use the notation $\hat{h}^{ab}$ for the inverse of the metric $\hat{h}_{ab}$. Indices have been raised and lowered with $\hat{h}^{ab}$ and $\hat{h}_{ab}$, respectively. We have also denoted by $^3\hat{R}$ the Ricci scalar of curvature of the metric $\hat{h}_{ab}$. Finally, recall that the constraint Eqs. (2.1)-(2.2) are indeed equations on $\hat{h}_{ab}$ and $\hat{k}_{ab}$ due to the matter fields satisfying the energy condition $-\hat{\rho}^2 + \hat{\jmath}_a\hat{\jmath}^a < 0$, which is implied by the dominant energy condition on the stress-energy tensor $T^{\mu\nu}$ in spacetime.

Let ϕ be a positive scalar field on $\mathcal{M}$, and decompose the extrinsic curvature tensor $\hat{k}_{ab} = \hat{s}_{ab} + \hat{h}_{ab}\hat{\tau}/3$, where $\hat{\tau} := \hat{k}_{ab}\hat{h}^{ab}$ is the trace and then $\hat{s}_{ab}$ is the traceless part of the extrinsic curvature tensor. Then, introduce now the following conformal rescaling:

$$\hat{h}_{ab} =: \phi^4\, h_{ab}, \qquad \hat{s}^{ab} =: \phi^{-10}\, s^{ab}, \qquad \hat{\tau} =: \tau, \tag{2.3}$$

$$\hat{\jmath}^a =: \phi^{-10}\, j^a, \qquad \hat{\rho} =: \phi^{-8}\, \rho. \tag{2.4}$$

We have introduced the Riemannian metric h_{ab} on the 3-dimensional manifold $\mathcal{M}$, which determines the Levi-Civita connection D_a, and so we have that $D_a h_{bc} = 0$. We have also introduced the symmetric, traceless tensor s_{ab}, and the non-physical matter sources j^a and ρ. The different powers of the conformal rescaling above are carefully chosen so that the constraint Eqs. (2.1)-(2.2) transform into the following equations

$$-8\Delta\phi + {}^3R\phi + \frac{2}{3}\,\tau^2\phi^5 - s_{ab}s^{ab}\,\phi^{-7} - 2\kappa\rho\,\phi^{-3} = 0, \tag{2.5}$$

$$-D_b s^{ab} + \frac{2}{3}\phi^6 D^a\tau + \kappa j^a = 0, \tag{2.6}$$

where in equation above, and from now on, indices of unhatted fields are raised and lowered with h^{ab} and h_{ab} respectively. We have also introduced the **Laplace-Beltrami** operator with respect to the metric h_{ab}, acting on smooth scalar fields; it is defined as follows

$$\Delta\phi := h^{ab}D_aD_b\phi.$$

Eqs. (2.5)-(2.6) can be obtained by a long, but otherwise straightforward computation. In order to perform this calculation it is useful to recall that both $\hat{D}_a$ and D_a are connections on the manifold $\mathcal{M}$, and so they differ on a tensor field $C_{ab}{}^c$, which can be computed explicitly in terms of ϕ, and has the form

$$C_{ab}{}^c = 4\delta_{(a}{}^c D_{b)}\ln(\phi) - 2h_{ab}h^{cd}D_d\ln(\phi).$$

We remark that the power four on the rescaling of the metric $\hat{h}_{ab}$ and $\mathcal{M}$ being 3-dimensional imply that ${}^3\hat{R} = \phi^{-5}({}^3R\phi - 8\hat{\Delta}\phi)$, and for any other power in the rescaling, terms proportional to $h^{ab}(D_a\phi)(D_b\phi)/\phi^2$ appear in the transformation. Similar reasons force the power negative ten on the rescaling of the tensor $\hat{s}^{ab}$ and $\hat{\jmath}^a$, so terms proportional to $(D_a\phi)/\phi$ cancel out in Eq. (2.6). Finally, the ratio between the conformal rescaling powers of $\hat{\rho}$ and $\hat{\jmath}^a$ is chosen such that the inequality $-\rho^2 + h_{ab}j^aj^b < 0$ implies the inequality $-\hat{\rho}^2 + \hat{h}_{ab}\hat{\jmath}^a\hat{\jmath}^b < 0$.

There is one more step to convert the original constraint Eq. (2.1)-(2.2) into a determined elliptic system of equations. This step is the following: Decompose the symmetric, traceless tensor s_{ab} into a divergence-free part σ_{ab}, and the symmetrized and traceless gradient of a vector, that is, $s^{ab} =: \sigma^{ab} + (\mathcal{L}w)^{ab}$, where $D_a\sigma^{ab} = 0$ and we have introduced the **conformal Killing** operator $\mathcal{L}$ acting on smooth vector fields and defined as follows

$$(\mathcal{L}w)^{ab} := D^aw^b + D^bw^a - \frac{2}{3}\,(D_cw^c)\,h^{ab}. \tag{2.7}$$

Therefore, the constraint Eqs. (2.1)-(2.2) are transformed by the conformal rescaling into the following equations

$$-8\Delta\phi + {}^3R\phi + \frac{2}{3}\,\tau^2\phi^5 - \big(\sigma_{ab} + (\mathcal{L}w)_{ab}\big)\big(\sigma^{ab} + (\mathcal{L}w)^{ab}\big)\,\phi^{-7} - 2\kappa\rho\,\phi^{-3} = 0, \tag{2.8}$$

$$-D_b(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6 D^a\tau + \kappa j^a = 0. \tag{2.9}$$

In the next section we interpret these equations above as partial differential equations for the scalar field ϕ and the vector field w^a, while the rest of the fields are considered as given fields. Given a solution ϕ and w^a of Eqs. (2.8)-(2.9), the physical metric $\hat{h}_{ab}$ and extrinsic curvature $\hat{k}^{ab}$ of the hypersurface $\mathcal{M}$ are given by

$$\hat{h}_{ab} = \phi^4\,h_{ab}, \qquad \hat{k}^{ab} = \phi^{-10}\big[\sigma^{ab} + (\mathcal{L}w)^{ab}\big] + \frac{1}{3}\,\phi^{-4}\,\tau\,h^{ab},$$

while the matter fields are given by Eq (2.4).

2.2. **Classical formulation.** Beginning in this section, we will change the notation slightly from the classical notation used to introduce the conformal method in §2.1. In particular, the Levi-Civita connection of the metric h_{ab} on the 3-dimensional manifold $\mathcal{M}$ will denoted by ∇_a rather than D_a, and the Ricci scalar of h_{ab} will be denoted by R instead of ${}^3R$. This change will simplify the presentation in the remainder of the paper.

Let $(\mathcal{M}, h)$ be a 3-dimensional Riemannian manifold, where $\mathcal{M}$ is a smooth, compact manifold with smooth boundary $\partial\mathcal{M}$, and $h \in C^\infty(\overline{\mathcal{M}}, 2)$ is a positive definite metric. Let $L : C^\infty \to C^\infty$ and $I\!\!L : \boldsymbol{C}^\infty \to \boldsymbol{C}^\infty$ be the **Laplace-Beltrami** and **momentum** operators, respectively, with actions on a scalar field $\phi \in C^\infty$ and a vector field $\boldsymbol{w} \in \boldsymbol{C}^\infty$ given by

$$L\phi := -\Delta\phi, \tag{2.10}$$

$$(I\!\!L\boldsymbol{w})^a := -\nabla_b(\mathcal{L}\boldsymbol{w})^{ab}, \tag{2.11}$$

where $\Delta\phi := \nabla_a\nabla^a\phi$, and $\mathcal{L}$ denotes the conformal Killing operator defined in Eq. (2.7). We will also use the index-free notation $I\!\!L\boldsymbol{w}$ and $\mathcal{L}\boldsymbol{w}$. Assume that $\partial\mathcal{M}$ is divided according to Eqs. (1.7)-(1.8). Then, consider boundary conditions for the scalar equation of Dirichlet type on $\partial\mathcal{M}_D$ and of Robin type on $\partial\mathcal{M}_N$. Analogously, consider boundary conditions for the vector equation of Dirichlet type on $\partial\mathcal{M}_{I\!\!D}$ and of Robin type on $\partial\mathcal{M}_{I\!\!N}$. The conditions

$$\overline{\partial\mathcal{M}}_D \cap \overline{\partial\mathcal{M}}_N = \emptyset, \quad \overline{\partial\mathcal{M}}_{I\!\!D} \cap \overline{\partial\mathcal{M}}_{I\!\!N} = \emptyset,$$

are needed to simplify the proofs regarding the regularity at the intersection points of the two types of boundaries of solutions of elliptic equations with Dirichlet-Robin boundary conditions. In what follows we include the cases given by $\partial\mathcal{M}_D = \emptyset$ or $\partial\mathcal{M}_N = \emptyset$, and $\partial\mathcal{M}_{I\!\!D} = \emptyset$ or $\partial\mathcal{M}_{I\!\!N} = \emptyset$.

The freely specifiable functions of the problem are a scalar function τ, interpreted as the trace of the physical extrinsic curvature; a symmetric, traceless, and divergence-free, contravariant, two-index tensor σ; the non-physical energy density ρ and the non-physical momentum current density vector $\boldsymbol{j}$ subject to the requirement $-\rho^2 + h_{ab}j^aj^b < 0$. The term non-physical refers here to a conformal rescaled field, while physical refers to a conformally non-rescaled field. The requirement on ρ and $\boldsymbol{j}$ mentioned above and the particular conformal rescaling used in the semi-decoupling decomposition imply that the same inequality is satisfied by the physical energy and momentum current densities. Introduce the nonlinear operators $F : C^\infty \times \boldsymbol{C}^\infty \to C^\infty$ and $I\!\!F : C^\infty \to \boldsymbol{C}^\infty$ given by

$$F(\phi, \boldsymbol{w}) := a_\tau\phi^5 + a_R\phi - a_\rho\phi^{-3} - a_w\phi^{-7}, \tag{2.12}$$

$$I\!\!F(\phi) := \boldsymbol{b}_\tau\,\phi^6 + \boldsymbol{b}_j, \tag{2.13}$$

where the coefficient functions are defined as follows

$$a_\tau := \frac{\tau^2}{12}, \qquad a_R = \frac{R}{8}, \qquad a_\rho := \frac{\kappa}{4}\rho, \tag{2.14}$$

$$a_w := \frac{1}{8}\,(\sigma + \mathcal{L}\boldsymbol{w})_{ab}(\sigma + \mathcal{L}\boldsymbol{w})^{ab}, \qquad b_\tau^a := \frac{2}{3}\,\nabla^a\tau, \qquad b_j^a := \kappa j^a. \tag{2.15}$$

Notice that the scalar coefficients a_τ, a_w, and a_ρ are non-negative, while there is no sign restriction on a_R.

The **classical Dirichlet-Robin boundary value formulation** for the semi-decoupling Einstein constraint equations is the following: Given the freely specifiable smooth fields τ, σ, ρ, and $\boldsymbol{j}$ in $\mathcal{M}$ and the smooth Dirichlet boundary data $\hat{\phi}_D$ on $\partial\mathcal{M}_D$ and $\hat{\boldsymbol{w}}_{I\!\!D}$ on $\partial\mathcal{M}_{I\!\!D}$, and smooth Robin boundary data $\hat{\phi}_N$, K, scalar fields on $\partial\mathcal{M}_N$ and $\hat{\boldsymbol{w}}_{I\!\!N}$, $I\!\!K$, a vector and a two-index tensor fields on $\partial\mathcal{M}_{I\!\!N}$, find a scalar field ϕ and a vector field $\boldsymbol{w}$ in $\mathcal{M}$ solution of the system

$$L\phi + F(\phi,\boldsymbol{w}) = 0 \text{ in } \mathcal{M}, \qquad \begin{cases} \phi = \hat{\phi}_D \text{ on } \partial\mathcal{M}_D, \\ \boldsymbol{n}\cdot\nabla\phi + K\phi = \hat{\phi}_N \text{ on } \partial\mathcal{M}_N, \end{cases} \tag{2.16}$$

$$I\!\!L\boldsymbol{w} + I\!\!F(\phi) = 0 \text{ in } \mathcal{M}, \qquad \begin{cases} \boldsymbol{w} = \hat{\boldsymbol{w}}_{I\!\!D} \text{ on } \partial\mathcal{M}_{I\!\!D}, \\ \boldsymbol{n}\cdot\nabla\boldsymbol{w} + I\!\!K\boldsymbol{w} = \hat{\boldsymbol{w}}_{I\!\!N} \text{ on } \partial\mathcal{M}_{I\!\!N}, \end{cases} \tag{2.17}$$

where $I\!\!K\boldsymbol{w}$ denotes the vector field $(I\!\!K\boldsymbol{w})^a = I\!\!K^{ab}w_b$, and $\boldsymbol{n}\cdot\nabla\boldsymbol{w}$ denotes the vector $n^b\nabla_b w^a$.

The classical formulation has been done on spaces of smooth fields, which are not complete spaces under any known norm defined on them. This is inconvenient for finding solutions to PDE, because these solutions are usually found as limits of appropriate approximations. If a normed vector space is not complete, then a Cauchy sequence may not converge. In the next section we introduce the weak formulation of the equations above, where we rewrite the classical formulation in appropriate normed vector spaces which are also complete.

2.3. **Weak formulation.** We present the weak formulation associated with the classical formulation with Eqs. (2.16)-(2.17). We introduce one of the weakest forms of the constraint equations, that is, we assume the weakest regularity of the equation coefficients such that the equation itself is well-defined. We will be able to obtain existence and uniqueness results for the momentum constraint using either variational methods in §3.3 or Riesz-Schauder methods in §3.4. We will also be able to obtain existence (and when possible, uniqueness) for the Hamiltonian constraint in §4.5 using variational methods in this weakest setting. However, the barrier-based existence and uniqueness results for the Hamiltonian constraint equation in §4.6, and the compactness argument in §5.1 giving existence for the coupled system of constraints, require higher regularity on the equation coefficients, but we still obtain some non-CMC results for the coupled system in weaker settings and in more general physical situations than have been previously obtained. These additional assumptions are clearly stated in those sections.

Let $(\mathcal{M},h)$ be a 3-dimensional Riemannian manifold, where $\mathcal{M}$ is a smooth, compact manifold with Lipschitz boundary $\partial\mathcal{M}$, and $h\in C^2(\overline{\mathcal{M}},2)$ is a positive definite metric. Introduce the bilinear forms

$$a_L : W^{1,2}\times W^{1,2}\to\mathbb{R}, \qquad a_L(\phi,\underline{\phi}) := (\nabla\phi,\nabla\underline{\phi}) + (K\,\mathsf{tr}_N\phi,\mathsf{tr}_N\underline{\phi})_N, \tag{2.18}$$

$$a_{I\!\!L} : \boldsymbol{W}^{1,2}\times\boldsymbol{W}^{1,2}\to\mathbb{R}, \quad a_{I\!\!L}(\boldsymbol{w},\underline{\boldsymbol{w}}) := (\mathcal{L}\boldsymbol{w},\mathcal{L}\underline{\boldsymbol{w}}) + (I\!\!K\,\mathsf{tr}_{I\!\!N}\boldsymbol{w},\mathsf{tr}_{I\!\!N}\underline{\boldsymbol{w}})_{I\!\!N}, \tag{2.19}$$

where the Robin scalar field $K\in L^\infty(\partial\mathcal{M}_N,0)$ and the two-index, symmetric tensor field $I\!\!K\in L^\infty(\partial\mathcal{M}_{I\!\!N},2)$ satisfy the bounds

$$\hat{\mathtt{k}}\,\|\mathsf{tr}_N\phi\|_N^2 \leqslant (K\mathsf{tr}_N\phi,\mathsf{tr}_N\phi)_N, \qquad \forall\phi\in W^{1,2}, \tag{2.20}$$

$$\hat{\mathtt{K}}\,\|\mathsf{tr}_{I\!\!N}\boldsymbol{w}\|_{I\!\!N}^2 \leqslant (I\!\!K\mathsf{tr}_{I\!\!N}\boldsymbol{w},\mathsf{tr}_{I\!\!N}\boldsymbol{w})_{I\!\!N}, \qquad \forall\boldsymbol{w}\in\boldsymbol{W}^{1,2}, \tag{2.21}$$

with $\hat{\mathtt{k}}$ and $\hat{\mathtt{K}}$ being non-negative constants. In this Section set the number $p = 12/5$, and then fix the source functions

$$\tau \in L^p, \quad \rho^* \in W^{-1,p}_{D+}, \quad \sigma \in L^p(\mathcal{M}, 2), \quad \boldsymbol{j}^* \in \boldsymbol{W}^{-1,p}_{I\!\!D}, \tag{2.22}$$

where σ is symmetric, traceless and divergence-free in weak sense, that is, it satisfies $(\sigma, \mathcal{L}\underline{\boldsymbol{\omega}}) = 0$ for all $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_0$. The asterisk on the matter fields is to emphasize that they are elements of spaces of linear functionals. That is, $\rho^* : W^{1,p'}_D \to \mathbb{R}$ is linear and bounded, and an analogous definition holds for $\boldsymbol{j}^*$. In the Appendix it is shown that the spaces $W^{1,q'}_D \subset L^2 \equiv [L^2]^* \subset W^{-1,q}_D$, with $1 < q < \infty$, form a Gelfand triple, so given any element $\rho^* \in W^{-1,p}_{D+}$ there exists a sequence $\{\rho_n\} \subset L^2$ such that

$$\rho^*(\underline{\varphi}) := \lim_{n\to\infty} (\rho_n, \underline{\varphi}), \qquad \forall\, \underline{\varphi} \in W^{1,p'}_D.$$

An analogous statement holds for $\boldsymbol{j}^*$. We say that the matter fields ρ^* and $\boldsymbol{j}^*$ satisfy the **energy condition in weak sense** iff there exist sequences $\{\rho_n\} \subset L^2$ and $\{\boldsymbol{j}_n\} \subset \boldsymbol{L}^2$ such that

$$[\rho_n^2 - \boldsymbol{j}_n \cdot \boldsymbol{j}_n] \in L^1_+ \qquad \forall\, n \in \mathbb{N}. \tag{2.23}$$

(We have required $\rho^* \in W^{-1,p}_{D+}$ in Eq. (2.22) instead of $\rho^* \in W^{-1,p/2}_{D+}$ because $\boldsymbol{j}^*$ must belong to $\boldsymbol{W}^{-1,p}_{I\!\!D}$ and Eq. (2.23) must hold.) Given any function $\tau \in L^p$, then it is known that $(\nabla\tau)^* \in \boldsymbol{W}^{-1,p}_0$, where the asterisk, we repeat for the last time, is added only to reinforce the idea that $(\nabla\tau)^*$ is a linear functional on elements in $W^{1,p'}_0$, and it is not meant to indicate the adjoint operator of ∇. We assume here that $(\nabla\tau)^* \in \boldsymbol{W}^{-1,p}_{I\!\!D}$, which is indeed an extra assumption due to $\boldsymbol{W}^{-1,p}_{I\!\!D} \subset \boldsymbol{W}^{-1,p}_0$; this assumption is needed because the functional $(b^a_\tau)^* := (2/3)(\nabla^a\tau)^*$ is the source in the momentum constraint equation, which requires this particular type of boundary conditions. The assumptions above on τ and σ imply that for every $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ the functions a_τ and a_w belong to $L^{p/2}$. The assumption on the background metric implies that a_R is a continuous function on $\overline{\mathcal{M}}$. These three functions define the elements a^*_τ, a^*_R and a^*_w in the space $W^{-1,2}_D$ as follows

$$a^*_\tau(\underline{\varphi}) := (a_\tau, \underline{\varphi}), \quad a^*_R(\underline{\varphi}) := (a_R, \underline{\varphi}), \quad a^*_w(\underline{\varphi}) := (a_w, \underline{\varphi}), \quad \forall\, \underline{\varphi} \in W^{1,2}_D. \tag{2.24}$$

The proof that these functionals are well-defined is based on Hölder inequality, for example, consider the functional a^*_τ, then

$$|(a_\tau, \underline{\varphi})| \leqslant \|a_\tau\|_{\frac{6}{5}}\, \|\underline{\varphi}\|_6 \leqslant c_s\, \|a_\tau\|_{\frac{6}{5}}\, \|\underline{\varphi}\|_{1,2}.$$

These functionals above belong to a particular class of elements in $W^{-1,2}_D$, while the functionals $a^*_\rho := (\kappa/4)\rho^*$ and $\boldsymbol{b}^*_j := \kappa\boldsymbol{j}^*$ are not restricted to such a particular form, and they can be any element in $W^{-1,p}_{D+}$ and $\boldsymbol{W}^{-1,p}_{I\!\!D}$, respectively compatible with the energy condition. Given any two functions ϕ_1, $\phi_2 \in L^\infty$ with $\phi_1 \leqslant \phi_2$, define the interval

$$[\phi_1, \phi_2] := \{\phi \in L^\infty : \phi_1 \leqslant \phi \leqslant \phi_2\} \subset L^\infty,$$

which is a closed, bounded set in L^∞. Assume $\phi_1 > 0$, and then introduce the nonlinear operators

$$f_F : [\phi_1, \phi_2] \subset L^2 \times \boldsymbol{W}^{1,p} \to W^{-1,2}_D, \qquad \boldsymbol{f}_{I\!\!F} : [\phi_1, \phi_2] \subset L^2 \to \boldsymbol{W}^{-1,p}_{I\!\!D},$$

$$f_F(\phi, \boldsymbol{w}) := (a_\tau \phi^5)^* + (a_R \phi)^* - (a_\rho \phi^{-3})^* - (a_w \phi^{-7})^*, \tag{2.25}$$
$$\boldsymbol{f}_{I\!\!F}(\phi) := (\boldsymbol{b}_\tau \phi^6)^* + \boldsymbol{b}_j^*, \tag{2.26}$$

where the product of an element $\phi \in L^\infty$ by an element in $a^* \in W^{-1,q}$, with $1 < q < \infty$, denoted by $(a\phi)^*$, is a well-defined element in $W^{-1,q}$. The proof of this statement is given in the Appendix using appropriate Gelfand triple structures. The functionals f_F and $\boldsymbol{f}_{I\!\!F}$ are the generalizations of the functionals F and $I\!\!F$ defined in Eq. (2.12)-(2.13). We remark that the operators defined in Eqs. (2.25)-(2.26) are continuous but *not Gâteaux differentiable*. They have Gâteaux derivatives only along directions in L^∞, not on the whole space L^2. This fact introduces some technical complexity with the use of variational methods for the individual Hamiltonian and momentum constraints (see §3.3 and §4.5). Recall that the trace operators

$$\mathsf{tr}_D : W^{1,2} \to W^{\frac{1}{2},2}(\partial\mathcal{M}_D, 0), \qquad \mathsf{tr}_{I\!\!D} : \boldsymbol{W}^{1,p} \to W^{\frac{1}{p'},p}(\partial\mathcal{M}_{I\!\!D}, 1),$$

satisfy the following property: given any element $\hat{\phi}_D \in W^{\frac{1}{2},2}(\partial\mathcal{M}_D, 0)$, there exists an element $\phi_D \in W^{1,2}$ such that $\mathsf{tr}_D \phi_D = \hat{\phi}_D$; analogously, given a boundary data element $\hat{\boldsymbol{w}}_{I\!\!D} \in W^{\frac{1}{p'},p}(\partial\mathcal{M}_{I\!\!D}, 1)$, there exists an element $\boldsymbol{w}_{I\!\!D} \in \boldsymbol{W}^{1,p}$ such that $\mathsf{tr}_{I\!\!D} \boldsymbol{w}_{I\!\!D} = \hat{\boldsymbol{w}}_{I\!\!D}$. The elements ϕ_D and $\boldsymbol{w}_{I\!\!D}$ are called here extensions of $\hat{\phi}_D$ and $\hat{\boldsymbol{w}}_{I\!\!D}$, respectively. They are not uniquely determined by the boundary data.

The **weak Dirichlet-Robin boundary value formulation** which is associated with Eqs. (2.16)-(2.17) is the following: Fix Dirichlet boundary data

$$0 < \operatorname*{ess\,inf}_{\partial\mathcal{M}_D} \hat{\phi}_D \leqslant \hat{\phi}_D \in L^\infty(\partial\mathcal{M}_D, 0) \cap W^{\frac{1}{2},2}(\partial\mathcal{M}_D, 0), \quad \hat{\boldsymbol{w}}_{I\!\!D} \in W^{\frac{1}{p'},p}(\partial\mathcal{M}_{I\!\!D}, 1),$$

with extensions $\inf_{\partial\mathcal{M}_D} \hat{\phi}_D \leqslant \phi_D \in W^{1,2}$ and $\boldsymbol{w}_{I\!\!D} \in \boldsymbol{W}^{1,p}$, respectively. Choose the extension function ϕ_D as a harmonic extension of the Dirichlet boundary data using the Laplace-Beltrami operator on $\mathcal{M}$:

$$\Delta\phi_D = 0 \text{ in } \mathcal{M}, \quad \mathsf{tr}_D \phi_D = \hat{\phi}_D > 0 \text{ on } \partial\mathcal{M}_D, \quad \mathsf{tr}_N \phi_D = \inf_{\partial\mathcal{M}_D} \hat{\phi}_D > 0 \text{ on } \partial\mathcal{M}_N.$$

The maximum principle for the Laplace-Beltrami operator [4] implies that $0 < \inf_{\partial\mathcal{M}} \hat{\phi}_D \leqslant \phi_D(x)$, a.e. in $\mathcal{M}$. Fix Robin boundary data functionals

$$\hat{\phi}_N^* \in W^{-\frac{1}{2},2}(\partial\mathcal{M}_N, 0), \qquad \hat{\boldsymbol{w}}_{I\!\!N}^* \in W^{-\frac{1}{p},p}(\partial\mathcal{M}_{I\!\!N}, 1);$$

Given the extension ϕ_D of the Dirichlet data $\hat{\phi}_D$ chosen above, fix any two functions $\phi_1,\ \phi_2 \in L^\infty \cap W^{1,2}$, with the property that $0 < \phi_1 \leqslant \phi_2$ and satisfying $\phi_D \in [\phi_1, \phi_2] \cap W^{1,2}$. Introduce the non-principal part operators including the Robin boundary conditions,

$$f : [\phi_1, \phi_2] \subset L^2 \times \boldsymbol{W}^{1,p} \to W_D^{-1,2}, \quad f(\phi, \boldsymbol{w})(\underline{\varphi}) := f_F(\phi, \boldsymbol{w})(\underline{\varphi}) - \hat{\phi}_N^*(\mathsf{tr}_N \underline{\varphi}), \tag{2.27}$$

$$\boldsymbol{f} : [\phi_1, \phi_2] \subset L^2 \to \boldsymbol{W}_{I\!\!D}^{-1,p}, \qquad \boldsymbol{f}(\phi)(\underline{\boldsymbol{\omega}}) := \boldsymbol{f}_{I\!\!F}(\phi)(\boldsymbol{\omega}) - \hat{\boldsymbol{w}}_{I\!\!N}^*(\mathsf{tr}_{I\!\!N} \underline{\boldsymbol{\omega}}), \tag{2.28}$$

where f_F and $\boldsymbol{f}_{I\!\!F}$ are given by Eqs. (2.25)-(2.26). Introduce the affine spaces $A^{1,2}$ and $\boldsymbol{A}^{1,p}$, which include the Dirichlet boundary conditions, as follows,

$$A^{1,2} := \phi_D + W_D^{1,2} := \{\phi \in W^{1,2} : \phi - \phi_D \in W_D^{1,2}\}, \tag{2.29}$$
$$\boldsymbol{A}^{1,p} := \boldsymbol{w}_{I\!\!D} + \boldsymbol{W}_{I\!\!D}^{1,p} := \{\boldsymbol{w} \in \boldsymbol{W}^{1,p} : \boldsymbol{w} - \boldsymbol{w}_{I\!\!D} \in \boldsymbol{W}_{I\!\!D}^{1,p}\}. \tag{2.30}$$

Then, find elements $\phi \in [\phi_1, \phi_2] \cap A^{1,2}$ and $\boldsymbol{w} \in \boldsymbol{A}^{1,p}$ solutions of

$$a_L(\phi, \underline{\varphi}) + f(\phi, \boldsymbol{w})(\underline{\varphi}) = 0 \qquad \forall\, \underline{\varphi} \in W_D^{1,2}, \tag{2.31}$$

$$a_{I\!L}(\boldsymbol{w}, \underline{\boldsymbol{\omega}}) + \boldsymbol{f}(\phi)(\underline{\boldsymbol{\omega}}) = 0 \qquad \forall\, \underline{\boldsymbol{\omega}} \in \boldsymbol{W}_{I\!D}^{1,p'}. \tag{2.32}$$

It will be convenient later on to express Eqs. (2.31)-(2.32) in terms of operators instead of bilinear forms. Introduce the operators

$$A_L : W^{1,2} \to W_D^{-1,2}, \qquad A_L\phi(\underline{\varphi}) := a_L(\phi, \underline{\varphi}), \tag{2.33}$$

$$A_{I\!L} : \boldsymbol{W}^{1,p} \to \boldsymbol{W}_{I\!D}^{-1,p}, \qquad A_{I\!L}\boldsymbol{w}(\underline{\boldsymbol{\omega}}) := a_{I\!L}(\boldsymbol{w}, \underline{\boldsymbol{\omega}}). \tag{2.34}$$

Also recall that given any $\phi \in [\phi_1, \phi_2]$ and $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ then $f(\phi, \boldsymbol{w}) \in W_D^{-1,2}$ and $\boldsymbol{f}(\phi) \in \boldsymbol{W}_{I\!D}^{-1,p}$. Hence, the Eqs. (2.31)-(2.32) written in terms of operators is the following: Find elements $\phi \in [\phi_1, \phi_2] \cap A^{1,2}$ and $\boldsymbol{w} \in \boldsymbol{A}^{1,p}$ solutions of

$$A_L\phi + f(\phi, \boldsymbol{w}) = 0, \tag{2.35}$$

$$A_{I\!L}\boldsymbol{w} + \boldsymbol{f}(\phi) = 0. \tag{2.36}$$

Lemma 1. *Every smooth solution ϕ, $\boldsymbol{w}$ of the classical problem with Eqs. (2.16)-(2.17) is also a solution of the weak problem with Eqs. (2.31)-(2.32).*

Proof. *(Lemma 1.)* Given any smooth fields ϕ, $\boldsymbol{w}$ solutions of Eqs. (2.16)-(2.17), then the proof consists in multiplying these equations by test functions $\underline{\varphi} \in W_D^{1,2}$ and $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}_{I\!D}^{1,p'}$, respectively, and then integrating by parts. In the case of Eq. (2.16) one gets

$$(-\Delta\phi, \underline{\varphi}) + \big(F(\phi, \boldsymbol{w}), \underline{\varphi}\big) = 0. \tag{2.37}$$

The first term on the left hand side can be rewritten as follows,

$$\begin{aligned}
\big(-\Delta\phi, \underline{\varphi}\big) &= (\nabla\phi, \nabla\underline{\varphi}) - \big(\mathsf{tr}_N(\boldsymbol{n}\cdot\nabla\phi), \mathsf{tr}_N\underline{\varphi}\big)_N \\
&= (\nabla\phi, \nabla\underline{\varphi}) + \big([K\,\mathsf{tr}_N\phi - \hat{\phi}_N], \mathsf{tr}_N\underline{\varphi}\big)_N \\
&= a_L(\phi, \underline{\varphi}) - (\hat{\phi}_N, \mathsf{tr}_N\underline{\varphi})_N,
\end{aligned}$$

which holds for all $\underline{\varphi} \in W_D^{1,2}$ where in the second line we introduce the Robin boundary condition on $\partial\mathcal{M}_N$, and note the integral on $\partial\mathcal{M}_D$ vanishes because the test function $\underline{\varphi}$ vanishes on this part of the boundary, in the third line we introduce the definition of the bilinear form a_L. Now, replace this expression into Eq. (2.37) and one obtains Eq. (2.31). Finally, since ϕ is a solution of the classical problem, it can be written as $\phi = \phi_D + \varphi$ for some smooth extension ϕ_D of the boundary data $\hat{\phi}_D$, therefore $\phi \in A^{1,2}$. (Again, ϕ_D can be constructed e.g. by harmonic extension.) In the case of Eq. (2.17) one gets

$$\big(-\nabla\cdot(\mathcal{L}\boldsymbol{w}), \underline{\boldsymbol{\omega}}\big) + \big(I\!F(\phi), \underline{\boldsymbol{\omega}}\big) = 0. \tag{2.38}$$

The first term on the left hand side can be rewritten as follows,

$$\begin{aligned}
\big(-\nabla\cdot(\mathcal{L}\boldsymbol{w}), \underline{\boldsymbol{\omega}}\big) &= \big(\mathcal{L}\boldsymbol{w}, \nabla\underline{\boldsymbol{\omega}}\big) - \big(\mathsf{tr}_{I\!N}[\boldsymbol{n}\cdot(\mathcal{L}\boldsymbol{w})], \mathsf{tr}_{I\!N}\underline{\boldsymbol{\omega}}\big)_{I\!N} \\
&= \big(\mathcal{L}\boldsymbol{w}, \mathcal{L}\underline{\boldsymbol{\omega}}\big) + \big([I\!K\,\mathsf{tr}_{I\!N}\boldsymbol{w} - \hat{\boldsymbol{w}}_{I\!N}], \mathsf{tr}_{I\!N}\underline{\boldsymbol{\omega}}\big)_{I\!N} \\
&= a_{I\!L}(\boldsymbol{w}, \underline{\boldsymbol{\omega}}) - \big(\hat{\boldsymbol{w}}_{I\!N}, \mathsf{tr}_{I\!N}\underline{\boldsymbol{\omega}}\big)_{I\!N},
\end{aligned}$$

which holds for all $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}_{I\!D}^{1,p'}$, where the first term in the second line comes from the symmetries of $\mathcal{L}$, and the second term in that line comes from the Robin boundary conditions; the definition of $a_{I\!L}$ is used to obtain the third line. Now, replace this expression into Eq. (2.38) and one obtains Eq. (2.32). Finally, since $\boldsymbol{w}$ is a solution of classical problem, it can be written as $\boldsymbol{w} = \boldsymbol{w}_{I\!D} + \boldsymbol{\omega}$ for some smooth extension $\boldsymbol{w}_{I\!D}$ of the boundary data $\hat{\boldsymbol{w}}_{I\!D}$, therefore $\boldsymbol{w} \in \boldsymbol{A}^{1,p}$. □

Let us recall here that the space $W^{1,2}_D$ is an ordered Banach space with order cone $W^{1,2}_{D+}$ defined as follows:

$$W^{1,2}_{D+} := \{\phi \in W^{1,2}_D : \phi \geqslant 0 \text{ a.e. in } \mathcal{M}\}.$$

The order relation is then $\phi \geqslant \underline{\phi}$ iff $\phi - \underline{\phi} \in W^{1,2}_{D+}$. In the Appendix we discuss the main properties of ordered Banach spaces. In particular, we show that the order structure implied by $W^{1,2}_{D+}$ can be translated to the dual space $W^{-1,2}_D$ as follows,

$$W^{-1,2}_{D+} := \big\{\phi^* \in W^{-1,2}_D : \phi^*(\underline{\phi}) \geqslant 0 \quad \forall\, \underline{\phi} \in W^{1,2}_{D+}\big\}.$$

Given two ordered Banach spaces X, X_+ and Y, Y_+ an operator $A : D_A \subset X \to Y$ satisfies the **maximum principle** iff for every elements u, $v \in D_A$ such that $Au - Av \in Y_+$ it holds that $u - v \in X_+$. In the particular case that the operator (A, D_A) is linear, then it satisfies the maximum principle iff for every element $u \in X_+$ such that $Au \in Y_+$ it holds that $u \in X_+$. If an operator A satisfies the maximum principle and is invertible, then the inverse is a monotone increasing operator, a result shown in the Appendix. This last property is useful to solve nonlinear equations of the form $Au = f(u)$, in the case that there exist sub- and super-solutions to that equation (see below for the definition). In this case there is a well-known existence proof technique that works for many equations of this type, and has been one of the main techniques used previously for the Hamiltonian constraint [6, 29, 30, 38]. While we will exploit the fact that the construction of sub- and super-solutions can be done in a very weak setting, the use of the existence proof based directly on barriers requires additional regularity beyond what is needed for the barrier construction. This additional regularity assumption can be avoided by combining barriers with variational techniques, which we do in §4.5.

The following properties are of interest to us below. Firstly, in the Appendix we review results from the literature showing that the operator A_L defined above satisfies a maximum principle. Secondly, we can show that there exist sub- and super-solutions to Eq. (2.35). Given any function $u \in W^{1,2}$, introduce the notation

$$u^+ := \text{ess max}\{u, 0\}, \qquad u^- := -\text{ess min}\{u, 0\}.$$

An element $\phi_- \in W^{1,2}$ is called a **sub-solution** of Eq. (2.35) iff the function ϕ_- satisfies the inequalities

$$(\phi_D - \phi_-)^- \in W^{1,2}_D \text{ and } -\big[A_L\phi_- + f(\phi_-, \boldsymbol{w})\big] \in W^{-1,2}_{D+}. \tag{2.39}$$

An element $\phi_+ \in W^{1,2}$ is called a **super-solution** of Eq. (2.35) iff the scalar function ϕ_+ satisfies the inequalities

$$(\phi_D - \phi_+)^+ \in W^{1,2}_D \text{ and } \big[A_L\phi_+ + f(\phi_+, \boldsymbol{w})\big] \in W^{-1,2}_{D+}. \tag{2.40}$$

The sub and super-solutions of Eq. (2.35) may depend on the choice of the vector field $\boldsymbol{w}$ that appears in the functional a^*_w. A sub-solution ϕ_- of Eq. (2.35) is called **global** iff Eq. (2.39) holds for every vector field $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ solution of the momentum constraint Eq. (2.36) with any source function ϕ satisfying $(\phi - \phi_-) \in W^{1,2}_{D+}$, and it is called **local** iff it is not global. Analogous definitions are introduced for super-solutions. While it will be sufficient to derive only local sub- and super-solutions to produce the existence and uniqueness results for the Hamiltonian constraint using variational methods in §4.5 and using barrier methods in §4.6, proving results for the coupled system rests critically on deriving *global* sub- and super-solutions for this coupled system; we come back to this in §5.

3. The momentum constraint

In this section we fix a particular scalar function $\phi \in L^\infty$ and consider the momentum constraint equation (2.36) for the vector valued function $\boldsymbol{w} \in \boldsymbol{W}^{1,2}$. The result is a linear elliptic system of equations for this variable $\boldsymbol{w}$. We first develop the weak formulation of the momentum constraint more precisely in §3.1. In §3.2 we establish generalized Korn inequalities for the conformal Killing operator on compact manifolds with boundary under several boundary condition scenarios; the results do not appear to be in the literature. We then briefly summarize here the main ideas for solving the Dirichlet-Robin problems for the momentum constraint equation, for an appropriately given ϕ. We use two different methods, namely variational methods [33, 48, 57], and Riesz-Schauder theory for compact operators [51]. Both methods yield essentially the same results, since the momentum constraint equation is linear in the variable $\boldsymbol{w}$.

The variational approach is taken in §3.3 when the Dirichlet part of the boundary is non-empty, giving existence and uniqueness of weak solutions to the momentum constraint in $W^{1,2}$. The weak assumptions on the data do not allow for the use of standard techniques to establish additional regularity. While the variational approach has no real advantage over Riesz-Schauder theory for the momentum constraint, it will give us some insight in its use for the Hamiltonian constraint, for which it will be critical. In addition, some of the supporting results are of interest in their own right, so we include the analysis using variational methods here along with the Riesz-Schauder arguments. In §3.4 we establish existence and uniqueness of solutions to the Dirichlet-Robin problem for the momentum constraint equation using Riesz-Schauder theory for compact operators. The literature on Riesz-Schauder theory for systems of elliptic equations is not so clearly presented as it is for scalar equations, so we summarize it here. The main ideas in this method include establishing a Gårding inequality for a bilinear form associated to the principal part of the equation in the appropriate function spaces, and then transforming the problem into one involving a Fredholm operator. Finally, regularity of solutions to the momentum constraint is discussed briefly in §3.5.

3.1. **Weak formulation.** Let $(\mathcal{M}, h)$ be a 3-dimensional Riemannian manifold, where $\mathcal{M}$ is a smooth, compact manifold with Lipschitz boundary $\partial\mathcal{M}$, and $h \in C^2(\overline{\mathcal{M}}, 2)$ is a positive definite metric. Introduce the bilinear form

$$a_{\mathbb{L}} : \boldsymbol{W}^{1,2} \times \boldsymbol{W}^{1,2} \to \mathbb{R}, \qquad a_{\mathbb{L}}(\boldsymbol{w}, \underline{\boldsymbol{w}}) := (\mathcal{L}\boldsymbol{w}, \mathcal{L}\underline{\boldsymbol{w}}) + \big(\mathbb{K}\, \mathsf{tr}_{\mathbb{N}}\, \boldsymbol{w}, \mathsf{tr}_{\mathbb{N}}\, \underline{\boldsymbol{w}}\big)_{\mathbb{N}}, \tag{3.1}$$

where the Robin tensor field $\mathbb{K} \in L^\infty(\partial\mathcal{M}_{\mathbb{N}}, 2)$ is symmetric and satisfies the bound

$$\hat{\mathsf{K}}\, \|\mathsf{tr}_{\mathbb{N}}\, \boldsymbol{w}\|^2_{\mathbb{N}} \leqslant (\mathbb{K}\, \mathsf{tr}_{\mathbb{N}}\, \boldsymbol{w}, \mathsf{tr}_{\mathbb{N}}\, \boldsymbol{w})_{\mathbb{N}}, \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,2}, \tag{3.2}$$

and where $\hat{\mathsf{K}}$ is a non-negative constant. Fix the functionals $\boldsymbol{b}^*_\tau$, $\boldsymbol{b}^*_j \in \boldsymbol{W}^{-1,2}_{\mathbb{D}}$. Fix a function $\phi \in L^\infty$ and introduce the linear functional

$$\boldsymbol{f}_{\phi F} \in \boldsymbol{W}^{-1,2}_{\mathbb{D}}, \qquad \boldsymbol{f}_{\phi F} := (\boldsymbol{b}_\tau \phi^6)^* + \boldsymbol{b}^*_j, \tag{3.3}$$

We used the subscript ϕ in $\boldsymbol{f}_{\phi F}$ to emphasize that ϕ is not a variable of the problem. The functional $\boldsymbol{f}_{\phi F}$ is a generalization of the functional $\mathbb{F}$ defined in Eq. (2.13).

The **weak Dirichlet-Robin boundary value formulation** for the momentum constraint is the following: Fix Dirichlet and Robin boundary data

$$\hat{\boldsymbol{w}}_{\mathbb{D}} \in W^{\frac{1}{2},2}(\partial\mathcal{M}_{\mathbb{D}},1), \quad \hat{\boldsymbol{w}}^*_N \in W^{-\frac{1}{2},2}(\partial\mathcal{M}_{\mathbb{N}},1), \tag{3.4}$$

and introduce an extension $\boldsymbol{w}_{\mathbb{D}}$ of the Dirichlet boundary data as described in §2.3; Introduce the non-principal part operator including the Robin boundary conditions,

$$\boldsymbol{f}_\phi \in \boldsymbol{W}^{-1,2}_{\mathbb{D}}, \qquad \boldsymbol{f}_\phi(\underline{\boldsymbol{\omega}}) := \boldsymbol{f}_{\phi F}(\underline{\boldsymbol{\omega}}) - \hat{\boldsymbol{w}}^*_{\mathbb{N}}(\mathsf{tr}_{\mathbb{N}}\underline{\boldsymbol{\omega}}), \tag{3.5}$$

where $\boldsymbol{f}_{\phi F}$ is given by Eq. (3.3); Let $\boldsymbol{A}^{1,2}$ be the affine space given in Eq. (2.30) for the case $p=2$; Then, find an element $\boldsymbol{w} \in \boldsymbol{A}^{1,2}$ solution of

$$a_{\mathbb{L}}(\boldsymbol{w},\underline{\boldsymbol{\omega}}) + \boldsymbol{f}_\phi(\underline{\boldsymbol{\omega}}) = 0 \qquad \forall\, \underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_{\mathbb{D}}. \tag{3.6}$$

It is convenient to express Eq. (3.6) in terms of operators instead of bilinear forms. Introduce the operator

$$A_{\mathbb{L}} : \boldsymbol{W}^{1,2} \to \boldsymbol{W}^{-1,2}_{\mathbb{D}}, \qquad A_{\mathbb{L}}\boldsymbol{w}(\underline{\boldsymbol{\omega}}) := a_{\mathbb{L}}(\boldsymbol{w},\underline{\boldsymbol{\omega}}).$$

Hence, Eq. (3.6) written in terms of operators is the following: find an element $\boldsymbol{w} \in \boldsymbol{A}^{1,2}$ solution of

$$A_{\mathbb{L}}\boldsymbol{w} + \boldsymbol{f}_\phi = 0. \tag{3.7}$$

Lemma 2. *Every smooth solution $\boldsymbol{w}$ of the classical Eq. (2.17) for a given smooth function ϕ is also a solution of the Eqs. (3.6).*

Proof. *(Lemma 2.)* The proof is similar to the proof of Lemma 1, and we do not reproduce it here. □

3.2. **Generalized Korn's inequalities.** The **Korn inequalities** are a fundamental step in proving existence of solutions to the linearized displacement-traction equations in elasticity. The inequalities involve the Killing operator $\ell : \boldsymbol{W}^{1,2} \to L^2(\mathcal{M},2)$ with action $(\ell\boldsymbol{u})_{ab} := \nabla_a u_b + \nabla_b u_a$. There are two main inequalities, called "without" or "with boundary conditions", which can be described in terms of the bilinear form $a_\ell : \boldsymbol{W}^{1,2} \times \boldsymbol{W}^{1,2} \to \mathbb{R}$ with action $a_\ell(\boldsymbol{u},\boldsymbol{v}) := (\ell\boldsymbol{u},\ell\boldsymbol{v})$. The former inequality says that the bilinear form a_ℓ satisfies **Gårding's inequality**, that is, there exists $k_0 > 0$ such that

$$k_0\,\|\boldsymbol{u}\|^2_{1,2} \leqslant \|\boldsymbol{u}\|^2 + \|\ell\boldsymbol{u}\|^2 \qquad \forall\,\boldsymbol{u} \in \boldsymbol{W}^{1,2}.$$

The latter inequality says that the bilinear form a_ℓ is **coercive** in the space $\boldsymbol{W}^{1,2}_{\mathbb{D}}$ in the case that $\text{meas}(\partial\mathcal{M}_{\mathbb{D}}) \neq \emptyset$, that is, there exists a constant $k_0 > 0$ such that

$$k_0\,\|\boldsymbol{u}\|^2_{1,2} \leqslant \|\ell\boldsymbol{u}\|^2 \qquad \forall\,\boldsymbol{u} \in \boldsymbol{W}^{1,2}_{\mathbb{D}}.$$

These inequalities were first established in the case that the manifold $\mathcal{M} \subset \mathbb{R}^3$ and the metric h_{ab} is the Euclidean metric [35, 36], with new proofs given in [21]. A review of elasticity theory is nicely presented in [14] with Korn's inequalities discussed on Volume II, pages 10-13. See also [44]. Both types of Korn's inequalities for the Killing operator have been generalized to Riemannian manifolds in [12].

We just mention here that the Gårding type inequality on the particular case of the spaces $W^{1,2}_0(\mathcal{M},n)$ can be proven for a general class of bilinear forms called strongly elliptic. See [58], exercise 22.7b, page 396. A bilinear form $a : W_0^{1,2}(\mathcal{M}, n) \times W_0^{1,2}(\mathcal{M}, n) \to \mathbb{R}$ with action

$$a(u, v) = \int_{\mathcal{M}} a_{ac_1\cdots c_n bd_1\cdots d_n} \nabla^a u^{c_1\cdots c_n} \nabla^b v^{d_1\cdots d_n} \, dx + \int_{\mathcal{M}} b_{c_1\cdots c_n d_1\cdots d_n} u^{c_1\cdots c_n} v^{d_1\cdots d_n} \, dx$$

is **strongly elliptic** iff there exists a positive constant α_0 such that

$$a_{ac_1\cdots c_n bd_1\cdots d_n} \zeta^a \zeta^b u^{c_1\cdots c_n} u^{d_1\cdots d_n} \geqslant \alpha_0 \, \zeta_a \zeta^a \, u_{c_1\cdots c_n} u^{c_1\cdots c_n}$$

for all vectors $\zeta \in \mathbb{R}^3$ and all tensors $u_{c_1\cdots c_n} \in \mathbb{R}^{3n}$. An example of a strongly elliptic form is the bilinear form a_ℓ.

The role played in elasticity theory by the Killing operator ℓ is played in the momentum constraint Eq. (2.32) by the conformal Killing operator $\mathcal{L}$, which is defined in Eq.(2.7). Inequalities similar to those satisfied by the Killing operator can be obtained for the conformal Killing operator, called here **generalized Korn's inequalities**. First notice that the bilinear form $a_{\mathcal{L}} : \boldsymbol{W}_0^{1,2} \times \boldsymbol{W}_0^{1,2} \to \mathbb{R}$ given by $a_{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{\mathcal{L}u}, \boldsymbol{\mathcal{L}v})$ is strongly elliptic, as the following calculation shows:

$$\begin{aligned}
&\big[\zeta^a u^c + \zeta^c u^a - \frac{2}{3} h^{ac} (\zeta_d u^d)\big] \big[\zeta_a u_c + \zeta_c u_a - \frac{2}{3} h_{ac} (\zeta_e u^e)\big] \\
&\qquad = 2(\zeta_a \zeta^a)(u_b u^b) + \frac{2}{3} \, (\zeta_a u^a)^2 \geqslant 2(\zeta_a \zeta^a)(u_b u^b).
\end{aligned}$$

Hence, a Gårding type inequality is satisfied by the bilinear form $a_{\mathcal{L}}$ on the Hilbert space $\boldsymbol{W}_0^{1,2}$. However, this space is too small in our case where we need the same inequality on the space $\boldsymbol{W}^{1,2}$. In addition, later we will need the coercivity type inequality for the bilinear form $a_{\mathcal{L}}$ on the space $\boldsymbol{W}_{I\!D}^{1,2}$.

We first review the generalized Korn inequality without boundary conditions, which has been proven in [18] in the case where $\mathcal{M} \subset \mathbb{R}^n$, with $n \geqslant 3$, and h_{ab} is the Euclidean metric. It is also shown in [18] that the inequality does not hold for $n = 2$ where the null space of the conformal Killing operator is infinite dimensional. It is also mentioned in that article that the same arguments given in [12] imply that the generalized Korn inequality without boundary conditions also holds on a Riemannian manifold. We summarize these ideas in the following result.

Lemma 3. (Gårding's inequality for $\mathcal{L}$) *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact, Riemannian manifold, with Lipschitz boundary, and with a metric $h \in C^2(\overline{\mathcal{M}}, 2)$. Then, there exists a positive constant k_0 such that the following inequality holds*

$$k_0 \, \|\boldsymbol{u}\|_{1,2}^2 \leqslant \|\boldsymbol{u}\|^2 + \|\boldsymbol{\mathcal{L}u}\|^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}. \tag{3.8}$$

Proof. *(Lemma 3.)* See [18] for the proof. □

Using Lemma 3 it is not difficult to establish that the same type of inequality is satisfied by the bilinear form $a_{I\!L}$.

Corollary 1. (Gårding's inequality for $a_{I\!L}$) *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact, Riemannian manifold, with Lipschitz boundary and with a metric $h \in C^2(\overline{\mathcal{M}}, 2)$. Let $a_{I\!L}$ be the bilinear form defined in Eq. (3.1) for any tensor $I\!K \in L^\infty(\partial\mathcal{M}_{I\!N}, 2)$. Then, there exists a positive constant k_1 such that the following inequality holds*

$$k_1 \, \|\boldsymbol{u}\|_{1,2}^2 \leqslant \|\boldsymbol{u}\|^2 + a_{I\!L}(\boldsymbol{u}, \boldsymbol{u}) \qquad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}. \tag{3.9}$$

Remark. This result holds for both cases $\partial\mathcal{M}_{I\!D} \neq \emptyset$ and $\partial\mathcal{M}_{I\!D} = \emptyset$, and also notice that the Robin tensor field $I\!K$ is arbitrary; we do not require this tensor to be positive definite.

Proof. *(Corollary 1.)* The definition of the bilinear form in Eq. (3.1) implies

$$a_{I\!L}(\boldsymbol{u},\boldsymbol{u}) = \|\mathcal{L}\boldsymbol{u}\|^2 + (I\!K\,\mathsf{tr}_{I\!N}\,\boldsymbol{u}, \mathsf{tr}_{I\!N}\,\boldsymbol{u})_{I\!N} \quad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}.$$

Recalling that $I\!K \in L^\infty(\partial\mathcal{M}_{I\!N}, 2)$, then the second term on the right hand side can be bounded as follows:

$$\begin{aligned}(I\!K\,\mathsf{tr}_{I\!N}\,\boldsymbol{u}, \mathsf{tr}_{I\!N}\,\boldsymbol{u})_{I\!N} &\leqslant \|I\!K\|_\infty \,\|\mathsf{tr}_{I\!N}\,\boldsymbol{u}\|^2_{I\!N}\\ &\leqslant \|I\!K\|_\infty \,\|\boldsymbol{u}\|\;\|\nabla\boldsymbol{u}\|\\ &\leqslant \frac{1}{2}\|I\!K\|_\infty \Big(\frac{1}{\epsilon^2}\|\boldsymbol{u}\|^2 + \epsilon^2\,\|\nabla\boldsymbol{u}\|^2\Big),\end{aligned}$$

for every non-zero number ϵ. Let $\tilde{k}_1 := \|I\!K\|_\infty/2$, and then compute

$$\begin{aligned}a_{I\!L}(\boldsymbol{u},\boldsymbol{u}) &\geqslant \|\mathcal{L}\boldsymbol{u}\|^2 - \tilde{k}_1\Big(\frac{1}{\epsilon^2}\,\|\boldsymbol{u}\|^2 + \epsilon^2\|\nabla\boldsymbol{u}\|^2\Big)\\ &\geqslant -\|\boldsymbol{u}\|^2 + k_0\|\boldsymbol{u}\|^2_{1,2} - \tilde{k}_1\Big(\frac{1}{\epsilon^2}\,\|\boldsymbol{u}\|^2 + \epsilon^2\|\boldsymbol{u}\|^2_{1,2}\Big)\\ &\geqslant -\big(1+\frac{\tilde{k}_1}{\epsilon^2}\big)\|\boldsymbol{u}\|^2 + (k_0 - \tilde{k}_1\epsilon^2)\|\boldsymbol{u}\|^2_{1,2}.\end{aligned}$$

Choose the number ϵ such that $k_0 - \tilde{k}_1\epsilon^2 = k_0/2$, then

$$a_{I\!L}(\boldsymbol{u},\boldsymbol{u}) \geqslant -\big(1+\frac{3\tilde{k}_1^2}{2k_0}\big)\|\boldsymbol{u}\|^2 + \frac{k_0}{2}\,\|\boldsymbol{u}\|^2_{1,2},$$

and so,

$$\begin{aligned}\frac{k_0}{2}\,\|\boldsymbol{u}\|^2_{1,2} &\leqslant \big(1+\frac{3\tilde{k}_1^2}{2k_0}\big)\|\boldsymbol{u}\|^2 + a_{I\!L}(\boldsymbol{u},\boldsymbol{u})\\ &\leqslant \big(1+\frac{3\tilde{k}_1^2}{2k_0}\big)\Big[\|\boldsymbol{u}\|^2 + a_{I\!L}(\boldsymbol{u},\boldsymbol{u})\Big].\end{aligned}$$

Divide by $\big(1+\frac{3\tilde{k}_1^2}{2k_0}\big)$ and set $k_1 = \frac{k_0^2}{(2k_0+3\tilde{k}_1^2)}$ and the result is the inequality (3.9). $\square$

We have not found in the literature the generalized Korn inequality with boundary conditions on only part of the manifold boundary, neither in Euclidean space nor in an arbitrary Riemannian manifold. This type of inequality is crucial in §3.3, so we proceed to establish this result.

Lemma 4. (Coercivity of $\mathcal{L}$) *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact, Riemannian manifold, with Lipschitz boundary such that meas$(\partial\mathcal{M}_{I\!D}) > 0$, and the metric $h \in C^2(\overline{\mathcal{M}}, 2)$. Then, there exists a positive constant k_0 such that the following inequality holds*

$$k_0\,\|\boldsymbol{u}\|^2_{1,2} \leqslant \|\mathcal{L}\boldsymbol{u}\|^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}_{I\!D}. \tag{3.10}$$

Proof. *(Lemma 4.)* The proof has two main parts: The first one is to show that the null space of the operator $\mathcal{L} : \boldsymbol{W}^{1,2}_{I\!D} \to L^2(\mathcal{M}, 2)$ is trivial when meas$(\partial\mathcal{M}_{I\!D}) > 0$; the second part uses the Gårding type inequality satisfied by the operator $\mathcal{L}$ and presented in Lemma 3 together with a well known argument by contradiction to show Eq. (3.10).

The first part mentioned above also consists of two steps. We first step is to show that any vector field belonging to the null space of $\mathcal{L}$, vectors called conformal Killing vectors, must satisfy a particular set of ordinary differential equations (ODE). Indeed, assume that u^a is a conformal Killing vector, so $\boldsymbol{\mathcal{L}u} = 0$, and introduce the fields

$$\alpha_{ab} := \nabla_{[a}u_{b]}, \qquad \beta := \nabla_a u^a, \qquad \gamma_a := \nabla_a \beta,$$

where we introduced the notation $\nabla_{[a}u_{b]} := (\nabla_a u_b - \nabla_b u_a)/2$, and similarly we will denote $\nabla_{(a}u_{b)} := (\nabla_a u_b + \nabla_b u_a)/2$. A straightforward albeit long computation commuting derivatives shows that a conformal Killing vector u^a and its derivatives introduced above must satisfy the following equations,

$$\nabla_a u_b = \alpha_{ab} + \frac{1}{3}\,\beta h_{ab}, \tag{3.11}$$

$$\nabla_a \beta = \gamma_a, \tag{3.12}$$

$$\nabla_a \alpha_{bc} = -R_{bca}{}^d u_d + \frac{2}{3}\,\gamma_{[b}h_{c]a}, \tag{3.13}$$

$$\nabla_a \gamma_b = -3u^c \nabla_c L_{ab} - 2\beta L_{ab} - 6R_{c(a}\alpha_{b)}{}^c, \tag{3.14}$$

where the tensor $R_{abc}{}^d$ is the Riemann tensor of the metric connection ∇_a, the tensor $R_{ab} = R_{acb}{}^c$ is the Ricci tensor, and we have introduced the tensor $L_{ab} := R_{ab} - Rh_{ab}/4$. The first two equations above are the definitions of the fields α_{ab}, β and γ_a. The other two equations are obtained by commuting second and third derivatives of the conformal Killing vector u^a. They are generalizations of the well-known formulas for Killing vectors (where $\beta = 0$, $\gamma_a = 0$) which can be found for example in [50], page 443. These formulas in the case of Lorentzian metrics have been used in [23]. Contract Eqs. (3.11)-(3.14) on index a with any vector field v^a, and the result is a system of ODE for the fields u^a, α_{ab}, β and γ_a. From this system of ODE we conclude the following: If these four fields vanish at a single point in $\mathcal{M}$, then they vanish identically on $\mathcal{M}$.

The second step is to show the following: If u^a is a conformal Killing vector that vanishes on a two-dimensional hypersurface $\partial\mathcal{M}_{I\!D}$ with $\text{meas}(\partial\mathcal{M}_{I\!D}) > 0$, then the vector u^a, and the fields α_{ab}, β and γ_a vanish at any point on the hypersurface $\partial\mathcal{M}_{I\!D}$. This statement and the conclusion of the paragraph above will imply that u^a vanishes identically on the manifold $\mathcal{M}$. Denote by n^a the unit vector field normal to the tangent space at each point in the manifold $\partial\mathcal{M}_{I\!D}$, and introduce the first and second fundamental forms of the hypersurface $\partial\mathcal{M}_{I\!D}$ as follows,

$$l_{ab} := h_{ab} - n_a n_b, \qquad \kappa_{ab} := -l_a{}^c \nabla_c n_b,$$

where the tensor κ_{ab} is symmetric. Denote by D_a the Levi-Civita connection associated with the two-metric l_{ab} defined on the 2-dimensional hypersurface $\partial\mathcal{M}_{I\!D}$, so the connection satisfies the property that $D_a l_{bc} = 0$. Extend the vector field n^a to a neighborhood of the hypersurface $\partial\mathcal{M}_{I\!D}$ in the manifold $\mathcal{M}$ as the tangent vector solution to the geodesic equation $n^a \nabla_a n^b = 0$ with initial data n^a on $\partial\mathcal{M}_{I\!D}$. Hence, the resulting vector field satisfies $n^a \nabla_a n^b = 0$ in a neighborhood of the hypersurface $\partial\mathcal{M}_{I\!D}$. Then, decompose the conformal Killing field u^a as follows $u^a = u_n n^a + \hat{u}^a$, with $n_a \hat{u}^a = 0$. Denote by $|_{I\!D}$ evaluation at the hypersurface $\partial\mathcal{M}_{I\!D}$, then the condition $u^a|_{I\!D} = 0$ implies

$$u_n|_{I\!D} = 0, \qquad \hat{u}^a|_{I\!D} = 0.$$

The latter condition means that $(D_a u_n)|_{\mathbb{D}} = 0$ and $(D_a \hat{u}^b)|_{\mathbb{D}} = 0$, while the latter equation together with the equation $n^a n^b (\mathcal{L}u)_{ab} = 0$, which also holds on $\partial\mathcal{M}_{\mathbb{D}}$, imply that $(\nabla_n u_n)|_{\mathbb{D}} = 0$, where we use the notation $\nabla_n := n^a \nabla_a$. Therefore, from expression $\nabla_a u^a = \nabla_n u_n + D_a \hat{u}^a + \kappa_a{}^a u_n$ we then conclude that

$$\beta|_{\mathbb{D}} = 0.$$

The equation $n^a l_c{}^b (\mathcal{L}u)_{ab} = 0$ implies, after a short calculation, that the equation $l_c{}^b \nabla_n \hat{u}_b + \kappa_{cb} \hat{u}^b + D_c u_n = 0$ holds in the manifold $\mathcal{M}$, and so it also holds on the hypersurface $\partial\mathcal{M}_{\mathbb{D}}$. This result together with our previous results establish the condition $(l_c{}^b \nabla_n \hat{u}_b)|_{\mathbb{D}} = 0$. The decomposition

$$\nabla_{[a} u_{b]} = D_{[a} \hat{u}_{b]} + n_{[b} \kappa_{a]c} \hat{u}^c + n_{[b} D_{a]} u_n + n_{[a} l_{b]}{}^c \nabla_n \hat{u}_c$$

and our previous results then imply that

$$\alpha_{ab}|_{\mathbb{D}} = 0.$$

We still have to show that the vector field $\gamma_a = \nabla_a \beta$ vanishes on the hypersurface $\partial\mathcal{M}_{\mathbb{D}}$. Since β is a scalar field and vanishes on $\partial\mathcal{M}_{\mathbb{D}}$, we conclude that the field $(D_a \beta)|_{\mathbb{D}} = 0$, which implies $(D_a \nabla_n u_n)|_{\mathbb{D}} = 0$. We now only need to compute the field $(\nabla_n \beta)|_{\mathbb{D}}$. The identity $n^a l_c{}^b \nabla_{[a} \nabla_{b]} u_n = 0$ when evaluated on the hypersurface $\partial\mathcal{M}_{\mathbb{D}}$ together with our previous results imply the equation $(D_c \nabla_n u_n)|_{\mathbb{D}} = (l_c{}^b \nabla_n D_b u_n)|_{\mathbb{D}}$. But we just showed that the left hand side vanishes, and so then does the right hand side $(l_c{}^b \nabla_n D_b u_n)|_{\mathbb{D}} = 0$. Finally, from the equation $n^a l_d{}^b \big(2\nabla_{[a} \nabla_{b]} \hat{u}_c - R_{abc}{}^e \hat{u}_e\big) = 0$ evaluated on the hypersurface $\partial\mathcal{M}_{\mathbb{D}}$ we conclude that

$$(l_c{}^a l_d{}^b \nabla_n D_a \hat{u}_b)|_{\mathbb{D}} - \big[D_c (l_d{}^b \nabla_n \hat{u}_b)\big]|_{\mathbb{D}} = 0.$$

The result $(l_c{}^b \nabla_n \hat{u}_b)|_{\mathbb{D}} = 0$ implies that the second term on the left hand side above vanishes, so we conclude that $(l_c{}^a l_d{}^b \nabla_n D_a \hat{u}_b)|_{\mathbb{D}} = 0$, and from this equation one gets $\big[\nabla_n (D_a \hat{u}^a)\big]|_{\mathbb{D}} = 0$. Therefore, in order to show that the field $(\nabla_n \beta)|_{\mathbb{D}}$ vanishes we only have left to prove that the field $(\nabla_n \nabla_n u_n)|_{\mathbb{D}}$ vanishes. That this is the case follows from the equation $\nabla_n \big[n^a n^b (\mathcal{L}u)_{ab}\big] = 0$, which holds in the manifold $\mathcal{M}$, and so on the hypersurface $\partial\mathcal{M}_{\mathbb{D}}$, and an explicit computation shows that $(\nabla_n \nabla_n u_n)|_{\mathbb{D}} = 0$. We then conclude that

$$\gamma_a|_{\mathbb{D}} = 0.$$

Let us now recall that the ODE equations obtained by contracting Eqs. (3.11)-(3.14) on index a with any vector field v^a are homogeneous on the fields u^a, α_{ab}, β and γ_a with vanishing initial data on the hypersurface $\partial\mathcal{M}_{\mathbb{D}}$. The solution vanish identically in a neighborhood of this hypersurface $\partial\mathcal{M}_{\mathbb{D}}$ in the manifold $\mathcal{M}$. Repeating this procedure we conclude that the conformal Killing vector field vanishes identically in $\mathcal{M}$. This result establishes that the null space of the operator $\mathcal{L}$ is trivial on the space $\boldsymbol{W}^{1,2}_{\mathbb{D}}$.

We now consider the second part of the proof of Lemma 4 using a well-known argument by contradiction. Assume that there exists a sequence $\{\boldsymbol{u}_n\} \subset \boldsymbol{W}^{1,2}_{\mathbb{D}}$ such that

$$\|\boldsymbol{u}_n\|_{1,2} = 1, \quad \text{and} \quad \lim_{n\to\infty} \|\mathcal{L}\boldsymbol{u}\| = 0.$$

The sequence $\{\boldsymbol{u}_n\}$ is bounded in $\boldsymbol{W}^{1,2}_{\mathbb{D}}$ which is a reflexive Banach space, so there exists a subsequence, also denoted as $\{\boldsymbol{u}_n\}$, such that

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}_0 \quad \text{in} \quad \boldsymbol{W}^{1,2}_{\mathbb{D}}, \quad \text{and} \quad \boldsymbol{u}_n \to \boldsymbol{u}_0 \quad \text{in} \quad \boldsymbol{L}^2,$$

the latter statement following from the imbedding $\boldsymbol{W}^{1,2}_{I\!D} \subset \boldsymbol{L}^2$ being compact. So $\{\boldsymbol{u}_n\}$ is a Cauchy sequence in $\boldsymbol{L}^2$, and by assumption the sequence $\{\mathcal{L}\boldsymbol{u}_n\} \subset \boldsymbol{L}^2$ is also a Cauchy sequence. The Gårding inequality in Lemma 3 implies that

$$k_0 \,\|\boldsymbol{u}_n - \boldsymbol{u}_m\|^2_{1,2} \leqslant \|\boldsymbol{u}_n - \boldsymbol{u}_m\|^2 + \|\mathcal{L}\boldsymbol{u}_n - \mathcal{L}\boldsymbol{u}_m\|^2 \to 0 \quad \text{as} \quad n, m \to \infty,$$

and so the sequence $\{\boldsymbol{u}_n\}$ is also a Cauchy sequence in $\boldsymbol{W}^{1,2}_{I\!D}$. We then conclude that

$$\boldsymbol{u}_n \to \boldsymbol{u}_0 \quad \text{in} \quad \boldsymbol{W}^{1,2}_{I\!D} \quad \Rightarrow \quad \mathcal{L}\boldsymbol{u}_n \to 0 = \mathcal{L}\boldsymbol{u}_0.$$

But the null space of the operator $\mathcal{L}$ is trivial on the space $\boldsymbol{W}^{1,2}_{I\!D}$, therefore we conclude that the element $\boldsymbol{u}_0 = 0$. However, this leads us to a contradiction from the hypothesis that $\|\boldsymbol{u}_n\|_{1,2} = 1$ which implies that $\|\boldsymbol{u}_0\|_{1,2} = 1$ so the element $\boldsymbol{u}_0 \neq 0$. Therefore, such sequence $\{\boldsymbol{u}_n\}$ does not exist, which then establishes the Lemma. □

Corollary 2. **(Coercivity of $a_{I\!L}$)** *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact, Riemannian manifold, with Lipschitz boundary such that meas$(\partial\mathcal{M}_{I\!D}) > 0$, and the metric $h \in C^2(\overline{\mathcal{M}}, 2)$. Let $a_{I\!L}$ be the bilinear form defined in Eq. (3.1), and assume that the Robin tensor $I\!K \in L^\infty(\partial\mathcal{M}_{I\!N}, 2)$ is positive definite. Then, there exists a positive constant k_1 such that the following inequality holds*

$$k_1 \,\|\boldsymbol{u}\|^2_{1,2} \leqslant a_{I\!L}(\boldsymbol{u}, \boldsymbol{u}) \qquad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}_{I\!D}. \tag{3.15}$$

Proof. *(Corollary 2.)* Since the Robin tensor $I\!K$ is positive definite, then the result is straightforward from Eq. (3.10), due to the following inequalities,

$$\begin{aligned} k_0 \,\|\boldsymbol{u}\|^2_{1,2} &\leqslant \|\mathcal{L}\boldsymbol{u}\|^2 \\ &\leqslant \|\mathcal{L}\boldsymbol{u}\|^2 + \hat{\mathsf{K}} \,\|\mathsf{tr}_{I\!N}\boldsymbol{u}\|^2_{I\!N} \\ &\leqslant \|\mathcal{L}\boldsymbol{u}\|^2 + (I\!K\,\mathsf{tr}_{I\!N}\boldsymbol{u}, \mathsf{tr}_{I\!N}\boldsymbol{u})_{I\!N} \\ &\leqslant a_{I\!L}(\boldsymbol{u}, \boldsymbol{u}), \qquad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}_{I\!D}. \end{aligned}$$

This inequality establishes the Corollary. □

Remark. It can be shown that the result in the Corollary 2 remains valid if the Robin tensor field $I\!K$ is slightly negative definite.

3.3. **Results using variational methods.** The momentum constraint Eq. (3.6) can be written as the Euler condition for stationarity of a real-valued functional on a Banach space. Direct methods in the calculus of variations can be used to find the points that minimize this functional in the Banach space in the case that the hypersurface $\partial\mathcal{M}_{I\!D} \neq \emptyset$. The main concepts needed from the calculus of variations are summarized in the Appendix, where we also explain the part of the notation used in this Section. Since the momentum constraint is linear, we will achieve similar (in fact, slightly more general) results in §3.4 using Riesz-Schauder Theory. However, the presentation here is a guide for our variational treatment of the Hamiltonian constraint equation in §4.5, and the results we assemble in this section are of interest in their own right.

Let $a_{I\!L} : \boldsymbol{W}^{1,2}_{I\!D} \times \boldsymbol{W}^{1,2}_{I\!D} \to \mathbb{R}$ be a bilinear form with action defined in Eq. (3.1), and fix the functionals $\boldsymbol{b}^*_\tau$, $\boldsymbol{b}^*_j \in \boldsymbol{W}^{-1,2}_{I\!D}$. Let $\boldsymbol{w}_{I\!D} \in \boldsymbol{W}^{1,2}$ be the extension of the Dirichlet data $\hat{\boldsymbol{w}}_{I\!D}$, and $\hat{\boldsymbol{w}}^*_{I\!N}$ be the Robin data functional, both data defined in Eq. (3.4). Introduce the functional

$$J_{I\!L} : \boldsymbol{W}^{1,2}_{I\!D} \to \mathbb{R}, \qquad J_{I\!L}(\boldsymbol{\omega}) := \frac{1}{2}\, a_{I\!L}(\boldsymbol{\omega}, \boldsymbol{\omega}) + G_\phi(\boldsymbol{\omega}), \tag{3.16}$$

where the functional G_ϕ is given by

$$G_\phi(\boldsymbol{\omega}) := g_\phi(\boldsymbol{w}_{I\!D} + \boldsymbol{\omega}) - \hat{\boldsymbol{w}}^*_{I\!N}(\mathsf{tr}_{I\!N}\boldsymbol{\omega}) + a_{I\!L}(\boldsymbol{w}_{I\!D}, \boldsymbol{\omega}),$$

with the functional $g_\phi(\boldsymbol{w})$ having the form

$$g_\phi(\boldsymbol{w}) := (\boldsymbol{b}_\tau \phi^6)^*(\boldsymbol{w}) + \boldsymbol{b}^*_j(\boldsymbol{w}), \tag{3.17}$$

and we will use the notation $\boldsymbol{w} = \boldsymbol{w}_{I\!D} + \boldsymbol{\omega}$.

Theorem 1. (Existence of a minimizer) *Let $J_{I\!L} : \boldsymbol{W}^{1,2}_{I\!D} \to \mathbb{R}$ be the functional defined in Eq. (3.16). Assume that the hypersurface $\partial\mathcal{M}_{I\!D} \neq \emptyset$, and fix an extension of the Dirichlet boundary data $\boldsymbol{w}_{I\!D} \in \boldsymbol{W}^{1,2}$ and the Robin boundary data $\hat{\boldsymbol{w}}^*_{I\!N} \in W^{-\frac{1}{2},2}(\partial\mathcal{M}_{I\!N}, 1)$. Fix the functionals $\boldsymbol{b}^*_\tau$, $\boldsymbol{b}^*_j \in \boldsymbol{W}^{-1,2}_{I\!D}$, and the tensor $I\!K \in L^\infty(\partial\mathcal{M}_{I\!N}, 2)$ satisfying the inequality in Eq. (3.2) with $\hat{\mathtt{K}} \geqslant 0$. Then, there exists a unique element $\boldsymbol{\omega} \in \boldsymbol{W}^{1,2}_{I\!D}$ minimizer of the functional $J_{I\!L}$ on $\boldsymbol{W}^{1,2}_{I\!D}$, that is,*

$$J_{I\!L}(\boldsymbol{\omega}) = \inf_{\underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_{I\!D}} J_{I\!L}(\underline{\boldsymbol{\omega}}).$$

Proof. *(Theorem 1.)* We start showing that the functional $J_{I\!L}$ is coercive, and the first step is the following inequality

$$J_{I\!L}(\boldsymbol{\omega}) \geqslant \frac{1}{2}\, a_{I\!L}(\boldsymbol{\omega}, \boldsymbol{\omega}) - |G_\phi(\boldsymbol{\omega})|.$$

The linear terms in G_ϕ can be bounded as follows: recall the notation $\boldsymbol{w} = \boldsymbol{w}_{I\!D} + \boldsymbol{\omega}$, then

$$\begin{aligned} |g_\phi(\boldsymbol{w})| &\leqslant \big|(\boldsymbol{b}_\tau \phi^6)^*(\boldsymbol{w})\big| + \big|\boldsymbol{b}^*_j(\boldsymbol{w})\big| \\ &\leqslant \Big[\|\phi\|^6_\infty \,\|\boldsymbol{b}^*_\tau\|_{-1,2} + \|\boldsymbol{b}^*_j\|_{-1,2}\Big]\, \|\boldsymbol{w}\|_{1,2}, \end{aligned}$$

introducing the constant $c_g := \big[\|\phi\|^6_\infty \,\|\boldsymbol{b}^*_\tau\|_{-1,2} + \|\boldsymbol{b}^*_j\|_{-1,2}\big]/2$, we then obtain

$$|g_\phi(\boldsymbol{w})| \leqslant 2c_g \,\|\boldsymbol{w}\|_{1,2} \leqslant \frac{1}{\epsilon}\, c_g^2 + \epsilon\, \|\boldsymbol{w}\|^2_{1,2}, \leqslant \frac{1}{\epsilon}\, c_g^2 + 2\epsilon\, \|\boldsymbol{w}_{I\!D}\|^2_{1,2} + 2\epsilon\, \|\boldsymbol{\omega}\|^2_{1,2}, \tag{3.18}$$

where ϵ is any positive constant. The second term in the functional G_ϕ can be bounded as follows

$$\begin{aligned} \big|-\hat{\boldsymbol{w}}^*_{I\!N}(\mathsf{tr}_{I\!N}\boldsymbol{\omega})\big| &\leqslant \|\hat{\boldsymbol{w}}^*_{I\!N}\|_{-\frac{1}{2},2,I\!N}\, \|\mathsf{tr}_{I\!N}\boldsymbol{\omega}\|_{\frac{1}{2},2,I\!N} \\ &\leqslant c_0\, \|\hat{\boldsymbol{w}}^*_{I\!N}\|_{-\frac{1}{2},2,I\!N}\, \|\boldsymbol{\omega}\|_{1,2} \\ &\leqslant \frac{c_0^2}{2\epsilon}\, \|\hat{\boldsymbol{w}}^*_{I\!N}\|^2_{-\frac{1}{2},2,I\!N} + \frac{\epsilon}{2}\, \|\boldsymbol{\omega}\|^2_{1,2}, \end{aligned} \tag{3.19}$$

where c_0 is a positive constant. The third term in G_ϕ can be bounded as follows

$$\begin{aligned} |a_{I\!L}(\boldsymbol{w}_{I\!D}, \boldsymbol{\omega})| &\leqslant c_{I\!L}\, \|\boldsymbol{w}_{I\!D}\|_{1,2}\, \|\boldsymbol{\omega}\|_{1,2} \\ &\leqslant \frac{c_{I\!L}^2}{2\epsilon}\, \|\boldsymbol{w}_{I\!D}\|^2_{1,2} + \frac{\epsilon}{2}\, \|\boldsymbol{\omega}\|^2_{1,2}. \end{aligned} \tag{3.20}$$

By adding the inequalities in Eqs. (3.18)-(3.20) we obtain the bound on G_ϕ,

$$|G_\phi(\boldsymbol{\omega})| \leqslant c_G + 3\epsilon\, \|\boldsymbol{\omega}\|^2_{1,2}, \tag{3.21}$$

$$c_G := \frac{1}{\epsilon}\Big[c_g^2 + 2\epsilon^2\, \|\boldsymbol{w}_{I\!D}\|^2_{1,2} + \frac{c_0^2}{2}\, \|\hat{\boldsymbol{w}}^*_{I\!N}\|^2_{-\frac{1}{2},2,I\!N} + \frac{c_{I\!L}^2}{2}\, \|\boldsymbol{w}_{I\!D}\|^2_{1,2}\Big].$$

We now consider the bilinear form $a_{\mathbb{L}}$. First, the tensor $\mathbb{K}$ satisfies the inequality in Eq. (3.2) with $\hat{\mathtt{K}} \geqslant 0$, so we have

$$a_{\mathbb{L}}(\boldsymbol{\omega},\boldsymbol{\omega}) \geqslant (\mathcal{L}\boldsymbol{\omega},\mathcal{L}\boldsymbol{\omega}).$$

The assumption that the hypersurface $\partial\mathcal{M}_{\mathbb{D}} \neq \emptyset$ and the generalized Korn inequality in Lemma 4 imply that there exists a positive constant k_0 such that $(\mathcal{L}\boldsymbol{\omega},\mathcal{L}\boldsymbol{\omega}) \geqslant k_0\,\|\boldsymbol{\omega}\|_{1,2}^2$, which together with the inequality above imply

$$a_{\mathbb{L}}(\boldsymbol{\omega},\boldsymbol{\omega}) \geqslant k_0\,\|\boldsymbol{\omega}\|_{1,2}^2. \tag{3.22}$$

Therefore, from the inequalities (3.21)-(3.22) we obtain

$$J_{\mathbb{L}}(\boldsymbol{\omega}) \geqslant (k_0 - 3\epsilon)\|\boldsymbol{\omega}\|_{1,2}^2 - c_G.$$

Choosing ϵ small enough we have established that $J_{\mathbb{L}}$ is coercive in $\boldsymbol{W}_{\mathbb{D}}^{1,2}$.

We now show that the functional $J_{\mathbb{L}}$ is lsc_w. Let $\{\boldsymbol{\omega}_n\} \subset \boldsymbol{W}_{\mathbb{D}}^{1,2}$ be a sequence such that $\boldsymbol{\omega}_n \rightharpoonup \boldsymbol{\omega}_0$ in $\boldsymbol{W}_{\mathbb{D}}^{1,2}$, which then implies that $\boldsymbol{\omega}_n \to \boldsymbol{\omega}_0$ in $\boldsymbol{L}^2$. We again start with the functional G_ϕ, which is linear on its variable $\boldsymbol{\omega} \in \boldsymbol{W}_{\mathbb{D}}^{1,2}$, therefore it is continuous under weak convergence (by definition of weak convergence). So it is also lsc_w, and the following equation holds

$$G_\phi(\boldsymbol{\omega}_0) = \liminf_{n\to\infty} G_\phi(\boldsymbol{\omega}_n).$$

We only have to show that the functional $\boldsymbol{\omega} \mapsto a_{\mathbb{L}}(\boldsymbol{\omega},\boldsymbol{\omega})$ given in Eq. (3.1) is also lsc_w. The first term in the bilinear form $a_{\mathbb{L}}$ defines a norm in $\boldsymbol{W}_{\mathbb{D}}^{1,2}$, since the generalized Korn inequality given in Lemma 4 and the fact that the conformal Killing operator is bounded in $\boldsymbol{W}_{\mathbb{D}}^{1,2}$ imply that there exist positive constants k_0, K_0 such that

$$k_0\,\|\boldsymbol{\omega}\|_{1,2}^2 \leqslant \|\mathcal{L}\boldsymbol{\omega}\|^2 \leqslant K_0\,\|\boldsymbol{\omega}\|_{1,2}^2, \qquad \forall\,\boldsymbol{\omega} \in \boldsymbol{W}_{\mathbb{D}}^{1,2}.$$

This last inequality means that the map $\boldsymbol{\omega} \mapsto \|\mathcal{L}\boldsymbol{\omega}\|$ defines a norm in $\boldsymbol{W}_{\mathbb{D}}^{1,2}$, and so it is lsc_w, a result proven in the Appendix. Therefore, the following inequality holds,

$$\|\mathcal{L}\boldsymbol{\omega}_0\|^2 \leqslant \liminf_{n\to\infty} \|\mathcal{L}\boldsymbol{\omega}_n\|^2.$$

The second term in the definition of the bilinear form $a_{\mathbb{L}}$ contains the two-index tensor $\mathbb{K}$, which is positive definite and symmetric, therefore the function

$$\boldsymbol{\omega} \mapsto (\mathbb{K}\,\mathsf{tr}_{\mathbb{N}}\boldsymbol{\omega}, \mathsf{tr}_{\mathbb{N}}\boldsymbol{\omega})_{\mathbb{N}}$$

is a continuous and convex functional, and so it is lsc_w, a result also proven in the Appendix. We then conclude that the functional $J_{\mathbb{L}}$ is lsc_w. Therefore, Theorem 13 in the Appendix in the case $U = X$ shows that there exists a minimizer for $J_{\mathbb{L}}$ in $\boldsymbol{W}_{\mathbb{D}}^{1,2}$.

The uniqueness of the minimizer is a consequence of the strict convexity of the functional $J_{\mathbb{L}}$, which is a general result that, once again, is established in the Appendix. We have to show that for all non-zero $\hat{\boldsymbol{\omega}}$, $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}_{\mathbb{D}}^{1,2}$ and all $t \in (0,1)$ holds

$$J_{\mathbb{L}}\big(t\hat{\boldsymbol{\omega}} + (1-t)\underline{\boldsymbol{\omega}}\big) < tJ_{\mathbb{L}}(\hat{\boldsymbol{\omega}}) + (1-t)J_{\mathbb{L}}(\underline{\boldsymbol{\omega}}),$$

or equivalently, as it is explained in the Appendix, we only have to show that for all non-zero $\tilde{\boldsymbol{\omega}}$, $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}_{\mathbb{D}}^{1,2}$ holds

$$DJ_{\mathbb{L}}(\underline{\boldsymbol{\omega}})(\tilde{\boldsymbol{\omega}}) < J_{\mathbb{L}}(\tilde{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}) - J_{\mathbb{L}}(\underline{\boldsymbol{\omega}}).$$

A straightforward calculation shows that for all $\tilde{\boldsymbol{\omega}}$, $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_{\mathbb{D}}$ holds

$$\begin{aligned} J_{\mathbb{L}}(\tilde{\boldsymbol{\omega}}+\underline{\boldsymbol{\omega}}) - J_{\mathbb{L}}(\underline{\boldsymbol{\omega}}) &= \frac{1}{2}\, a_{\mathbb{L}}(\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}) + a_{\mathbb{L}}(\underline{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}) + G_{\phi}(\tilde{\boldsymbol{\omega}}) \\ &= DJ_{\mathbb{L}}(\underline{\boldsymbol{\omega}})(\tilde{\boldsymbol{\omega}}) + \frac{1}{2}\, a_{\mathbb{L}}(\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}) \\ &\geqslant DJ_{\mathbb{L}}(\underline{\boldsymbol{\omega}})(\tilde{\boldsymbol{\omega}}) + \frac{1}{2}\, (\mathcal{L}\tilde{\boldsymbol{\omega}},\mathcal{L}\tilde{\boldsymbol{\omega}}) \\ &> DJ_{\mathbb{L}}(\underline{\boldsymbol{\omega}})(\tilde{\boldsymbol{\omega}}), \end{aligned}$$

where the symmetry of the Robin two-tensor field $\mathbb{K}$ is used to establish the first line, and the last line is obtained from the generalized Korn's inequality Eq. (3.10). Therefore, the functional $J_{\mathbb{L}}$ is strictly convex, hence, Theorem 14 in the Appendix implies that the minimizer $\boldsymbol{\omega}$ is unique. This establishes the Theorem. □

The next result shows that the minimum $\boldsymbol{\omega}$ of the functional $J_{\mathbb{L}}$ on the space $\boldsymbol{W}^{1,2}_{\mathbb{D}}$ found in Theorem 1 is a solution of the Euler equation $DJ_{\mathbb{L}}(\boldsymbol{\omega}) = 0$.

Theorem 2. (Momentum constraint) *Assume the hypotheses in Theorem 1. Then, the functional $J_{\mathbb{L}}$ is Gâteaux differentiable on $\boldsymbol{W}^{1,2}_{\mathbb{D}}$ and the minimizer $\boldsymbol{\omega} \in \boldsymbol{W}^{1,2}_{\mathbb{D}}$ is solution of the Euler equation*

$$DJ_{\mathbb{L}}(\boldsymbol{\omega})(\underline{\boldsymbol{\omega}}) = 0, \qquad \forall\, \underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_{\mathbb{D}},$$

where the equation above is the momentum constraint Eq. (3.6).

Proof. *(Theorem 2.)* It is straightforward to verify that the functional $J_{\mathbb{L}}$ is Gâteaux differentiable, and its derivative at an arbitrary element $\hat{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_{\mathbb{D}}$ is given by

$$DJ_{\mathbb{L}}(\hat{\boldsymbol{\omega}})(\underline{\boldsymbol{\omega}}) = a_{\mathbb{L}}(\hat{\boldsymbol{\omega}},\underline{\boldsymbol{\omega}}) + G_{\phi}(\underline{\boldsymbol{\omega}}) = a_{\mathbb{L}}(\hat{\boldsymbol{w}},\underline{\boldsymbol{\omega}}) + \boldsymbol{f}_{\phi}(\underline{\boldsymbol{\omega}}),$$

with $\hat{\boldsymbol{w}} := \boldsymbol{w}_{\mathbb{D}} + \hat{\boldsymbol{\omega}}$. Therefore, the Gâteaux derivative $DJ_{\mathbb{L}}$ is the left hand side in Eq. (3.6). Let $\boldsymbol{\omega} \in \boldsymbol{W}^{1,2}_{\mathbb{D}}$ be the minimizer of the functional $J_{\mathbb{L}}$ on the space $\boldsymbol{W}^{1,2}_{\mathbb{D}}$. Then the following inequality holds,

$$DJ_{\mathbb{L}}(\boldsymbol{\omega})(\underline{\boldsymbol{\omega}}) \geqslant 0, \qquad \forall\, \underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_{\mathbb{D}}. \tag{3.23}$$

For the proof, write down the Gâteaux derivative of the functional $J_{\mathbb{L}}$ at the minimizer $\boldsymbol{\omega}$,

$$DJ_{\mathbb{L}}(\boldsymbol{\omega})(\underline{\boldsymbol{\omega}}) = \lim_{t\to 0^+} \big[J_{\mathbb{L}}(\boldsymbol{\omega}+t\underline{\boldsymbol{\omega}}) - J_{\mathbb{L}}(\boldsymbol{\omega})\big].$$

The element $\boldsymbol{\omega}$ is a minimizer of $J_{\mathbb{L}}$, so $J_{\mathbb{L}}(\boldsymbol{\omega}+t\underline{\boldsymbol{\omega}}) \geqslant J_{\mathbb{L}}(\boldsymbol{\omega})$, which establishes Eq. (3.23). This Eq. (3.23) holds for $-\underline{\boldsymbol{\omega}}$, so we conclude that $DJ_{\mathbb{L}}(\boldsymbol{\omega}) = 0$. This establishes the Theorem. □

3.4. **Results using Riesz-Schauder theory.** We present here a proof of existence and uniqueness of solutions of the weak Dirichlet-Robin boundary value problem for the momentum constraint Eq. (3.7). The proof is based on the Riesz-Schauder theory for compact operators, see [51]. The proof is more general than the one given in §3.3 because it includes the case where $\operatorname{meas}(\partial\mathcal{M}_{\mathbb{D}}) = 0$, that is, the pure Robin case. Riesz-Schauder theory was used for the momentum constraint in [28] to develop an approximation theory and corresponding error estimates for numerical approximations.

Theorem 3. (Momentum constraint) *Consider the weak formulation for the momentum constraint Eq. (3.7). Assume that the Robin tensor field $I\!K$ satisfies Eq. (3.2) with positive constant $\hat{\mathtt{K}}$. Then, there exists a unique solution $\boldsymbol{w} \in \boldsymbol{A}^{1,2}$ to the momentum constraint Eq. (3.7), and there exist positive constants c_1 and c_2 such that the following estimate holds,*

$$\|\boldsymbol{w}\|_{1,2} \leqslant \|\phi\|_\infty^6 \,\|\boldsymbol{b}_\tau^*\|_{-1,2} + \|\boldsymbol{b}_j^*\|_{-1,2} + c_1 \,\|\hat{\boldsymbol{w}}_{I\!N}^*\|_{-\frac{1}{2},2,I\!N} + c_2 \,\|\boldsymbol{w}_{I\!D}\|_{1,2}. \tag{3.24}$$

Proof. *(Theorem 3.)* First translate the problem from the affine space $\boldsymbol{A}^{1,2}$ into a problem on the vector space $\boldsymbol{W}_{I\!D}^{1,2}$ with the change of variable $\boldsymbol{w} = \boldsymbol{w}_{I\!D} + \boldsymbol{\omega}$. Then, Eq. (3.7) has the form: Find $\boldsymbol{\omega} \in \boldsymbol{W}_{I\!D}^{1,2}$ solution of

$$A_{I\!L}\boldsymbol{\omega} + G_\phi = 0 \tag{3.25}$$

where $G_\phi(\underline{\boldsymbol{\omega}}) := \boldsymbol{f}_\phi(\underline{\boldsymbol{\omega}}) + a_{I\!L}(\boldsymbol{w}_{I\!D}, \underline{\boldsymbol{\omega}})$, and we now consider the operator $A_{I\!L} : \boldsymbol{W}_{I\!D}^{1,2} \to \boldsymbol{W}_{I\!D}^{-1,2}$. To find a solution of Eq. (3.25) is equivalent to show that this operator $A_{I\!L}$ is invertible. Lemma 3 says that the conformal Killing operator $\mathcal{L}$ satisfies Gårding's inequality Eq. (3.8). This implies that the bilinear form $a_{I\!L}$ also satisfies a Gårding inequality, which was proven in Corollary 1. Then, Theorem 12 in the Appendix implies that the operator $A_{I\!L}$ is Fredholm with index zero. That means $\dim N_{A_{I\!L}} = \operatorname{codim} R_{A_{I\!L}}$, which can be described saying that the operator $A_{I\!L}$ is bijective iff it is injective. This property is described in the PDE literature as "uniqueness implies existence". So, in order to show that $A_{I\!L}$ is invertible we only have to show that its null space is trivial. Consider an element $\boldsymbol{u} \in \boldsymbol{W}_{I\!D}^{1,2}$ such that $A_{I\!L}\boldsymbol{u} = 0$. In particular $A_{I\!L}\boldsymbol{u}(\boldsymbol{u}) = 0$, which is equivalent to

$$0 = \|\mathcal{L}\boldsymbol{u}\|^2 + (I\!K \,\mathsf{tr}_{I\!N}\boldsymbol{u}, \mathsf{tr}_{I\!N}\boldsymbol{u})_{I\!N} \geqslant \|\mathcal{L}\boldsymbol{u}\|^2 + \hat{\mathtt{K}}\,\|\mathsf{tr}_{I\!N}\boldsymbol{u}\|_{I\!N}^2.$$

Both terms must vanish, since the tensor $I\!K$ is strictly positive definite and so the constant $\hat{\mathtt{K}}$ is positive. From the first term one obtains that $\boldsymbol{u}$ is a conformal Killing vector, and from the second term together with $\boldsymbol{u} \in \boldsymbol{W}_{I\!D}^{1,2}$ one obtains that $\mathtt{tr}\boldsymbol{u} = 0$ on the whole boundary $\partial\mathcal{M}$ and so, $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}$. Therefore, $\boldsymbol{u} = 0$ in the manifold $\mathcal{M}$ since the bilinear form $a_{I\!L}$ is strongly elliptic. So, the null space of the operator $A_{I\!L}$ is trivial, and then $A_{I\!L}$ is invertible. Finally, it is not difficult to check that the estimate given in Eq. (A.6) of the Appendix applied to $\boldsymbol{\omega} = \boldsymbol{w} - \boldsymbol{w}_{I\!D}$ implies the estimate on $\boldsymbol{w}$ given in Eq. (3.24). This establishes the Theorem. □

Remark. In the case that the hypersurface $\partial\mathcal{M}_{I\!D} \neq \emptyset$ the Robin tensor $I\!K$ need not to be strictly positive. There exists a unique solution to the momentum constraint in the case that the constant $\hat{\mathtt{K}}$ in Eq. (3.2) is slightly negative, that is, $\hat{\mathtt{K}} > -\hat{\mathtt{K}}_0$ for small enough $\hat{\mathtt{K}}_0 > 0$. The proof uses the coercivity of the conformal Killing operator $\mathcal{L}$, the inequality (3.10) in Lemma 4, instead of the Gårding inequality. It can be shown that the assumption that $\hat{\mathtt{K}}$ is negative but not too negative implies that the bilinear form $a_{I\!L}$ itself is strictly positive on $\boldsymbol{W}_{I\!D}^{1,2}$. Recalling that the linear form $\hat{\boldsymbol{f}}_\phi : \boldsymbol{W}_{I\!D}^{1,2} \to \mathbb{R}$ is bounded, then the Riesz representation Theorem says that there exists a unique $\boldsymbol{u} \in \boldsymbol{W}_{I\!D}^{1,2}$ solution to the weak problem with Eq. (3.6). In terms of the operator $A_{I\!L}$, this statement means that $A_{I\!L} : \boldsymbol{W}_{I\!D}^{1,2} \to \boldsymbol{W}_{I\!D}^{-1,2}$ is invertible.

3.5. **Regularity of solutions.** In this Section we state without proof regularity results, which can be obtained from the literature, and are applied to the weak solutions of the momentum constraint.

Theorem 4. (Regularity $\mathbf{W}^{1,p}$) *Assume the hypotheses in Theorem 3, and in addition assume that the boundary set $\partial\mathcal{M}$ is $C^{1,1}$. Assume that the source functional $\boldsymbol{f}_{\phi F}$ and the boundary data satisfy the following conditions,*

$$\boldsymbol{b}^*_\tau, \boldsymbol{b}^*_j \in \boldsymbol{W}^{-1,p}, \quad \boldsymbol{w}_{I\!D} \in \boldsymbol{W}^{1,p}, \quad \hat{\boldsymbol{w}}^*_{I\!N} \in W^{-\frac{1}{p},p}(\partial\mathcal{M}_{I\!N}, 1), \qquad p \geqslant 2,$$

then, the solution $\boldsymbol{w}$ to the momentum constraint Eq. (3.7) satisfies that $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ and there exist positive constants c_1 and c_2 such that the following estimate holds,

$$\|\boldsymbol{w}\|_{1,p} \leqslant \|\phi\|^6_\infty \|\boldsymbol{b}^*_\tau\|_{-1,p} + \|\boldsymbol{b}^*_j\|_{-1,p} + c_1 \|\hat{\boldsymbol{w}}^*_{I\!N}\|_{-\frac{1}{p},p,I\!N} + c_2 \|\boldsymbol{w}_{I\!D}\|_{1,p}. \tag{3.26}$$

Proof. *(Theorem 4.)* We only describe a sketch of the proof. See for example [26]. See also [11] for interior estimates only, Theorems in §7 and §8. These results can be extended up to the boundary for smooth enough boundaries. □

We also present here a result from [14], stating higher regularity of the weak solution of the momentum constraint Eq. (3.7) in the case that the data and the source function also possess additional regularity.

Theorem 5. (Regularity $\mathbf{W}^{2,p}$) *Assume the hypotheses in Theorem 3, and in addition assume that the boundary set $\partial\mathcal{M}$ is C^2. Assume that the source functionals have the form $\boldsymbol{b}^*_\tau(\underline{\boldsymbol{\omega}}) = (\boldsymbol{b}_\tau, \underline{\boldsymbol{\omega}})$, and $\boldsymbol{b}^*_j(\underline{\boldsymbol{\omega}}) = (\boldsymbol{b}_j, \underline{\boldsymbol{\omega}})$, while the boundary data have the form $\hat{\boldsymbol{w}}^*_{I\!N}(\mathsf{tr}_{I\!N}\underline{\boldsymbol{\omega}}) = (\hat{\boldsymbol{w}}_{I\!N}, \mathsf{tr}_{I\!N}\underline{\boldsymbol{\omega}})_{I\!N}$, for all $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,2}_{I\!D}$. If the following conditions hold*

$$\boldsymbol{w}_{I\!D} \in \boldsymbol{W}^{2,p}, \quad \hat{\boldsymbol{w}}_{I\!N} \in W^{\frac{1}{p'},p}(\partial\mathcal{M}_{I\!N}, 1), \quad \boldsymbol{b}_\tau, \boldsymbol{b}_j \in \boldsymbol{L}^p, \quad p \geqslant \frac{6}{5},$$

then, the solution $\boldsymbol{w}$ to the momentum constraint Eq. (3.7) satisfies that $\boldsymbol{w} \in \boldsymbol{W}^{2,p}$ and there exist positive constants c_1 and c_2 such that the following estimate holds,

$$\|\boldsymbol{w}\|_{2,p} \leqslant \|\phi\|^6_\infty \|\boldsymbol{b}_\tau\|_p + \|\boldsymbol{b}_j\|_p + c_1 \|\hat{\boldsymbol{w}}_{I\!N}\|_{\frac{1}{p'},p,I\!N} + c_2 \|\boldsymbol{w}_{I\!D}\|_{2,p}. \tag{3.27}$$

Proof. *(Theorem 5.)* We only describe a sketch of the proof, which follows [14], Vol. II, page 296. It is based on the fact that the momentum constraint bilinear form $a_{I\!L}$ is strongly elliptic and satisfies the supplementary and complementing conditions given in [2]. □

4. The Hamiltonian constraint

In this section we fix a particular functional a^*_w in an appropriate space and we then look for weak solutions only of the Hamiltonian constraint Eq. (2.35). We first develop the weak formulation more precisely in §4.1, and as in §2.3 we assume the weakest regularity of the equation coefficients such that the equation itself is well-defined. As was the case for the momentum constraint, we will be able to use variational methods to obtain existence (and when possible, uniqueness) results for the Hamiltonian constraint in this weakest setting. First, we establish some preliminary results on generalized local and global barriers (constant sub- and super-solutions) for weak solutions in §4.2. The term local means that the barrier does not depend on the coefficient a^*_w, while global means the barrier does depend on this coefficient. We summarize the generalized local and global barriers in §4.3. In §4.4, we establish some related *a priori* L^∞-bounds on any $W^{1,2}$-solution to the Hamiltonian constraint. In §4.5, we then use the barriers from §4.2, together with a variational argument, to establish existence, and when possible uniqueness, of solutions to the Hamiltonian constraint in the weakest possible setting of $L^\infty \cap W^{1,2}$.

Due to the lack of Gâteaux-differentiability of the nonlinearity in $W^{1,2}$, the connection between the energy used for the variational argument and the Hamiltonian constraint as its Euler condition is non-trivial, and is established through several Lemmas. In §4.6 we give a second (non-variational) argument for existence, using a barriers approach as in most of the earlier work [29, 30], which requires additional regularity on the equation coefficients. Regularity of solutions is discussed briefly in §4.7.

The results obtained using variational methods in §4.5 can be viewed as lowering the regularity of the recent result of Maxwell on "rough" CMC solutions in $W^{k,2}$ for $k > 3/2$ down to $L^\infty \cap W^{1,2}$. We note that the barrier-based existence results for the Hamiltonian constraint equation in §4.6, and the compactness argument in §5.1 giving existence for the coupled non-CMC system, require higher regularity on the equation coefficients. However, we still end up with some non-CMC results for the coupled system in weaker settings and in more general physical situations than have been previously obtained. These additional assumptions are clearly stated in those sections.

4.1. **Weak formulation.** Let $(\mathcal{M}, h)$ be a 3-dimensional Riemannian manifold, where $\mathcal{M}$ is a smooth, compact manifold with Lipschitz boundary $\partial\mathcal{M}$, and $h \in C^2(\overline{\mathcal{M}}, 2)$ is a positive definite metric. Introduce the bilinear form

$$a_L : W^{1,2} \times W^{1,2} \to \mathbb{R}, \qquad a_L(\phi, \underline{\phi}) := (\nabla\phi, \nabla\underline{\phi}) + (K\, \mathsf{tr}_N\phi, \mathsf{tr}_N\underline{\phi})_N, \tag{4.1}$$

where the Robin function $K \in L^\infty(\partial\mathcal{M}_N, 0)$ satisfies the bound

$$\hat{\mathtt{k}}\, \|\mathsf{tr}_N\phi\|_N^2 \leqslant (K\,\mathsf{tr}_N\phi, \mathsf{tr}_N\phi)_N, \qquad \forall\phi \in W^{1,2}, \tag{4.2}$$

with $\hat{\mathtt{k}}$ being a non-negative constant. Fix the functionals

$$a^*_\tau \in W_D^{-1,2}, \quad a^*_\rho \in W_{D+}^{-1,2}, \quad a^*_w \in W_D^{-1,2}. \tag{4.3}$$

The assumption on the background metric implies that the function a_R is continuous on the manifold $\overline{\mathcal{M}}$, so the functional $a^*_R \in W_D^{-1,2}$ given by

$$a^*_R(\underline{\varphi}) := (a_R, \underline{\varphi}), \qquad \forall\, \underline{\varphi} \in W_D^{1,2} \tag{4.4}$$

is well-defined. Given any two functions ϕ_1, $\phi_2 \in L^\infty$ with $0 < \phi_1 \leqslant \phi_2$, define the interval

$$[\phi_1, \phi_2] := \{\phi \in L^\infty : \phi_1 \leqslant \phi \leqslant \phi_2\},$$

which is a closed, bounded set in L^∞, and also in L^2. Introduce the nonlinear operator

$$f_{wF} : [\phi_1, \phi_2] \subset L^2 \to W_D^{-1,2},$$
$$f_{wF}(\phi) := (a_\tau\phi^5)^* + (a_R\phi)^* - (a_\rho\phi^{-3})^* - (a_w\phi^{-7})^*. \tag{4.5}$$

We used the subscript $\boldsymbol{w}$ in f_{wF} to emphasize that $\boldsymbol{w}$ is not a variable for the analysis of the Hamiltonian constraint in isolation from the momentum constraint. The functional f_{wF} is the generalization of the functional F defined in Eq. (2.12). We remark that the operator defined in Eq. (4.5) is continuous but *not Gâteaux differentiable*. It has Gâteaux derivatives only along directions in L^∞, not in the whole space L^2. This property of the functional f_{wF} will introduce some technical complexity in the use of variational methods for the Hamiltonian constraint (see §4.5).

The **weak Dirichlet-Robin boundary value formulation** for the Hamiltonian constraint is the following: Fix Dirichlet and Robin boundary data

$$0 < \operatorname{ess}\inf_{\mathcal{M}_D} \hat{\phi}_D \leqslant \hat{\phi}_D \in L^{\infty}(\partial\mathcal{M}_D, 0) \cap W^{\frac{1}{2},2}(\partial\mathcal{M}_D, 0), \quad \hat{\phi}^*_N \in W^{-\frac{1}{2},2}(\partial\mathcal{M}_N, 0); \tag{4.6}$$

Introduce an extension ϕ_D of the Dirichlet boundary data as explained in §2.3, in particular, given $\hat{\phi}_D > 0$ on $\partial\mathcal{M}_D$, we can use the Laplace-Beltrami operator to harmonically extend $\hat{\phi}_D$ to ϕ_D such that $\phi_D > 0$ a.e. in $\mathcal{M}$. Given such extension function ϕ_D, fix any two functions ϕ_1, $\phi_2 \in L^{\infty} \cap W^{1,2}$, with the property that $0 < \phi_1 \leqslant \phi_2$ and such that $\phi_D \in [\phi_1, \phi_2] \cap W^{1,2}$; Introduce the non-principal part operator including the Robin boundary conditions,

$$f_w : [\phi_1, \phi_2] \subset L^2 \to W^{-1,2}_D, \qquad f_w(\phi)(\underline{\varphi}) := f_{wF}(\phi)(\underline{\varphi}) - \hat{\phi}^*_N(\mathsf{tr}_N\underline{\varphi}), \tag{4.7}$$

where the functional f_{wF} is given by Eq. (4.5); Let $A^{1,2}$ be the affine space defined in Eq. (2.29), which includes the Dirichlet boundary condition; Then, find an element $\phi \in [\phi_1, \phi_2] \cap A^{1,2}$ solution of the equation

$$a_L(\phi, \underline{\varphi}) + f_w(\phi)(\underline{\varphi}) = 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_D. \tag{4.8}$$

As was the case earlier for analysis of the momentum constraint, it is convenient to express Eq. (4.8) in terms of operators instead of bilinear forms. Introduce the operator

$$A_L : W^{1,2} \to W^{-1,2}_D, \qquad A_L\phi(\underline{\varphi}) := a_L(\phi, \underline{\varphi}).$$

Also recall that, if given any $\phi \in [\phi_1, \phi_2]$, then $f_w(\phi) \in W^{-1,2}_D$. Hence, Eq. (4.8) written in terms of operators is the following: find an element $\phi \in [\phi_1, \phi_2] \cap A^{1,2}$ solution of

$$A_L\phi + f_w(\phi) = 0. \tag{4.9}$$

Lemma 5. *Given a smooth vector field $\boldsymbol{w}$, every smooth function ϕ solution of the classical Dirichlet-Robin boundary value formulation for the Hamiltonian constraint Eq. (2.16) is also a solution of the weak formulation with Eq. (4.8) corresponding to the equation coefficients and Robin data function given by the following expressions, which hold for all $\underline{\varphi} \in W^{1,2}_D$,*

$$a^*_\tau(\underline{\varphi}) := (a_\tau, \underline{\varphi}), \qquad a^*_\rho(\underline{\varphi}) := (a_\rho, \underline{\varphi}),$$
$$a^*_w(\underline{\varphi}) := (a_w, \underline{\varphi}), \qquad \hat{\phi}^*_N(\mathsf{tr}_N\underline{\varphi}) := (\hat{\phi}_N, \mathsf{tr}_N\underline{\varphi})_N.$$

Proof. *(Lemma 5.)* The proof is similar to the proof of Lemma 1 and it is not reproduced here. □

Given any function $u \in W^{1,2}$, recall the notation

$$u^+ := \operatorname{ess}\max\{u, 0\}, \qquad u^- := -\operatorname{ess}\min\{u, 0\}.$$

An element $\phi_- \in W^{1,2}$ is called a **sub-solution** of Eq. (4.9) iff the function ϕ_- satisfies the inequalities

$$(\phi_D - \phi_-)^- \in W^{1,2}_D \text{ and } -\big[A_L\phi_- + f_w(\phi_-)\big] \in W^{-1,2}_{D+}. \tag{4.10}$$

An element $\phi_+ \in W^{1,2}$ is called a **super-solution** of Eq. (4.9) iff the scalar function ϕ_+ satisfies the inequalities

$$(\phi_D - \phi_+)^+ \in W^{1,2}_D \text{ and } \big[A_L\phi_+ + f_{\boldsymbol{w}}(\phi_+)\big] \in W^{-1,2}_{D+}. \tag{4.11}$$

The sub and super-solutions of Eq. (4.9) may depend on the choice of a^*_w. A sub-solution is called **global** iff Eq. (4.10) holds for every functional $a^*_w \in W_D^{-1,2}$, and it is called **local** iff it is not global.

4.2. **Global and local barriers.** In this section we show that there exist sub- and super-solutions to the Hamiltonian constraint equation (4.9) for different assumptions on the equation coefficients. The results in Lemmas 6-10 are generalizations to the weak problem of the barriers found in [30] in the case of closed manifolds, scalar curvature $R = -1$, and equation coefficients with higher regularity. The main idea of this generalization is to look at candidates for sub- and super-solutions only among the constant functions, and not among all functions in $[\phi_1, \phi_2] \subset L^\infty$, where $0 < \phi_1 \leqslant \phi_2$. This type of approach is reasonable, since in the smooth coefficient case there exist sub- and super-solutions which are indeed constants.

Assume that the background metric h belongs to $C^2(\overline{\mathcal{M}}, 2)$, then the Ricci scalar of curvature R is a continuous function on the manifold $\overline{\mathcal{M}}$. Introduce the constants

$$a^\wedge_\tau := \sup_{0\neq\underline{\varphi}\in W^{1,2}_{D+}} \frac{a^*_\tau(\underline{\varphi})}{\|\underline{\varphi}\|_{1,2}}, \qquad a^\wedge_R := \sup_{0\neq\underline{\varphi}\in W^{1,2}_{D+}} \frac{(|a_R|, \underline{\varphi})}{\|\underline{\varphi}\|_{1,2}}, \tag{4.12}$$

$$a^\wedge_\rho := \sup_{0\neq\underline{\varphi}\in W^{1,2}_{D+}} \frac{a^*_\rho(\underline{\varphi})}{\|\underline{\varphi}\|_{1,2}}, \qquad a^\wedge_w := \sup_{0\neq\underline{\varphi}\in W^{1,2}_{D+}} \frac{a^*_w(\underline{\varphi})}{\|\underline{\varphi}\|_{1,2}}. \tag{4.13}$$

In order that the Lemmas below also hold for the particular case when the Ricci scalar R vanishes identically, we introduce the constant $\bar{a}^\wedge_R := 1 + a^\wedge_R$. Given a two-index tensor $\sigma \in L^p(\mathcal{M}, 2)$ and a vector field $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$, with $p = 12/5$, introduce the functionals a^*_σ and $a^*_{\mathcal{L}w}$ given by $a^*_\sigma(\underline{\varphi}) = (\sigma^2, \underline{\varphi})/8$ and $a^*_{\mathcal{L}w}(\underline{\varphi}) = \big((\mathcal{L}\boldsymbol{w})^2, \underline{\varphi}\big)/8$, where $\underline{\phi} \in W^{1,2}_D$. Now introduce the further constants

$$a^\wedge_\sigma := \sup_{0\neq\underline{\varphi}\in W^{1,2}_{D+}} \frac{a^*_\sigma(\underline{\varphi})}{\|\underline{\varphi}\|_{1,2}}, \qquad K^\wedge := \sup_{0\neq\underline{\varphi}\in W^{1,2}_{D+}} \frac{(K, \mathsf{tr}_N\underline{\varphi})}{\|\underline{\varphi}\|_{1,2}},$$

$$\hat{\phi}^\wedge_N := \sup_{0\neq\underline{\varphi}\in W^{1,2}_{D+}} \frac{\hat{\phi}^*_N(\mathsf{tr}_N\underline{\varphi})}{\|\underline{\varphi}\|_{1,2}}, \qquad \phi^\wedge_D := \sup_{\mathcal{M}} \phi_D,$$

where we recall that the function ϕ_D is the harmonic extension of the Dirichlet boundary data $\hat{\phi}_D$ discussed in §2.3. In an analogous way, switching sup to inf, introduce the quantities $a^\vee_\tau$, $a^\vee_R$, $a^\vee_\rho$, $a^\vee_w$, $a^\vee_\sigma$, $K^\vee$, $\hat{\phi}^\vee_N$ and $\phi^\vee_D$.

Lemma 6. (Local super-solution R bounded) *Consider the weak formulation for the Hamiltonian constraint given in §4.1. Assume that the constants $a^\vee_\tau$, $K^\vee$ are positive, and denote by ϕ_{w+} the constant*

$$\phi_{w+} := \max\Big\{1,\ \Big[\frac{\bar{a}^\wedge_R + a^\wedge_\rho + a^\wedge_w}{a^\vee_\tau}\Big]^{1/4},\ \frac{\hat{\phi}^\wedge_N}{K^\vee},\ \phi^\wedge_D\Big\}. \tag{4.14}$$

Then, ϕ_{w+} is a local super-solution of Eq. (4.9).

Proof. *(Lemma 6.)* We look for a super-solution among the constant functions. Therefore, let ϕ_0 be any constant in $[\phi_1, \phi_2]$, with $\phi_1 > 0$, then the following inequalities hold for every element $\underline{\varphi} \in W^{1,2}_{D+}$,

$$
\begin{aligned}
f_{wF}(\phi_0)(\underline{\varphi}) &= (a_\tau \phi_0^5)^*(\underline{\varphi}) + (a_R\phi_0)^*(\underline{\varphi}) - (a_\rho \phi_0^{-3})^*(\underline{\varphi}) - (a_w\phi_0^{-7})^*(\underline{\varphi}) \\
&= a_\tau^*(\underline{\varphi})\,\phi_0^5 + a_R^*(\underline{\varphi})\,\phi_0 - a_\rho^*(\underline{\varphi})\,\phi_0^{-3} - a_w^*(\underline{\varphi})\,\phi_0^{-7} \\
&\geqslant \big(a_\tau^\vee\,\phi_0^5 - \bar{a}_R^\wedge\,\phi_0 - a_\rho^\wedge\,\phi_0^{-3} - a_w^\wedge\,\phi_0^{-7}\big)\,\|\underline{\varphi}\|_{1,2}.
\end{aligned}
$$

Introduce the polynomial on ϕ_0 given by

$$
q(\phi_0) := a_\tau^\vee\,\phi_0^5 - \bar{a}_R^\wedge\,\phi_0 - a_\rho^\wedge\,\phi_0^{-3} - a_w^\wedge\,\phi_0^{-7}. \tag{4.15}
$$

The assumptions that the constants $a_\tau^\vee$ and $\bar{a}_R^\wedge$ are strictly positive, while the constants $a_\rho^\wedge$ and $a_w^\wedge$ are non-negative imply that there exists a unique positive root of this polynomial. The proof consists of three steps. First, there exists at least one positive root of the polynomial q, because for ϕ_0 large enough $q(\phi_0)$ is positive, and for ϕ_0 close to zero from positive values $q(\phi_0)$ is negative, as it can be seen from the following expression,

$$
q(\phi_0) = \phi_0^{-7}\big[a_\tau^\vee\,\phi_0^{12} - \bar{a}_R^\wedge\,\phi_0^8 - a_\rho^\wedge\,\phi_0^4 - a_w^\wedge\big],
$$

where the term between brackets becomes negative for small enough ϕ_0. Second, this positive root is unique, since the function q is increasing for all $\phi_0 > \alpha_0 := [\bar{a}_R^\wedge/(5a_\tau^\vee)]^{1/4}$ (the proof is to verify that $q' > 0$ for $\phi_0 > \alpha_0$); and the function q satisfies the inequality $q(\phi_0) \leqslant r(\phi_0) := a_\tau^\vee\,\phi_0^5 - \bar{a}_R^\wedge\,\phi_0$ for all positive numbers ϕ_0. Since $r(\alpha_1) = 0$ for $\alpha_1 := [\bar{a}_R^\wedge/a_\tau^\vee]^{1/4}$, and $\alpha_1 > \alpha_0$ (so the root of the polynomial q must belong to the interval where q is increasing), we then conclude that the root of the polynomial q is unique. Denote by $\bar{\phi}_0$ the unique positive root of the polynomial q. Since $\bar{\phi}_0 > \alpha_0$, then $q(\phi_0) > q(\bar{\phi}_0) = 0$ for any $\phi_0 > \bar{\phi}_0$. The idea now is to find an upper bound for the root $\bar{\phi}_0$. The result is going to be the first two expressions on the right hand side in Eq. (4.14); the remaining two terms on the right hand side of Eq. (4.14) will account for the boundary contributions.

In the case that $\bar{\phi}_0 \leqslant 1$ (which could be verified, for example, by explicit evaluation), then choose a candidate for super-solution to be $\tilde{\phi}_{w+} = 1$. In the case that $\bar{\phi}_0 > 1$, then there exists an upper bound for this root, as can be seen from the following argument. Given any $\phi_0 \geqslant 1$, then the following inequalities hold,

$$
(\phi_0)^{n+1} \geqslant 1 \quad\Rightarrow\quad -(\phi_0)^{-(n+1)} \geqslant -1 \quad\Rightarrow\quad -(\phi_0)^{-n} \geqslant -\phi_0,
$$

therefore, this inequality for $n = 3$ and $n = 7$ implies that for all $\phi_0 \geqslant 1$ holds

$$
q(\phi_0) \geqslant s(\phi_0) := a_\tau^\vee\,(\phi_0)^5 - \big(\bar{a}_R^\wedge + a_\rho^\wedge + a_w^\wedge\big)\phi_0.
$$

This new polynomial s vanishes at

$$
\bar{\phi}_1 := \Big[\frac{\bar{a}_R^\wedge + a_\rho^\wedge + a_w^\wedge}{a_\tau^\vee}\Big]^{1/4},
$$

and the inequality $\bar{\phi}_1 > \alpha_0$ says that $\bar{\phi}_1$ belongs to the interval where the polynomial q is increasing. Therefore, we have that

$$
q(\bar{\phi}_1) \geqslant s(\bar{\phi}_1) = 0 \text{ and } q(\bar{\phi}_0) = 0 \quad\Rightarrow\quad \bar{\phi}_1 \geqslant \bar{\phi}_0.
$$

So, in this case $\bar{\phi}_0 > 1$, choose the candidate for super-solution to be $\tilde{\phi}_{w+} = \bar{\phi}_1$. Then, introducing the constant

$$
\tilde{\phi}_{w+} := \max\Big\{1, \Big[\frac{\bar{a}_R^\wedge + a_\rho^\wedge + a_w^\wedge}{a_\tau^\vee}\Big]^{1/4}\Big\},
$$

we have established that the following inequality holds

$$f_{wF}(\phi_0)(\underline{\varphi}) \geqslant 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}, \qquad \forall\, \phi_0 \geqslant \tilde{\phi}_{w+}. \tag{4.16}$$

We now account for the boundary contributions. The definitions of $\hat{\phi}^{\wedge}_N$ and $K^{\vee}$ imply that for any constant ϕ_0 holds

$$\big([K\phi_0 - \hat{\phi}_N], \mathrm{tr}_N \underline{\varphi}\big)_N \geqslant (K^{\vee}\phi_0 - \hat{\phi}^{\wedge}_N)\, \|\underline{\varphi}\|_{1,2} \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}.$$

In particular, defining the constant

$$\overline{\phi}_{w+} := \max\Big\{1,\ \Big[\frac{\bar{a}^{\wedge}_R + a^{\wedge}_{\rho} + a^{\wedge}_w}{a^{\vee}_{\tau}}\Big]^{1/4},\ \frac{\hat{\phi}^{\wedge}_N}{K^{\vee}}\Big\},$$

follows that $\overline{\phi}_{w+} \geqslant \tilde{\phi}_{w+}$ and the following inequality holds

$$\big([K\phi_0 - \hat{\phi}_N], \mathrm{tr}_N \underline{\varphi}\big)_N \geqslant 0 \qquad \forall\, \phi_0 \geqslant \overline{\phi}_{w+}. \tag{4.17}$$

Adding Eqs. (4.16) and (4.17) we conclude that

$$(K\phi_0, \mathrm{tr}_N \underline{\varphi})_N + f_w(\phi_0)(\underline{\varphi}) \geqslant 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}, \qquad \forall\, \phi_0 \geqslant \overline{\phi}_{w+}.$$

Recalling now that any constant ϕ_0 satisfies $(\nabla\phi_0, \nabla\underline{\varphi}) = 0$, we conclude that

$$A_L\phi_0(\underline{\varphi}) + f_w(\phi_0)(\underline{\varphi}) \geqslant 0 \qquad \forall \underline{\varphi} \in W^{1,2}_{D+}, \qquad \forall\, \phi_0 \geqslant \overline{\phi}_{w+}.$$

Finally, introduce the constant ϕ_{w+} as given by Eq. (4.14). In particular, this constant satisfies $(\phi_D - \phi_{w+})^+ = 0$ and $\phi_{w+} \geqslant \overline{\phi}_{w+}$, so the following inequality holds,

$$(\phi_D - \phi_{w+})^+ \in W^{1,2}_{D+} \text{ and } \big[A_L\phi_{w+} + f_w(\phi_{w+})\big] \in W^{-1,2}_{D+},$$

which establishes that $\phi_{\boldsymbol{w}+}$ is a super-solution of Eq. (4.9). □

We now find a global super-solution for the Hamiltonian and momentum constraint Eq. (2.35)-(2.36), where global means that the super-solution is independent of the vector field $\boldsymbol{w}$ solution of the momentum constraint Eq. (2.36). This global super-solution is a generalization suitable to our weak setting of the super-solution derived in [30]. We use the same idea as in §4.2, that is, we look for super-solutions only among the constant functions. The "near-CMC" assumption on the trace of the extrinsic curvature made in [30] to construct the super-solution is still present here, although in weaker norms.

Lemma 7. (Global super-solution R bounded) *Consider the weak formulation for the Hamiltonian and momentum constraints given in §2.3. Assume that the numbers $(a^{\vee}_{\tau} - \mathtt{K}_1)$ and $K^{\vee}$ are positive, where the constant $\mathtt{K}_1$ is defined in Eq. (4.21). Denote by ϕ_+ the constant*

$$\phi_+ := \max\Big\{1,\ \Big[\frac{\bar{a}^{\wedge}_R + a^{\wedge}_{\rho} + \mathtt{K}_2}{a^{\vee}_{\tau} - \mathtt{K}_1}\Big]^{1/4},\ \frac{\hat{\phi}^{\wedge}_N}{K^{\vee}},\ \hat{\phi}^{\wedge}_D\Big\}, \tag{4.18}$$

with the constant $\mathtt{K}_2$ defined in Eq. (4.22). Then, the constant ϕ_+ is a global super-solution of Eqs. (2.35)-(2.36).

Proof. *(Lemma 7.)* Consider the weak formulation in §2.3. Let ϕ_0 be any constant in $[\phi_1, \phi_2]$ with $\phi_1 > 0$, then the following inequalities hold,

$$\begin{aligned} f_F(\phi_0, \boldsymbol{w})(\underline{\varphi}) &= (a^*_{\tau}\phi_0^5)(\underline{\varphi}) + (a^*_R\phi_0)(\underline{\varphi}) - (a^*_{\rho}\phi_0^{-3})(\underline{\varphi}) - (a^*_w\phi_0^{-7})(\underline{\varphi}) \\ &= a^*_{\tau}(\underline{\varphi})\,\phi_0^5 + a^*_R(\underline{\varphi})\,\phi_0 - a^*_{\rho}(\underline{\varphi})\,\phi_0^{-3} - a^*_w(\underline{\varphi})\,\phi_0^{-7} \\ &\geqslant \big(a^{\vee}_{\tau}\,\phi_0^5 - \bar{a}^{\wedge}_R\,\phi_0 - a^{\wedge}_{\rho}\,\phi_0^{-3} - a^{\wedge}_w\,\phi_0^{-7}\big)\,\|\underline{\varphi}\|_{1,2}. \end{aligned} \tag{4.19}$$

The number $a_w^\wedge$ in the last term is bounded when $\boldsymbol{w}$ is solution of the momentum constraint Eq. (2.32) with any source $\phi \in [\phi_1, \phi_2]$. For the proof, start with the definition of $a_w^\wedge$ in §4.2, where $a_w^*(\underline{\varphi}) = (a_w, \underline{\varphi})$ with $a_w \in L^{6/5}$ and $\underline{\varphi} \in W_D^{1,2}$. Then, the following inequalities hold

$$(a_w, \underline{\varphi}) \leqslant \|a_w\|_{\frac{6}{5}} \, \|\underline{\varphi}\|_6 \leqslant c_s \, \|a_w\|_{\frac{6}{5}} \, \|\underline{\varphi}\|_{1,2},$$

where we used the imbedding $W^{1,2} \subset L^6$, and where c_s is the positive imbedding constant that relates the norm of these spaces. The inequality above holds for all $\underline{\varphi} \in W_{D+}^{1,2}$, and in particular holds for the supremum in that space, hence

$$a_w^\wedge \leqslant c_s \, \|a_w\|_{\frac{6}{5}}.$$

Now, the definition of the function a_w, standard inequalities and the notation $p = 12/6$ show that,

$$\|a_w\|_{\frac{6}{5}} = \frac{1}{8} \, \|\sigma + \mathcal{L}\boldsymbol{w}\|_p^2 \leqslant \frac{1}{4} \Big(\|\sigma\|_p^2 + \|\mathcal{L}\boldsymbol{w}\|_p^2 \Big) \leqslant \frac{1}{4} \Big(\|\sigma\|_p^2 + c_{\mathcal{L}} \, \|\boldsymbol{w}\|_{1,p}^2 \Big).$$

In §3.4 and §3.5 it is shown that there exist positive constants c_1 and c_2 such that

$$\|\boldsymbol{w}\|_{1,p} \leqslant \|\phi\|_\infty^6 \, \|\boldsymbol{b}_\tau^*\|_{-1,p} + \|\boldsymbol{b}_j^*\|_{-1,p} + c_1 \, \|\hat{\boldsymbol{w}}_{I\!N}^*\|_{-\frac{1}{p},p,I\!N} + c_2 \, \|\boldsymbol{w}_{I\!D}\|_{1,p}.$$

Then, the bound for the number $a_w^\wedge$ can be written as

$$a_w^\wedge \leqslant \mathtt{K}_1 \, \|\phi\|_\infty^{12} + \mathtt{K}_2, \tag{4.20}$$

where the constants $\mathtt{K}_1$ and $\mathtt{K}_2$ are given by

$$\mathtt{K}_1 := 4 c_s c_{\mathcal{L}} \|\boldsymbol{b}_\tau^*\|_{-1,p}^2, \tag{4.21}$$

$$\mathtt{K}_2 := \frac{c_s}{4} \, \|\sigma\|_p^2 + c_s c_{\mathcal{L}} \Big(\|\boldsymbol{b}_j^*\|_{-1,p}^2 + c_1^2 \, \|\hat{\boldsymbol{w}}_{I\!N}^*\|_{-\frac{1}{p},p,I\!N}^2 + c_2^2 \, \|\boldsymbol{w}_{I\!D}\|_{1,p}^2 \Big). \tag{4.22}$$

Introducing this expression in Eq. (4.19), one finds that for all ϕ_0 constant and all ϕ, both in $[\phi_1, \phi_2]$, it holds that

$$f_F(\phi_0, \boldsymbol{w}) \geqslant \Big[a_\tau^\vee \, \phi_0^5 - \bar{a}_R^\wedge \, \phi_0 - a_\rho^\wedge \, \phi_0^{-3} - \big(\mathtt{K}_1 \, \|\phi\|_\infty^{12} + \mathtt{K}_2 \big) \phi_0^{-7} \Big] \, \|\underline{\varphi}\|_{1,2}.$$

Now, evaluate this expression at $\phi_0 = \phi = \phi_2$. The result is

$$f_F(\phi_2, \boldsymbol{w}) \geqslant \Big[\big(a_\tau^\vee - \mathtt{K}_1 \big) \, \phi_2^5 - \bar{a}_R^\wedge \, \phi_2 - a_\rho^\wedge \, \phi_2^{-3} - \mathtt{K}_2 \, \phi_2^{-7} \Big] \, \|\underline{\varphi}\|_{1,2}. \tag{4.23}$$

The assumption in Lemma 7 implies that $a_\tau^\vee - \mathtt{K}_1 > 0$, so the polynomial

$$\tilde{q}(\phi_2) := \big(a_\tau^\vee - \mathtt{K}_1 \big) \, \phi_2^5 - \bar{a}_R^\wedge \, \phi_2 - a_\rho^\wedge \, \phi_2^{-3} - \mathtt{K}_2 \, \phi_2^{-7}$$

has the same form as the polynomial q introduced in Eq. (4.15). Therefore, the analysis done on the polynomial q in the proof of Lemma 6 holds for the polynomial $\tilde{q}$, in particular, $\tilde{q}$ has a unique positive root $\bar{\phi}_0$. The remainder of the proof involves finding an upper bound for $\bar{\phi}_0$, and the argument is almost identical to the one given in the proof of Lemma 6.

In the case that $\bar{\phi}_0 \leqslant 1$ (which could be verified, for example, by explicit evaluation), then choose a candidate for super-solution to be $\tilde{\phi}_+ = 1$. In the case that $\bar{\phi}_0 > 1$, then there exists an upper bound for this root, as can be seen from the following argument. Given any $\phi_0 \geqslant 1$, then the following inequalities hold,

$$(\phi_0)^{n+1} \geqslant 1 \quad \Rightarrow \quad -(\phi_0)^{-(n+1)} \geqslant -1 \quad \Rightarrow \quad -(\phi_0)^{-n} \geqslant -\phi_0,$$

therefore, this inequality for $n = 3$ and $n = 7$ implies that for all $\phi_0 \geqslant 1$ holds

$$\tilde{q}(\phi_0) \geqslant \tilde{s}(\phi_0) := \big(a_\tau^\vee - \mathtt{K}_1 \big) \, (\phi_0)^5 - \big(\bar{a}_R^\wedge + a_\rho^\wedge + \mathtt{K}_2 \big) \phi_0.$$

This new polynomial $\tilde{s}$ vanishes at the point

$$\bar{\phi}_1 := \Big[\frac{\bar{a}_R^{\wedge} + a_\rho^{\wedge} + \mathtt{K}_2}{a_\tau^{\vee} - \mathtt{K}_1}\Big]^{1/4},$$

which belongs to the interval where the polynomial $\tilde{q}$ is increasing. Therefore, we have that

$$\tilde{q}(\bar{\phi}_1) \geqslant \tilde{s}(\bar{\phi}_1) = 0 \quad \Rightarrow \quad \bar{\phi}_1 \geqslant \bar{\phi}_0.$$

So, in this case $\bar{\phi}_0 > 1$, choose the candidate for super-solution to be $\tilde{\phi}_+ = \bar{\phi}_1$. Then, introducing the constant

$$\tilde{\phi}_+ := \max\Big\{1, \Big[\frac{\bar{a}_R^{\wedge} + a_\rho^{\wedge} + \mathtt{K}_2}{a_\tau^{\vee} - \mathtt{K}_1}\Big]^{1/4}\Big\},$$

we have established that the following inequality holds

$$f_F(\phi_2, \boldsymbol{w})(\underline{\varphi}) \geqslant 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}, \qquad \forall\, \phi_2 \geqslant \tilde{\phi}_+, \tag{4.24}$$

and for all vector field $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ solution of the momentum constraint Eq. (2.36) with source function $\phi \in [0, \phi_2]$.

As in the proof of Lemma 6, what remains is to account for the boundary contributions. The definitions of $\hat{\phi}_N^{\wedge}$ and $K^{\vee}$ imply that for any constant ϕ_2 holds

$$\big([K\phi_2 - \hat{\phi}_N], \mathsf{tr}_N\underline{\varphi}\big)_N \geqslant (K^{\vee}\phi_2 - \hat{\phi}_N^{\wedge})\, \|\underline{\varphi}\|_{1,2} \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}.$$

In particular, defining the constant

$$\overline{\phi}_+ := \max\Big\{1,\ \Big[\frac{\bar{a}_R^{\wedge} + a_\rho^{\wedge} + \mathtt{K}_2}{a_\tau^{\vee} - \mathtt{K}_1}\Big]^{1/4},\ \frac{\hat{\phi}_N^{\wedge}}{K^{\vee}}\Big\},$$

follows that $\overline{\phi}_+ \geqslant \tilde{\phi}_+$ and the following inequality holds

$$\big([K\phi_2 - \hat{\phi}_N], \mathsf{tr}_N\underline{\varphi}\big)_N \geqslant 0 \qquad \forall\, \phi_2 \geqslant \overline{\phi}_+. \tag{4.25}$$

Adding Eqs. (4.24) and (4.25) we conclude that

$$(K\phi_2, \mathsf{tr}_N\underline{\varphi})_N + f(\phi_2, \boldsymbol{w})(\underline{\varphi}) \geqslant 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}, \qquad \forall\, \phi_2 \geqslant \overline{\phi}_+,$$

and for all $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ solution of the momentum constraint Eq. (2.36) with source function $\phi \in [0, \phi_2]$. Recalling now that any constant ϕ_2 satisfies $(\nabla\phi_2, \nabla\underline{\varphi}) = 0$, we conclude that

$$A_L\phi_2(\underline{\varphi}) + f(\phi_2, \boldsymbol{w})(\underline{\varphi}) \geqslant 0 \qquad \forall \underline{\varphi} \in W^{1,2}_{D+}, \qquad \forall\, \phi_2 \geqslant \overline{\phi}_+.$$

Finally, employ now the constant ϕ_+ as given by Eq. (4.18). This constant satisfies the conditions $(\hat{\phi}_D - \phi_+)^+ = 0$ and $\phi_+ \geqslant \overline{\phi}_+$, so the following inequality holds,

$$(\phi_D - \phi_+)^+ \in W^{1,2}_{D+} \text{ and } \big[A_L\phi_+ + f(\phi_+, \boldsymbol{w})\big] \in W^{-1,2}_{D+},$$

for all $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ solution of the momentum constraint Eq. (2.36) with source $\phi \in [0, \phi_+]$. This establishes that ϕ_+ is a global super-solution of Eqs. (2.35)-(2.36). $\square$

Consider now the particular case of a background metric $h \in C^2(\overline{\mathcal{M}}, 2)$ having a strictly negative Ricci scalar of curvature, that is,

$$-\, a_R^{\wedge} \leqslant \frac{(a_R, \underline{\varphi})}{\|\underline{\varphi}\|_{1,2}} \leqslant -a_R^{\vee} < 0 \qquad \forall \underline{\varphi} \in W^{1,2}_{D+}. \tag{4.26}$$

In this case it is possible to obtain a global sub-solution of Eq. (4.9).

Lemma 8. **(Global sub-solution for** $R<0$**)** *Consider the weak formulation for the Hamiltonian constraint given in §4.1. Assume that the constants* $a_\tau^\wedge$, $K^\wedge$, $\hat{\phi}_N^\vee$, *and* $\phi_D^\vee$ *are positive, and the Ricci scalar* R *satisfies Eq. (4.26). Denote by* ϕ_- *the constant*

$$\phi_- := \min\Big\{\Big(\frac{a_R^\vee}{a_\tau^\wedge}\Big)^{1/4},\ \frac{\hat{\phi}_N^\vee}{K^\wedge},\ \phi_D^\vee\Big\}. \tag{4.27}$$

Then, ϕ_- *is a global sub-solution of Eq. (4.9).*

Proof. *(Lemma 8.)* We look for the sub-solution among the constant functions. Therefore, let ϕ_0 be any constant in $[\phi_1,\phi_2]$, with $\phi_1>0$, then the following inequalities hold for every element $\underline{\varphi}\in W^{1,2}_{D+}$,

$$\begin{aligned} f_{wF}(\phi_0)(\underline{\varphi}) &= (a_\tau\phi_0^5)^*(\underline{\varphi}) + (a_R\phi_0)^*(\underline{\varphi}) - (a_\rho\phi_0^{-3})^*(\underline{\varphi}) - (a_w\phi_0^{-7})^*(\underline{\varphi})\\ &\leqslant (a_\tau\phi_0^5)^*(\underline{\varphi}) + (a_R\phi_0)^*(\underline{\varphi})\\ &\leqslant (a_\tau,\underline{\varphi})\,\phi_0^5 + (a_R,\underline{\varphi})\,\phi_0\\ &\leqslant \Big[a_\tau^\wedge\,\phi_0^5 - a_R^\vee\,\phi_0\Big]\,\|\underline{\varphi}\|_{1,2}, \end{aligned}$$

where we used that both functionals a_ρ^* and a_w^* belong to the space $W^{-1,2}_{D+}$, and the number $\phi_0>0$. Introduce the polynomial

$$q(\phi_0) := a_\tau^\wedge\,\phi_0^5 - a_R^\vee\,\phi_0.$$

There exists a unique positive root for q given by the number $\tilde{\phi}_- := \big(a_R^\vee/a_\tau^\wedge\big)^{1/4}$, and for all $0<\phi_0<\tilde{\phi}_-$ the corresponding values $q(\phi_0)$ are negative. Therefore, the following inequality holds

$$f_{wF}(\phi_0)(\underline{\varphi})\leqslant 0 \qquad \forall\,\underline{\varphi}\in W^{1,2}_{D+}, \qquad \forall\,\phi_0\in(0,\tilde{\phi}_-], \qquad \forall\,a_w^*\in \boldsymbol{W}^{-1,2}_{D+}. \tag{4.28}$$

We now account for the boundary contributions. The definitions of the numbers $\hat{\phi}_N^\vee$ and $K^\wedge$ imply that for any constant ϕ_0, it holds that

$$\big([K\phi_0-\hat{\phi}_N],\mathsf{tr}_N\underline{\varphi}\big)_N \leqslant (K^\wedge\phi_0-\hat{\phi}_N^\vee)\,\|\underline{\varphi}\|_{1,2} \qquad \forall\,\underline{\varphi}\in W^{1,2}_{D+}.$$

In particular, defining the constant

$$\overline{\phi}_- := \min\Big\{\Big(\frac{a_R^\vee}{a_\tau^\wedge}\Big)^{1/4},\ \frac{\hat{\phi}_N^\vee}{K^\wedge}\Big\},$$

it follows that $0<\overline{\phi}_-\leqslant\tilde{\phi}_-$ and the following inequality holds

$$\big([K\phi_0-\hat{\phi}_N],\mathsf{tr}_N\underline{\varphi}\big)_N\leqslant 0, \qquad \forall\,\phi_0\in(0,\overline{\phi}_-]. \tag{4.29}$$

Adding Eqs. (4.28) and (4.29) we conclude that

$$(K\phi_0,\mathsf{tr}_N\underline{\varphi})_N + f_w(\phi_0)(\underline{\varphi})\leqslant 0 \qquad \forall\,\underline{\varphi}\in W^{1,2}_{D+} \qquad \forall\,\phi_0\in(0,\overline{\phi}_-].$$

Recalling now that any constant ϕ_0 satisfies $(\nabla\phi_0,\nabla\underline{\varphi})=0$, we conclude that

$$A_L\phi_0(\underline{\varphi}) + f_w(\phi_0)(\underline{\varphi})\leqslant 0 \qquad \forall\underline{\varphi}\in W^{1,2}_{D+} \qquad \forall\,\phi_0\in(0,\overline{\phi}_-].$$

Finally, introduce the constant ϕ_- as given by Eq. (4.27). In particular, this constant satisfies $(\phi_D-\phi_-)^-=0$ and $0<\phi_-\leqslant\overline{\phi}_-$, so the following inequality holds,

$$(\phi_D-\phi_-)^-\in W^{1,2}_D \text{ and } -\big[A_L\phi_- + f_w(\phi_-)\big]\in W^{-1,2}_{D+}, \qquad \forall\,a_w^*\in\boldsymbol{W}^{-1,2}_{D+},$$

which establishes that ϕ_- is a global sub-solution of Eq. (4.9). $\square$

In the case that the Ricci scalar of curvature is non-negative, then it is not clear whether a constant and positive sub-solution to Eq. (4.9) exists. The latter exists when the conformally rescaled matter energy density ρ satisfies the condition $a_\rho^\vee > 0$. This result is summarized in the following two Lemmas below.

Lemma 9. (Global sub-solution for $R \geqslant 0$ and $a_\rho^\vee > 0$) *Consider the weak formulation for the Hamiltonian constraint given in §4.1. Assume that the constants $a_\tau^\wedge$, $a_\rho^\vee$, $\hat{\phi}_N^\vee$ $K^\wedge$, and $\phi_D^\vee$ are positive. Let ϕ_- be the constant*

$$\phi_- := \min\Big\{ \Big[\frac{1}{2a_\tau^\wedge} \Big(-a_R^\wedge + \sqrt{(a_R^\wedge)^2 + 4a_\tau^\wedge a_\rho^\vee} \Big) \Big]^{1/4},\ \frac{\hat{\phi}_N^\vee}{K^\wedge},\ \phi_D^\vee \Big\}. \tag{4.30}$$

Then, ϕ_- is a global sub-solution of Eq. (4.9).

Proof. *(Lemma 9.)* We look for the sub-solution among the constant functions. Therefore, let ϕ_0 be any constant in $[\phi_1, \phi_2]$ with $\phi_1 > 0$, then the following inequalities hold for every element $\underline{\varphi} \in W^{1,2}_{D+}$,

$$\begin{aligned} f_{wF}(\phi_0)(\underline{\varphi}) &= (a_\tau \phi_0^5)^*(\underline{\varphi}) + (a_R\phi_0)^*(\underline{\varphi}) - (a_\rho \phi_0^{-3})^*(\underline{\varphi}) - (a_w \phi_0^{-7})^*(\underline{\varphi}) \\ &\leqslant (a_\tau \phi_0^5)^*(\underline{\varphi}) + (a_R\phi_0)^*(\underline{\varphi}) - (a_\rho \phi_0^{-3})^*(\underline{\varphi}) \\ &\leqslant a_\tau^*(\underline{\varphi})\, \phi_0^5 + a_R^*(\underline{\varphi})\, \phi_0 - a_\rho^*(\underline{\varphi})\, \phi_0^{-3} \\ &\leqslant \Big[a_\tau^\wedge\, \phi_0^5 + a_R^\wedge\, \phi_0 - a_\rho^\vee\, \phi_0^{-3} \Big]\, \|\underline{\varphi}\|_{1,2}, \end{aligned}$$

where the used the assumption that the functional a_w^* is non-negative. Introduce the polynomial

$$q(\phi_0) := a_\tau^\wedge\, \phi_0^5 + a_R^\wedge\, \phi_0 - a_\rho^\vee\, \phi_0^{-3},$$

which is a non-decreasing function, because its derivative

$$q'(\phi_0) = 5\, a_\tau^\wedge\, \phi_0^4 + a_R^\wedge + 3\, a_\rho^\vee\, \phi_0^{-4},$$

is strictly positive for non-zero ϕ_0. Rewrite the polynomial q as follows,

$$q(\phi_0) \leqslant \phi_0^{-3}\, q_\rho(\phi_0) \quad \text{with} \quad q_\rho(\phi_0) := a_\tau^\wedge\, \phi_0^8 + a_R^\wedge\, \phi_0^4 - a_\rho^\vee. \tag{4.31}$$

There exists a unique positive root ϕ_ρ of the polynomial q_ρ given by

$$\phi_\rho = \Big[\frac{1}{2a_\tau^\wedge} \Big(-a_R^\wedge + \sqrt{(a_R^\wedge)^2 + 4a_\tau^\wedge a_\rho^\vee} \Big) \Big]^{1/4}.$$

Then, the inequality in Eq. (4.31) and the non-decreasing property of q imply that the polynomial q satisfies

$$q(\phi_0) \leqslant 0 \qquad \forall\, 0 < \phi_0 \leqslant \phi_\rho.$$

We then summarize the discussion above saying that the following inequality holds

$$f_{wF}(\phi_0)(\underline{\varphi}) \leqslant 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}, \qquad \forall\, \phi_0 \in (0, \phi_\rho], \tag{4.32}$$

and for all $a_w^* \in \boldsymbol{W}^{-1,2}_{D+}$. From this point forward, the proof is identical to the proof of Lemma 8. What remains is to account for the boundary contributions. The definitions of $\hat{\phi}_N^\vee$ and $K^\wedge$ imply that for any constant ϕ_0, it holds that

$$\big([K\phi_0 - \hat{\phi}_N], \mathsf{tr}_N \underline{\varphi} \big)_N \leqslant (K^\wedge \phi_0 - \hat{\phi}_N^\vee)\, \|\underline{\varphi}\|_{1,2} \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+}.$$

In particular, defining the constant $\overline{\phi}_{-} := \min\big\{\phi_\rho,\ \dfrac{\hat{\phi}_N^\vee}{K^\wedge}\big\}$, it follows that $0 < \overline{\phi}_{-} \leqslant \phi_\rho$, and the following inequality holds

$$\big([K\phi_0 - \hat{\phi}_N], \mathsf{tr}_N \underline{\varphi}\big)_N \leqslant 0, \qquad \forall\, \phi_0 \in (0, \overline{\phi}_{-}]. \tag{4.33}$$

Adding Eqs. (4.32) and (4.33) we conclude that

$$(K\phi_0, \mathsf{tr}_N\underline{\varphi})_N + f_w(\phi_0)(\underline{\varphi}) \leqslant 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+} \qquad \forall\, \phi_0 \in (0, \overline{\phi}_{-}].$$

Recalling now that any constant ϕ_0 satisfies $(\nabla\phi_0, \nabla\underline{\varphi}) = 0$, we conclude that

$$A_L\phi_0(\underline{\varphi}) + f_w(\phi_0)(\underline{\varphi}) \leqslant 0 \qquad \forall \underline{\varphi} \in W^{1,2}_{D+} \qquad \forall\, \phi_0 \in (0, \overline{\phi}_{-}].$$

Finally, introduce the constant ϕ_{-} as given by Eq. (4.30). In particular, this constant satisfies $(\phi_D - \phi_{-})^{-} = 0$ and $0 < \phi_{-} \leqslant \overline{\phi}_{-}$, so the following inequality holds,

$$(\phi_D - \phi_{-})^{-} \in W^{1,2}_D \text{ and } -\big[A_L\phi_{-} + f_w(\phi_{-})\big] \in W^{-1,2}_{D+}, \qquad \forall\, a_w^* \in W^{-1,2}_{D+},$$

which establishes that ϕ_{-} is a global sub-solution of Eq. (4.9). □

Again in the case that the Ricci scalar of curvature R is non-negative there exists a sub-solution to the Hamiltonian and momentum constraint Eqs. (2.35)-(2.36), when the trace-free divergence-free two-index tensor σ is big enough. By big we mean that $a_\sigma^\vee > \sigma_0$, with the positive constant σ_0 given in Eq. (4.35). This result requires the near-CMC hypotheses present in Lemma 7, and is summarized below.

Lemma 10. (Global sub-solution for $R \geqslant 0$ and $a_\sigma^\vee > \sigma_0$) *Consider the weak formulation for the Hamiltonian and momentum constraints given in §2.3 and assume that the hypotheses in Lemma 7 hold. Assume that the constants $a_\tau^\wedge$, $\hat{\phi}_N^\vee$ $K^\wedge$, and $\phi_D^\vee$ are positive, while the constant $a_\sigma^\vee > \sigma_0$, with the positive constant σ_0 given in Eq. (4.35). Denote by ϕ_σ the only positive root of the polynomial $q_\sigma(x) := a_\tau^\wedge x^3 + a_R^\wedge x^2 - a_\sigma^\vee/4$, where $x \in \mathbb{R}$. Let ϕ_{-} be the constant*

$$\phi_{-} := \min\Big\{\phi_\sigma,\ \frac{\hat{\phi}_N^\vee}{K^\wedge},\ \phi_D^\vee\Big\}. \tag{4.34}$$

Then, ϕ_{-} is a global sub-solution of Eq. (2.35).

Proof. *(Lemma 10.)* Consider the weak formulation for the Hamiltonian and momentum constraints given in §2.3. The a_w^* defined in Eq. (2.24) belongs to the space $W^{-1,2}_{D+}$, since the function $a_w = (\sigma + \mathcal{L}\boldsymbol{w})^2/8$ belongs to the space $L^{p/2}$, with $p = 12/5$. Given any positive number ϵ, the inequality $2|\sigma_{ab}(\mathcal{L}w)^{ab}| \leqslant \epsilon\sigma^2 + (\mathcal{L}\boldsymbol{w})^2/\epsilon$ implies that function a_w satisfies the following inequality,

$$8a_w = \sigma^2 + (\mathcal{L}\boldsymbol{w})^2 + 2\sigma_{ab}(\mathcal{L}w)^{ab} \geqslant (1-\epsilon)\,\sigma^2 - \Big(\frac{1}{\epsilon} - 1\Big)\,(\mathcal{L}\boldsymbol{w})^2,$$

hence, for any number $\epsilon \in (0,1)$ the functional a_w^* must fulfill the inequality

$$a_w^*(\underline{\varphi}) = (a_w, \underline{\varphi}) \geqslant (1-\epsilon)\, a_\sigma^\vee\, \|\underline{\varphi}\|_{1,2} - \frac{1}{8}\Big(\frac{1}{\epsilon} - 1\Big)\, \|\mathcal{L}\boldsymbol{w}\|_p^2\, \|\underline{\varphi}\|_6 \qquad \forall\ \underline{\varphi} \in W^{1,2}_{D+}.$$

For every vector field $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ solution of the momentum constraint Eq. (2.36) with source function ϕ holds the inequality in Eq. (3.26), therefore there exist positive constants $c_{\mathcal{L}}$, c_1 and c_2 such that

$$\|\mathcal{L}\boldsymbol{w}\|_p^2 \leqslant 4c_{\mathcal{L}}^2\Big[\|\phi\|_\infty^{12}\, \|\boldsymbol{b}_\tau^*\|_{-1,p}^2 + \|\boldsymbol{b}_j^*\|_{-1,p}^2 + c_1^2\, \|\hat{\boldsymbol{w}}_{I\!N}^*\|_{-\frac{1}{p},p,I\!N}^2 + c_2^2\, \|\boldsymbol{w}_{I\!D}\|_{1,p}^2\Big].$$

Hence, for all source functions $\phi \in [0, \phi_+]$, where the constant ϕ_+ is the positive super-solution found in Lemma 7, holds the inequality

$$\|\mathcal{L}\boldsymbol{w}\|_p^2 \leqslant 4c_{\mathcal{L}}^2\Big[\phi_+^{12}\,\|\boldsymbol{b}_\tau^*\|_{-1,p}^2 + \|\boldsymbol{b}_j^*\|_{-1,p}^2 + c_1^2\,\|\hat{\boldsymbol{w}}_{I\!N}^*\|_{-\frac{1}{p},p,I\!N}^2 + c_2^2\,\|\boldsymbol{w}_{I\!D}\|_{1,p}^2\Big].$$

Introducing the positive constant c_t defined by the inequality $\|\underline{\varphi}\|_6 \leqslant c_t\,\|\underline{\varphi}\|_{1,2}$, and the constant σ_0 given by

$$\sigma_0 := 2c_t c_{\mathcal{L}}^2\Big[\phi_+^{12}\,\|\boldsymbol{b}_\tau^*\|_{-1,p}^2 + \|\boldsymbol{b}_j^*\|_{-1,p}^2 + c_1^2\,\|\hat{\boldsymbol{w}}_{I\!N}^*\|_{-\frac{1}{p},p,I\!N}^2 + c_2^2\,\|\boldsymbol{w}_{I\!D}\|_{1,p}^2\Big], \tag{4.35}$$

we obtain that

$$a_w^*(\underline{\varphi}) \geqslant \Big[(1-\epsilon)\,a_\sigma^\vee - \frac{\sigma_0}{4}\Big(\frac{1}{\epsilon}-1\Big)\Big]\,\|\underline{\varphi}\|_{1,2}.$$

Choose the number $\epsilon = 1/2$, then we get $a_w^*(\underline{\varphi}) \geqslant (1/2)\big(a_\sigma^\vee - \sigma_0/2\big)\,\|\underline{\varphi}\|_{1,2}$. By assumption, we know that $a_\sigma^\vee > \sigma_0$, therefore, we conclude that

$$a_w^*(\underline{\varphi}) \geqslant \frac{1}{4}\,a_\sigma^\vee\,\|\underline{\varphi}\|_{1,2} \qquad \forall\,\underline{\varphi} \in W_{D+}^{1,2}. \tag{4.36}$$

Having established the inequality above we now start looking for the sub-solution among the constant functions. Therefore, let ϕ_0 be any constant in $[\phi_1, \phi_+]$ with $0 < \phi_1 \leqslant \phi_+$, then the following inequalities hold for every element $\underline{\varphi} \in W_{D+}^{1,2}$,

$$\begin{aligned} f_{wF}(\phi_0)(\underline{\varphi}) &= (a_\tau\phi_0^5)^*(\underline{\varphi}) + (a_R\phi_0)^*(\underline{\varphi}) - (a_\rho\phi_0^{-3})^*(\underline{\varphi}) - (a_w\phi_0^{-7})^*(\underline{\varphi}) \\ &\leqslant a_\tau^*(\underline{\varphi})\,\phi_0^5 + a_R^*(\underline{\varphi})\,\phi_0 - a_\rho^*(\underline{\varphi})\,\phi_0^{-3} - a_w^*(\underline{\varphi})\,\phi_0^{-7} \\ &\leqslant \Big[a_\tau^\wedge\,\phi_0^5 + a_R^\wedge\,\phi_0 - a_\rho^\vee\,\phi_0^{-3} - \frac{1}{4}\,a_\sigma^\vee\,\phi_0^{-7}\Big]\,\|\underline{\varphi}\|_{1,2}, \end{aligned}$$

where the used Eq. (4.36) to obtain the last line above. Introduce the polynomial

$$q(\phi_0) := a_\tau^\wedge\,\phi_0^5 + a_R^\wedge\,\phi_0 - a_\rho^\vee\,\phi_0^{-3} - \frac{1}{4}\,a_\sigma^\vee\,\phi_0^{-7},$$

which is a non-decreasing function, because its derivative

$$q'(\phi_0) = 5\,a_\tau^\wedge\,\phi_0^4 + a_R^\wedge + 3\,a_\rho^\vee\,\phi_0^{-4} + \frac{7}{4}\,a_\sigma^\vee\,\phi_0^{-8},$$

is strictly positive for non-zero ϕ_0. Rewrite the polynomial q as follows

$$q(\phi_0) = \phi_0^{-7}\,q_\sigma(\phi_0) \quad \text{with} \quad q_\sigma(\phi_0) := a_\tau^\wedge\,\phi_0^{12} + a_R^\wedge\,\phi_0^8 - \frac{1}{4}\,a_\sigma^\vee. \tag{4.37}$$

This polynomial q_σ has a unique positive root ϕ_σ, which exists because $q_\sigma(0) < 0$ and $\lim_{\phi_0\to\infty} q_\sigma(\phi_0) = \infty$, while the root is unique because the polynomial q_σ is an increasing function for positive ϕ_0. So, Eq. (4.37) implies that $q(\phi_\sigma) = 0$, and together with the property of the polynomial q being non-decreasing, we conclude that

$$q(\phi_0) \leqslant 0 \qquad \forall\, 0 < \phi_0 \leqslant \phi_\sigma.$$

We then summarize the discussion above saying that the following inequality holds

$$f_{wF}(\phi_0)(\underline{\varphi}) \leqslant 0 \qquad \forall\,\underline{\varphi} \in W_{D+}^{1,2}, \qquad \forall\,\phi_0 \in (0, \phi_\sigma], \tag{4.38}$$

and for all vector field $\boldsymbol{w} \in \boldsymbol{W}^{1,p}$ solution of the momentum constraint Eq. (2.36) with source function $\phi \in [0, \phi_+]$, where the constant ϕ_+ is the super-solution found in Lemma 7. From this point forward, the proof is identical to the proof of Lemma 8.

What remains is to account for the boundary contributions. The definitions of $\hat{\phi}_N^\vee$ and $K^\wedge$ imply that for any constant ϕ_0, it holds that

$$\big([K\phi_0 - \hat{\phi}_N], \mathsf{tr}_N\underline{\varphi}\big)_N \leqslant (K^\wedge\phi_0 - \hat{\phi}_N^\vee)\,\|\underline{\varphi}\|_{1,2} \qquad \forall\,\underline{\varphi}\in W^{1,2}_{D+}.$$

In particular, defining the constant $\overline{\phi}_- := \min\big\{\phi_\sigma,\ \dfrac{\hat{\phi}_N^\vee}{K^\wedge}\big\}$, it then follows that $0 < \overline{\phi}_- \leqslant \phi_\sigma$, and the following inequality holds

$$\big([K\phi_0 - \hat{\phi}_N], \mathsf{tr}_N\underline{\varphi}\big)_N \leqslant 0, \qquad \forall\,\phi_0\in(0,\overline{\phi}_-]. \tag{4.39}$$

Adding Eqs. (4.38) and (4.39) we conclude that

$$(K\phi_0, \mathsf{tr}_N\underline{\varphi})_N + f_w(\phi_0)(\underline{\varphi}) \leqslant 0 \qquad \forall\,\underline{\varphi}\in W^{1,2}_{D+} \qquad \forall\,\phi_0\in(0,\overline{\phi}_-].$$

Recalling now that any constant ϕ_0 satisfies $(\nabla\phi_0, \nabla\underline{\varphi}) = 0$, we conclude that

$$A_L\phi_0(\underline{\varphi}) + f_w(\phi_0)(\underline{\varphi}) \leqslant 0 \qquad \forall\underline{\varphi}\in W^{1,2}_{D+} \qquad \forall\,\phi_0\in(0,\overline{\phi}_-].$$

Finally, introduce the constant ϕ_- as given by Eq. (4.34). In particular, this constant satisfies $(\phi_D - \phi_-)^- = 0$ and $0 < \phi_- \leqslant \overline{\phi}_-$, so the following inequality holds,

$$(\phi_D - \phi_-)^- \in W^{1,2}_D \text{ and } -\big[A_L\phi_- + f_w(\phi_-)\big] \in W^{-1,2}_{D+},$$

and for all vector field $\boldsymbol{w}\in \boldsymbol{W}^{1,p}$ solution of the momentum constraint Eq. (2.36) with source function $\phi\in[0,\phi_+]$, where the constant ϕ_+ is the super-solution found in Lemma 7. This establishes that ϕ_- is a global sub-solution of Eq. (2.35) in the interval $[0,\phi_+]$. □

4.3. **Summary on barriers.** We present in this short Section a summary of the various results we have obtained in §4.2 for weak sub- and super-solutions. We state the constant sub- and super-solutions for each value of the Ricci scalar R. We do not state again the definition of the various constants that define the sub- and super-solutions, which can be found at the beginning of §4.2. Similarly, we do not state again the assumptions on the coefficients and data in the Hamiltonian and momentum constraint equations needed to construct the barriers. These assumptions can be found in §4.2, or they can be read out directly from the barriers expressions, because they are the sufficient conditions that guarantee that the barriers are positive and finite numbers.

$$R \text{ bounded} \qquad \begin{cases} \phi_{w+} := \max\Big\{1,\ \Big[\dfrac{\bar{a}_R^\wedge + a_\rho^\wedge + a_w^\wedge}{a_\tau^\vee}\Big]^{1/4},\ \dfrac{\hat{\phi}_N^\wedge}{K^\vee},\ \phi_D^\wedge\Big\}, \\[2ex] \phi_+ := \max\Big\{1,\ \Big[\dfrac{\bar{a}_R^\wedge + a_\rho^\wedge + \mathtt{K}_2}{a_\tau^\vee - \mathtt{K}_1}\Big]^{1/4},\ \dfrac{\hat{\phi}_N^\wedge}{K^\vee},\ \hat{\phi}_D^\wedge\Big\}, \end{cases}$$

$$R<0 \qquad \phi_- := \min\Big\{\Big(\frac{a_R^\vee}{a_\tau^\wedge}\Big)^{1/4},\ \frac{\hat{\phi}_N^\vee}{K^\wedge},\ \phi_D^\vee\Big\},$$

$$R\geqslant 0 \qquad \phi_- := \begin{cases} \min\Big\{\phi_\rho,\ \dfrac{\hat{\phi}_N^\vee}{K^\wedge},\ \phi_D^\vee\Big\} \text{ if } a_\rho^\vee > 0, \quad a_\sigma^\vee \geqslant 0, \\[2ex] \min\Big\{\phi_\sigma,\ \dfrac{\hat{\phi}_N^\vee}{K^\wedge},\ \phi_D^\vee\Big\} \text{ if } a_\rho^\vee \geqslant 0, \quad a_\sigma^\vee > \sigma_0. \end{cases}$$

Regarding the sub-solution for Ricci scalar $R \geqslant 0$, only the constant ϕ_ρ has been given explicitly by the expression $\phi_\rho := \Big[\Big(-a_R^\wedge + \sqrt{(a_R^\wedge)^2 + 4a_\tau^\wedge a_\rho^\vee}\Big)/(2a_\tau^\wedge)\Big]^{1/4}$, while the constant ϕ_σ has not been given explicitly, but it has been proven that ϕ_σ is a finite, positive number.

4.4. ***A priori* L^∞-bounds on $W^{1,2}$-solutions.** We now establish some related *a priori* L^∞-bounds on any $W^{1,2}$-solution to the Hamiltonian constraint equation. Although such results are standard for semi-linear scalar problems with monotone nonlinearities (for example, see [32]), the nonlinearity appearing in the Hamiltonian constraint becomes non-monotone when R becomes negative. Nonetheless, we are able to obtain *a priori* L^∞-bounds on solutions to the Hamiltonian constraint in all cases including the non-monotone case. The results are based on a new abstract result (Lemma 24 in the Appendix) which holds for general semi-linear problems in ordered Banach spaces, under very weak assumptions on the nonlinearity (see the second assumption in parts i and ii in Lemma 24). Monotone nonlinearities have the required property, but the property is much weaker than monotonicity and is satisfied for more general nonlinearities such as the one appearing in the Hamiltonian constraint. The results here generalize the *a priori* L^∞-bounds on weak solutions appearing previously in an earlier set of unpublished notes[1] and in a thesis[2] to the weakest possible assumptions on the coefficients appearing in the Hamiltonian nonlinearity.

Theorem 6. (*A priori* bounds on $W^{1,2}$-solutions) *Consider the weak formulation for the Hamiltonian constraint given in §4.1, and assume the hypotheses in Lemma 6, in Lemma 8 and either in Lemma 9 or in Lemma 10. Assume that given a functional $a_w^* \in W^{-1,2}_{D+}$, there exists a solution $\phi \in W^{1,2}$ of Hamiltonian constraint Eq. (4.9). Then, there exists positive numbers $\phi_\vee$, $\phi_\wedge$ with $\phi_\vee \leqslant \phi_\wedge$, such that*

$$0 < \phi_\vee \leqslant \phi \leqslant \phi_\wedge, \qquad \textit{a.e. in } \mathcal{M}. \tag{4.40}$$

Proof. *(Theorem 6.)* Let $\phi_\vee$ and $\phi_\wedge$ be the (constant) sub- and super-solutions, respectively, given in Lemmas 6-10. In the proofs of these Lemmas it is established that the function f_w given in Eq. (4.7) is monotone increasing for its argument $\tilde{\phi}$ satisfying $\tilde{\phi} \geqslant \phi_\wedge$ and for $0 < \tilde{\phi} \leqslant \phi_\vee$. Therefore, Lemma 24 implies that Eq. (4.40) holds. This establishes the Theorem. □

4.5. **Results using variational methods.** The Hamiltonian constraint Eq. (4.9) can be written as the Euler condition for stationarity of a real-valued functional in a Banach space. Direct methods in the calculus of variations can be used to find the points that minimize this functional in particular types of closed sets in Banach spaces: closed sets under weak convergence, which we denote here as closed$_w$. When the closed$_w$ set is chosen appropriately, and the barriers found in §4.2 are incorporated into the argument, we can show that the minimum is actually a solution of the Euler condition for the functional, that is, of the Hamiltonian constraint equation. The solution found with this approach requires fewer regularity assumptions on the data, and the resulting solution is weaker (has less regularity) than the solution found using barrier methods. The variational structure was exploited

[1] M. Holst, *Weak solutions to the Einstein constraint equations on manifolds with boundary.* Notes from the 2002–2003 Caltech Visitors Program in the Numerical Simulation of Gravitational Wave Sources.

[2] J. Kommemi, *Variational methods for weak solutions to the Einstein Hamiltonian constraint on finite domains with boundary.* Honors Thesis, Department of Mathematics, UC San Diego, 2007.

in [28] to develop an approximation theory and corresponding error estimates for numerical approximations to the Hamiltonian constraint.

Let $a_L : W_D^{1,2} \times W_D^{1,2} \to \mathbb{R}$ be a bilinear form with action defined in Eq. (4.1). Let a_τ^*, a_R^*, a_ρ^*, and a_w^* the functionals defined in Eq. (4.3)-(4.4). Let $\phi_D \in L^\infty \cap W^{1,2}$ be the positive, harmonic extension of the Dirichlet data function $\hat{\phi}_D$ discussed in §2.3, and $\hat{\phi}_N^*$ be the Robin data functional, both data defined in Eq. (4.6). Finally, let ϕ_1, $\phi_2 \in L^\infty$ be functions satisfying $0 < \phi_1 \leqslant \phi_2$, but otherwise arbitrary, and denote by $U := \big([\phi_1 - \phi_D, \phi_2 - \phi_D] \cap W_D^{1,2}\big) \subset W_D^{1,2}$. Then, introduce the functional

$$J_L : U \subset W_D^{1,2} \to \overline{\mathbb{R}}, \qquad J_L(\varphi) := \frac{1}{2}\, a_L(\varphi, \varphi) + G_w(\varphi), \tag{4.41}$$

where the functional G_w is given by

$$G_w(\varphi) := g_w(\phi_D + \varphi) - \hat{\phi}_N^*(\mathsf{tr}_N \varphi) + a_L(\phi_D, \varphi),$$

with the functional $g_w(\phi)$ having the form

$$g_w(\phi) := \frac{1}{6}\,(a_\tau \phi^5)^*(\phi) + \frac{1}{2}\,(a_R \phi)^*(\phi) + \frac{1}{2}\,(a_\rho \phi^{-3})^*(\phi) + \frac{1}{6}\,(a_w \phi^{-7})^*(\phi); \tag{4.42}$$

while the definition $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is explained in the Appendix.

Theorem 7. (Existence of a minimizer) *Consider the functional $J_L : U \subset W_D^{1,2} \to \overline{\mathbb{R}}$ defined in Eq. (4.41). Fix a positive extension $\phi_D \in L^\infty \cap W^{1,2}$ of the Dirichlet boundary data, and fix the Robin boundary data $\hat{\phi}_N^* \in W^{-\frac{1}{2},2}(\partial \mathcal{M}_N, 0)$. Fix the functionals a_τ^*, a_ρ^*, a_w^*, and the function K satisfying the inequalities $a_\tau^\vee > 0$, $a_\rho^\vee \geqslant 0$, $a_w^\vee \geqslant 0$, $K^\vee > 0$. Then, there exists an element $\varphi \in U$ minimizer of the functional J_L in U, that is,*

$$J_L(\varphi) = \inf_{\underline{\varphi} \in U} J_L(\underline{\varphi}).$$

Furthermore, if the Ricci scalar R is non-negative in $\overline{\mathcal{M}}$, then the minimizer φ is unique.

Proof. *(Theorem 7.)* The set U is closed in $W^{1,2}$, as the following argument shows: By contradiction, assume that there exists a sequence $\{\phi_n\} \in U$ such that $\phi_n \to \phi_0$ in $W^{1,2}$ but $\phi_0 \notin U$. In particular $\phi_n \to \phi_0$ in L^2 and the inequality $\big|\|\phi_n\| - \|\phi_0\|\big| \leqslant \|\phi_n - \phi_0\|$ implies that $\|\phi_n\| \to \|\phi_0\|$. The assumption that the limiting element $\phi_0 \notin U$ implies that there exists a set $\mathcal{M}_0 \subset \mathcal{M}$ with meas$(\mathcal{M}_0) \neq 0$ such that $\phi_0 > \phi_2$ or $\phi_0 < \phi_1$. We consider here only the first case, the second one is proven in an analogous way. Then, we have the following inequalities,

$$\begin{aligned} \|\phi_n\|_{\mathcal{M}}^2 &= \|\phi_n\|_{\mathcal{M}_0}^2 + \|\phi_n\|_{(\mathcal{M}\setminus\mathcal{M}_0)}^2 \\ &\leqslant \|\phi_2\|_{\mathcal{M}_0}^2 + \|\phi_n\|_{(\mathcal{M}\setminus\mathcal{M}_0)}^2, \end{aligned}$$

where we used the notation $\|\phi_n\|_{\mathcal{M}_0} := \|\phi_n\|_{L^2(\mathcal{M}_0)}$. Then taking the limit on both sides

$$\begin{aligned} \lim_{n\to\infty} \|\phi_n\|_{\mathcal{M}}^2 &\leqslant \|\phi_2\|_{\mathcal{M}_0}^2 + \|\phi_0\|_{(\mathcal{M}\setminus\mathcal{M}_0)}^2 \\ &< \|\phi_0\|_{\mathcal{M}_0}^2 + \|\phi_0\|_{(\mathcal{M}\setminus\mathcal{M}_0)}^2 = \|\phi_0\|_{\mathcal{M}}^2, \end{aligned}$$

from which we conclude that $\lim_{n\to\infty} \|\phi_n\| < \|\phi_0\|$, contradicting the assumption above that $\lim_{n\to\infty} \|\phi_n\| = \|\phi_0\|$. Therefore, the set U is closed. In addition, the set U is convex, therefore it is shown in the Appendix that U is closed$_w$.

We now show that the functional J_L is coercive. First notice that the condition $\phi_1 > 0$ implies that the functional with values $g_w(\phi)$ is continuous and bounded in $[\phi_1, \phi_2] \subset L^\infty_+$, therefore there exists a positive constant c_g such that

$$|g_w(\phi)| \leqslant c_g, \qquad \forall \phi \in [\phi_1, \phi_2].$$

Second, the Robin term is bounded as the following calculation shows

$$\begin{aligned}
|\hat{\phi}^*_N(\mathsf{tr}_N\varphi)| &\leqslant \|\hat{\phi}^*_N\|_{-\frac{1}{2},2,N}\, \|\mathsf{tr}_N\varphi\|_{\frac{1}{2},2,N} \\
&\leqslant c_t\, \|\hat{\phi}^*_N\|_{-\frac{1}{2},2,N}\, \|\varphi\|_{1,2} \\
&\leqslant \frac{c_t^2}{2\epsilon}\, \|\hat{\phi}^*_N\|^2_{-\frac{1}{2},2,N} + \frac{\epsilon}{2}\, \|\varphi\|^2_{1,2} \qquad \forall\, \varphi \in W^{1,2},
\end{aligned}$$

where ϵ is an arbitrary positive constant, and c_t is a positive constant that bounds the trace operator. Third, the Dirichlet term is also bounded because the bilinear form a_L is bounded, then the following inequalities hold,

$$\begin{aligned}
a_L(\phi_D, \varphi) &= (\nabla\phi_D, \nabla\varphi) + (K\,\mathsf{tr}_N\phi_D, \mathsf{tr}_N\varphi)_N \\
&\leqslant (1 + c_t\, \|K\|_\infty)\, \|\phi_D\|_{1,2}\, \|\varphi\|_{1,2} \\
&\leqslant \frac{1}{2\epsilon}(1 + c_t\, \|K\|_\infty)^2\, \|\phi_D\|^2_{1,2} + \frac{\epsilon}{2}\, \|\varphi\|^2_{1,2}, \qquad \forall\, \varphi \in W^{1,2}.
\end{aligned}$$

These calculations show that

$$\begin{aligned}
|G_w(\varphi)| &\leqslant c_G(\epsilon) + \epsilon\, \|\varphi\|^2_{1,2}, \qquad \forall\, \varphi \in U \subset W^{1,2}_D, \\
c_G(\epsilon) &= c_g + \frac{1}{2\epsilon}\left[c_t^2\, \|\hat{\phi}^*_N\|^2_{-\frac{1}{2},2,N} + (1 + c_t\, \|K\|_\infty)^2 \|\phi_D\|^2_{1,2}\right].
\end{aligned}$$

Fourth, the following argument, similar to the one used in the proof of Corollary 1, shows that a_L satisfies a Gårding inequality. Start with a bound on the Robin term in a_L,

$$\begin{aligned}
(K\,\mathsf{tr}_N\varphi, \mathsf{tr}_N\varphi)_N &\leqslant \|K\|_\infty\, \|\mathsf{tr}\varphi\|^2_N \\
&\leqslant c_t\, \|K\|_\infty\, \|\varphi\|\, \|\nabla\varphi\| \\
&\leqslant \frac{c_t^2}{2\epsilon}\, \|K\|^2_\infty\, \|\varphi\|^2 + \frac{\epsilon}{2}\, \|\nabla\varphi\|^2 \\
&\leqslant \frac{c_t^2}{2\epsilon}\, \|K\|^2_\infty\, \|\varphi\|^2 + \frac{\epsilon}{2}\, \|\nabla\varphi\|^2_{1,2};
\end{aligned}$$

then it is not difficult to derive the following inequality on a_L,

$$\begin{aligned}
a_L(\varphi, \varphi) &= (\nabla\varphi, \nabla\varphi) + (K\,\mathsf{tr}_N\varphi, \mathsf{tr}_N\varphi)_N \\
&\geqslant \|\varphi\|^2_{1,2} - \|\varphi\|^2 - \frac{c_t^2}{2\epsilon}\, \|K\|^2_\infty\, \|\varphi\|^2 - \frac{\epsilon}{2}\, \|\nabla\varphi\|^2_{1,2} \\
&\geqslant \big(1 - \frac{\epsilon}{2}\big)\, \|\varphi\|^2_{1,2} - \big(1 + \frac{c_t^2}{2\epsilon}\, \|K\|^2_\infty\big)\, \|\varphi\|^2 \\
&\geqslant \big(1 - \frac{\epsilon}{2}\big)\, \|\varphi\|^2_{1,2} - c_a(\epsilon),
\end{aligned}$$

where $c_a(\epsilon) := \big[1 + c_t^2 \, \|K\|_\infty^2/(2\epsilon)\big] c_s \, \|\phi_2\|_\infty^2$ and c_s is the positive constant in the imbedding $L^\infty \subset L^2$. Finally, the fifth step is to put all these inequalities together,

$$
\begin{aligned}
J_L(\varphi) &\geqslant \frac{1}{2}\big(1 - \frac{\epsilon}{2}\big) \, \|\varphi\|_{1,2}^2 - c_a(\epsilon) - c_G(\epsilon) - \epsilon \, \|\varphi\|_{1,2}^2 \\
&\geqslant \big(\frac{1}{2} - \frac{5}{4}\,\epsilon\big) \, \|\varphi\|_{1,2}^2 - c_J(\epsilon), \qquad \forall \varphi \in U \subset W^{1,2},
\end{aligned}
$$

where $c_J(\epsilon) = c_a(\epsilon) + c_G(\epsilon)$. By choosing ϵ positive and small enough the inequality above establishes that J_L is coercive. Therefore, J_L is proper, and is also trivially bounded below by $-c_J(\epsilon)$.

We now show that the functional J_L is lsc_w , and we do it term by term on J_L. We start with the term proportional to $a_L(\varphi,\varphi)$, which is lsc_w by the following three facts: First, the norm in a Banach space is a lsc_w functional (statement proved in the Appendix); Second, the compactness of the imbedding $W^{1,2} \subset L^2$. These two facts together imply the following: Given a sequence $\{\varphi_n\} \subset W^{1,2}$ such that $\varphi_n \rightharpoonup \varphi_0$ in $W_D^{1,2}$, and so $\varphi_n \to \varphi_0$ in L^2, we have that

$$
(\nabla\varphi, \nabla\varphi) = \|\varphi\|_{1,2}^2 - \|\varphi\|^2 \leqslant \liminf_{n\to\infty} \big(\|\varphi_n\|_{1,2}^2 - \|\varphi_n\|^2\big) = \liminf_{n\to\infty} \|\nabla\varphi_n\|^2,
$$

which establishes that this term is lsc_w . The third fact is that the remaining term in the bilinear form a_L has the form

$$
(K \, \mathsf{tr}_N \varphi, \mathsf{tr}_N \varphi)_N = (K, [\mathsf{tr}_N \varphi]^2)_N,
$$

which is continuous under strong convergence and convex (the latter because the function $K > 0$), therefore Lemma 12 in the Appendix implies that this term is lsc_w . These three facts then establish that the functional $\varphi \mapsto a_L(\varphi,\varphi)/2$ is lsc_w . We now consider the remaining terms in the functional J_L which are present in the functional G_w. The terms in G_w which are linear in the function φ are lsc_w because they are continuous under weak convergence, by definition of weak convergence. The nonlinear terms are gathered together in the functional g_w, and all of them except the term $(a_R\phi)^*(\phi)$ are continuous and strictly convex. Therefore, Lemma 12 implies they are lsc_w . The only remaining term, $(a_R\phi)^*(\phi)$, where a_R^* is not positive definite, is also lsc_w , since the imbedding $W^{1,2} \subset L^2$ is compact. The proof of this statement is the following calculation: Let $\{\phi_n\} \subset W^{1,2}$ such that $\phi_n \rightharpoonup \phi_0$ in $W^{1,2}$ which then implies that $\phi_n \to \phi_0$ in L^2. Then we have

$$
\begin{aligned}
\big|(a_R\phi_0)^*(\phi_0) - (a_R\phi_n)^*(\phi_n)\big| &= \big|\big(a_R, (\phi_0^2 - \phi_n^2)\big)\big| \\
&= \big|\big(a_R, (\phi_0 + \phi_n)(\phi_0 - \phi_n)\big)\big| \\
&= \Big(\|a_R\|_\infty \, \|\phi_0 + \phi_n\|\Big) \, \|\phi_0 - \phi_n\| \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

which then establishes that the functional $\phi \mapsto (a_R\phi)^*(\phi)$ is continuous under weak convergence, and so lsc_w . We conclude that the functional J_L is lsc_w . Hence, all hypotheses in Theorem 13 in the Appendix are satisfied by the functional J_L, therefore there exists a function $\varphi \in U$ minimizer of J_L. Furthermore, the minimizer φ is unique in the case that the Ricci scalar R is non-negative in $\overline{\mathcal{M}}$. The reason is that in this case the functional G_w is strictly convex, since all terms in G_w are convex and at least one of them is strictly convex. Therefore, Theorem 14 implies that the minimizer φ is unique. This establishes the Theorem. □

We now show that the functional J_L defined in Eq. (4.41) is Gâteaux differentiable on U along directions in U. We also show that this derivative can be extended along all directions on $W^{1,2}_D$, and that it coincides with the Hamiltonian constraint operator defined by the left hand side of Eq. (4.9). This technical Lemma is critical to connecting the minimizer of the functional J_L found in Theorem 7 to solutions of the Hamiltonian constraint, which we do below in Theorem 8.

Lemma 11. **(J_L Gâteaux differentiable)** *The functional $J_L : U \subset W^{1,2}_D \to \overline{\mathbb{R}}$ defined in Eq. (4.41) has Gâteaux derivative $DJ_L(\tilde{\varphi})(\underline{\varphi})$ for all $\tilde{\varphi} \in U$ along any direction $\underline{\varphi} \in U$. Furthermore, the map $DJ_L(\tilde{\varphi}) : U \to \mathbb{R}$ can be continuously extended for every $\tilde{\varphi} \in U$ into a map $DJ_L(\tilde{\varphi}) : W^{1,2}_D \to \mathbb{R}$, and this operator is precisely the left hand side in the Hamiltonian constraint Eq. (4.9).*

Proof. *(Lemma 11.)* The Gâteaux derivative of the functional J_L defined in Eq. (4.41) can be computed term by term. By definition of the Gâteaux derivative it is clear that

$$\begin{aligned} DJ_L(\tilde{\varphi})(\underline{\varphi}) &= a_L(\tilde{\varphi},\underline{\varphi}) + Dg_w(\phi_D+\tilde{\varphi})(\underline{\varphi}) - \hat{\phi}^*_N(\underline{\varphi}) + a_L(\phi_D,\underline{\varphi}) \\ &= a_L(\tilde{\phi},\underline{\varphi}) + Dg_w(\tilde{\phi})(\underline{\varphi}) - \hat{\phi}^*_N(\underline{\varphi}), \end{aligned}$$

where we introduced the notation $\tilde{\phi} := \phi_D + \tilde{\varphi}$. By the definition of the functional g_w given in Eq. (4.42), and by the definition of the Gelfand triple structure described in the Appendix, it is possible to compute Dg_w term by term. For example, this calculation on the first term is the following: Denote $g_\tau(\phi) := (1/6)(a_\tau\phi^5)^*(\phi)$, then

$$\begin{aligned} Dg_\tau(\tilde{\phi})(\underline{\varphi}) &= \frac{1}{6}\lim_{t\to 0^+}\frac{1}{t}\left[(a_\tau(\tilde{\phi}+t\underline{\varphi})^5)^*(\tilde{\phi}+t\underline{\varphi}) - (a_\tau\tilde{\phi}^5)^*(\tilde{\phi})\right] \\ &= \frac{1}{6}\lim_{t\to 0^+}\lim_{n\to\infty}\frac{1}{t}\left[\big(a_{\tau n},(\tilde{\phi}+t\underline{\varphi})^6\big) - \big(a_{\tau n},\tilde{\phi}^6\big)\right] \\ &= \frac{1}{6}\lim_{n\to\infty}\lim_{t\to 0^+}\frac{1}{t}\left[\big(a_{\tau n},(\tilde{\phi}+t\underline{\varphi})^6\big) - \big(a_{\tau n},\tilde{\phi}^6\big)\right] \\ &= \lim_{n\to\infty}\big(a_{\tau n}\,\tilde{\phi}^5,\underline{\varphi}\big) \\ &= (a_\tau\tilde{\phi}^5)^*(\underline{\varphi}), \end{aligned}$$

where $\tilde{\phi} = \phi_D + \tilde{\varphi}$, and this calculation holds for all $\tilde{\varphi}$, $\underline{\varphi} \in U \subset W^{1,2}_D$. The limits can be interchanged to obtain the third line in the equations above because the sequence $(a_{\tau n},\phi^6)$ is uniformly bounded in the index n for every element $\phi \in [\phi_1,\phi_2]$, that is,

$$|(a_{\tau n},\tilde{\phi}^6)| \leqslant \|\phi_2\|^5_\infty\,|(a_{\tau n},\tilde{\phi})| \leqslant \|\phi_2\|^5_\infty\,\|a^*_\tau\|_{-1,2}\,\|\tilde{\phi}\|_{1,2}.$$

We need the assumption $0 < \phi_1 \leqslant \phi_2$ in order we can do the same calculation above for the terms in the functional g_w that contain negative powers of the function $\tilde{\phi}$. Then, the principle of uniform boundness can be extended from sequences of linear functionals to the sequence that approximates the nonlinear functional g_τ above. Therefore, the limits in the second line of the expression for Dg_τ can be interchanged to obtain the third line in that expression. See [20], pages 52-53, and also see [46], pages 80-81 for a proof of the principle of uniform boundness. References about generalizations of this principle can be found in [20], page 82. Let us return to the proof of Lemma 11. The expression on the last line in the inequalities above can be continuously extended for all $\underline{\varphi} \in W^{1,2}_D$, as the following calculation shows

$$(a_\tau\tilde{\phi}^5)^*(\underline{\varphi}) \leqslant \|\tilde{\phi}\|^5_\infty\,|a^*_\tau(\underline{\varphi})| \leqslant \|\tilde{\phi}\|^5_\infty\,\|a^*_\tau\|_{-1,2}\,\|\underline{\varphi}\|_{1,2}.$$

Analogous calculations on the remaining terms in the functional g_w then show that

$$Dg_w(\tilde{\phi})(\underline{\varphi}) = f_{wF}(\tilde{\phi})(\underline{\varphi}), \qquad \forall\, \underline{\varphi} \in W_D^{1,2}.$$

Therefore, we have established that

$$DJ_L(\tilde{\phi} - \phi_D)(\underline{\varphi}) = A_L\tilde{\phi}(\underline{\varphi}) + f_w(\tilde{\phi})(\underline{\varphi}),$$

for all $\tilde{\phi} \in U$ and all $\underline{\varphi} \in W_D^{1,2}$. This last equation establishes the Lemma. □

So far the functions ϕ_1 and ϕ_2 that define the subset U in Theorem 7 and Lemma 11 can be any elements in L^∞, with only the condition that $0 < \phi_1 \leqslant \phi_2$. Theorem 7 above says that there always exists a minimizer φ of the functional J_L in the set U. The following result says that if the functions ϕ_1 and ϕ_2 are sub- and super-solutions of the Gâteaux derivative $DJ_L : U \subset W_D^{1,2} \to W_D^{-1,2}$, then the minimizer is actually the solution of the Euler equation $DJ_L(\varphi) = 0$, and is thus a weak solution to the Hamiltonian constraint Eq. (4.9).

Theorem 8. (Hamiltonian constraint) *Assume the hypotheses given in Theorem 7, and also assume that either the constant $a_\rho^\vee > 0$ or the constant $a_\sigma^\vee > \sigma_0$, where the positive constant σ_0 is defined in Eq. (4.35). Furthermore, assume that the subset $U \subset W_D^{1,2}$ is defined by $\phi_1 = \phi_-$ and $\phi_2 = \phi_+$, where ϕ_- and ϕ_+ are any of the sub- and super-solutions of the Hamiltonian constraint Eq. (4.9) found in §4.2. Then, the minimizer $\varphi \in U$ found in Theorem 7 is a solution of the Euler equation*

$$DJ_L(\varphi)(\underline{\varphi}) = 0, \qquad \forall\, \underline{\varphi} \in W_D^{1,2},$$

where DJ_L is the Gâteaux derivative of J_L, and the equation above is the Hamiltonian constraint Eq. (4.9).

Proof. *(Theorem 8.)* Let ϕ_-, ϕ_+ be sub- and super-solutions of the Hamiltonian constraint Eq. (4.9), respectively, and define $U := [\phi_- - \phi_D, \phi_+ - \phi_D] \cap W_D^{1,2}$. Let $\varphi \in U$ be a minimizer of J_L on U, whose existence was established in Theorem 7, and denote $\phi := \phi_D + \varphi$. We first establish the following result involving a minimizer φ of the functional J_L: Given any $\psi \in W_D^{1,2}$ such that $\varphi + t\psi \in U$ for small enough, positive number t, the following inequality holds,

$$DJ_L(\varphi)(\psi) \geqslant 0. \tag{4.43}$$

For the proof, compute the Gâteaux derivative of J_L at φ along ψ,

$$DJ_L(\varphi)(\psi) = \lim_{t\to 0^+} \frac{1}{t}\big[J_L(\varphi + t\psi) - J_L(\varphi)\big], \tag{4.44}$$

which is well-defined because we assume that for $0 < t$ small enough the element $\varphi + t\psi \in U$. The function φ is the minimum of the functional J_L in the set U, so $J_L(\varphi + t\psi) \geqslant J_L(\varphi)$, which establishes Eq. (4.43) when the limit $t \to 0^+$ is computed in Eq. (4.44).

We now use the inequality (4.43) to show that the function φ is solution of the Hamiltonian constraint Eq. (4.9). Let ζ be any scalar function in the space $C_D^\infty(\overline{\mathcal{M}}, 0)$, and then introduce the mono-parametric family of functions $\delta\varphi_\epsilon$ as a perturbation of the function φ inside the set U,

$$\delta\varphi_\epsilon := \min\big\{(\phi_+ - \phi_D), \max\{(\phi_- - \phi_D), \varphi + \epsilon\zeta\}\,\big\},$$

where ϵ is a positive, otherwise arbitrary real number. The construction above implies that $\delta\varphi_\epsilon \in U$, and also that the functional J_L is Gâteaux differentiable at the function φ along the function $\delta\varphi_\epsilon - \varphi$, where the latter statement follows from

$$
\begin{aligned}
DJ_L(\varphi)(\delta\varphi_\epsilon - \varphi) &= \lim_{t\to 0^+} \frac{1}{t}\left[J_L\big(\varphi + t\,[\delta\varphi_\epsilon - \varphi]\big) - J_L(\varphi)\right] \\
&= \lim_{t\to 0^+} \frac{1}{t}\left[J_L\big(t\,\delta\varphi_\epsilon + (1-t)\,\varphi]\big) - J_L(\varphi)\right],
\end{aligned}
$$

which is well-defined because the set U is convex. Therefore, we can choose the particular direction $\psi = (\delta\varphi_\epsilon - \varphi)$ in Eq. (4.43), which implies

$$
0 \leqslant DJ_L(\varphi)(\delta\varphi_\epsilon - \varphi). \tag{4.45}
$$

It is now be convenient to use the equivalent expression $\delta\varphi_\epsilon = (\varphi + \epsilon\zeta) + \zeta_\epsilon - \zeta^\epsilon$, where we have introduced the cut-off functions

$$
\zeta^\epsilon := \max\big\{0, (\varphi+\epsilon\zeta) - (\phi_+ - \phi_D)\big\}, \qquad \zeta_\epsilon := -\min\big\{0, (\varphi+\epsilon\zeta) - (\phi_- - \phi_D)\big\}.
$$

Notice that ζ^ϵ, ζ_ϵ are non-negative continuous functions belonging to $W^{1,2}_D$. The inequality in (4.45) implies

$$
DJ_L(\varphi)(\zeta) \geqslant \frac{1}{\epsilon}\left[DJ_L(\varphi)(\zeta^\epsilon) - DJ_L(\varphi)(\zeta_\epsilon)\right]. \tag{4.46}
$$

We now show that each term on the right hand side in the inequality above approaches zero as ϵ approaches zero. The first term on the right hand side in Eq. (4.46) satisfies the following inequalities,

$$
\begin{aligned}
DJ_L(\varphi)(\zeta^\epsilon) &= DJ_L(\varphi)(\zeta^\epsilon) - DJ_L(\phi_+ - \phi_D)(\zeta^\epsilon) + DJ_L(\phi_+ - \phi_D)(\zeta^\epsilon) \\
&\geqslant \big[DJ_L(\varphi) - DJ_L(\phi_+ - \phi_D)\big](\zeta^\epsilon) \\
&= A_L(\phi - \phi_+)(\zeta^\epsilon) + \big[f_w(\phi) - f_w(\phi_+)\big](\zeta^\epsilon),
\end{aligned} \tag{4.47}
$$

where the property that ϕ_+ is a super-solution of Eq. (4.9) was used to obtain the second line in the inequality above. We analyze the last inequality, term by term. We will need the subset $\mathcal{M}^\epsilon \subset \mathcal{M}$ defined as follows

$$
\mathcal{M}^\epsilon := \{x \in \mathcal{M} : \zeta^\epsilon > 0\}.
$$

This definition implies that $\mathrm{meas}(\mathcal{M}^\epsilon) \to 0$ as $\epsilon \to 0$. Then, the term involving the operator A_L satisfies the following inequalities,

$$
\begin{aligned}
A_L(\phi - \phi_+)(\zeta^\epsilon) &= \big(\nabla[\phi - \phi_+], \nabla\zeta^\epsilon\big) + \big(K\,\mathsf{tr}_N[\phi - \phi_+], \mathsf{tr}_N\zeta^\epsilon\big)_N \\
&= \big(\nabla[\phi - \phi_+], \nabla[\phi - \phi_+ + \epsilon\zeta]\big)_{\mathcal{M}^\epsilon} \\
&\quad + \big(K\,\mathsf{tr}_N[\phi - \phi_+], \mathsf{tr}_N[\phi - \phi_+ + \epsilon\zeta]\big)_{\partial\mathcal{M}^\epsilon_N} \\
&\geqslant \epsilon\,\big(\nabla[\phi - \phi_+], \nabla\zeta\big)_{\mathcal{M}^\epsilon} + \epsilon\,\big(K\,\mathsf{tr}_N[\phi - \phi_+], \mathsf{tr}_N\zeta\big)_{\partial\mathcal{M}^\epsilon_N},
\end{aligned}
$$

where the subscripts $\mathcal{M}^\epsilon$ and $\partial\mathcal{M}^\epsilon_N$ on the inner products mean the L^2 inner product on these domains, and where we used that $\zeta^\epsilon = (\phi - \phi_+) + \epsilon\zeta$ on $\mathcal{M}^\epsilon$. In order to analyze the term with the functional f_w, it is convenient to introduce a representation based on Gelfand triple structure, $W^{1,2}_D \subset L^2 \equiv \big[L^2\big]^* \subset W^{-1,2}_D$,

$$
f_w(\tilde{\phi})(\underline{\varphi}) = \lim_{n\to\infty} (f_{wn}(\tilde{\phi}), \underline{\varphi}), \qquad \forall\, \underline{\varphi} \in W^{1,2}_D,
$$

with the functions $f_{wn} \in L^2$ and $\tilde{\phi} \in U$. Using this representation it is not difficult to establish the following inequalities

$$\begin{aligned}
\big[f_w(\phi) - f_w(\phi_+)\big](\zeta^\epsilon) &= \lim_{n\to\infty} \big(\big[f_{wn}(\phi) - f_{wn}(\phi_+)\big], \zeta^\epsilon\big)_{\mathcal{M}^\epsilon} \\
&\geqslant - \lim_{n\to\infty} \big(|f_{wn}(\phi) - f_{wn}(\phi_+)|, \zeta^\epsilon\big)_{\mathcal{M}^\epsilon} \\
&\geqslant - \lim_{n\to\infty} \big(|f_{wn}(\phi) - f_{wn}(\phi_+)|, (\phi - \phi_+)\big)_{\mathcal{M}^\epsilon} \\
&\quad - \epsilon \lim_{n\to\infty} \big(|f_{wn}(\phi) - f_{wn}(\phi_+)|, \zeta\big)_{\mathcal{M}^\epsilon} \\
&\geqslant -\epsilon \lim_{n\to\infty} \big(|f_{wn}(\phi) - f_{wn}(\phi_+)|, \zeta\big)_{\mathcal{M}^\epsilon}.
\end{aligned}$$

Combining the inequalities obtained for the operator A_L and the functional f_w, and using them with Eq. (4.47), we obtain

$$\begin{aligned}
DJ_L(\varphi)(\zeta^\epsilon) \geqslant \epsilon\Big[&\big(\nabla[\phi - \phi_+], \nabla\zeta\big)_{\mathcal{M}^\epsilon} + \big(K \mathsf{tr}_N[\phi - \phi_+], \mathsf{tr}_N\zeta\big)_{\partial\mathcal{M}^\epsilon_N} \\
&- \lim_{n\to\infty} \big(|f_{wn}(\phi) - f_{wn}(\phi_+)|, \zeta\big)_{\mathcal{M}^\epsilon}\Big].
\end{aligned}$$

An analogous calculation can be performed on the second term on the right hand side in Eq. (4.46), and the result is

$$\begin{aligned}
DJ_L(\varphi)(\zeta_\epsilon) \leqslant -\epsilon\Big[&\big(\nabla[\phi - \phi_-], \nabla\zeta\big)_{\mathcal{M}_\epsilon} + \big(K \mathsf{tr}_N[\phi - \phi_-], \mathsf{tr}_N\zeta\big)_{\partial\mathcal{M}_{N\epsilon}} \\
&+ \lim_{n\to\infty} \big(|f_{wn}(\phi) - f_{wn}(\phi_-)|, \zeta\big)_{\mathcal{M}_\epsilon}\Big],
\end{aligned}$$

where now we have used the property that the function ϕ_- is a sub-solution of Eq. (4.9), and have introduced the analogous subset $\mathcal{M}_\epsilon \subset \mathcal{M}$ defined as

$$\mathcal{M}_\epsilon := \{x \in \mathcal{M} : \zeta_\epsilon > 0\}.$$

These last two inequalities used in Eq. (4.46) imply that

$$DJ_L(\phi)(\zeta) \geqslant o(\epsilon), \qquad \text{as } \epsilon \to 0,$$

since both meas$(\mathcal{M}^\epsilon)$ and meas$(\mathcal{M}_\epsilon)$ approach zero as ϵ approaches zero. The same calculation must hold for $-\zeta$, and therefore we conclude that $DJ_L(\phi)(\zeta) = 0$ for all $\zeta \in C^\infty_D(\mathcal{M}, 0)$. This space is dense in the space $W^{1,2}_D$, so we conclude that

$$DJ_L(\phi) = 0,$$

which establishes the Theorem. □

4.6. **Results using barrier methods.** We now use the barrier method to show that there exist weak solutions to the Dirichlet-Robin boundary value formulation for the Hamiltonian constraint equation. The barriers found in §4.2 are used to modify the original Eq. (4.9) into an equation with a monotone decreasing source. This allows us to construct an iteration which converges to a fixed point of a particular mapping, which is constructed so that the fixed point also solves the Hamiltonian constraint equation. The modification of the original Hamiltonian constraint equation is called here a shift of the equation, and imposes a restriction on the regularity of the equation coefficients. This shift is not needed in the variational method, which is a reason that variational methods are able to produce results with weaker regularity, requiring fewer assumptions on the data.

The barrier method and the variational method are both constructive, in the sense that they provide an algorithm to construct the solution, which could be of interest in numerical relativity. In the latter method, one can build algorithms based on gradient descent that can guarantee progress (descent) at each iteration. In fact, the most effective numerical algorithms for the constraints tend to be a combination of these two ideas: global inexact-Newton methods are basically highly-tuned fixed-point iterations that maximize their contraction rate, and which are "globalized" by enforcing descent in an associated energy functional [28].

We therefore include the barrier technique here to give the most complete picture of what can be shown using both techniques on compact manifolds with boundary.

Theorem 9. (Hamiltonian constraint) *Consider the weak formulation for the Hamiltonian constraint given in §4.1. Assume that the following conditions hold:*

(i) The coefficients functionals a^*_τ, a^*_ρ and a^*_w given in Eq. (4.3) have the form

$$a^*_\tau(\underline{\varphi}) = (a_\tau, \underline{\varphi}), \quad a^*_w(\underline{\varphi}) = (a_w, \underline{\varphi}), \quad a^*_\rho(\underline{\varphi}) = (a_\rho, \underline{\varphi}), \quad \forall\, \underline{\varphi} \in W^{1,2}_D,$$

where the functions a_τ, a_ρ and a_w belong to $L^{p/2}_+$ with $p = 3$. Fix a positive extension of the Dirichlet boundary data, as discussed in §2.3, and the Robin boundary data as follows

$$\phi_D \in L^\infty \cap W^{1,2}, \quad \hat{\phi}^*_N \in W^{-\frac{1}{2},2}(\partial\mathcal{M}_N, 0).$$

(ii) In the case that the Ricci scalar $R \geqslant 0$, then assume that either $a^\vee_\rho > 0$ or that $a^\vee_\sigma > \sigma_0$, where the positive constant σ_0 is defined in Eq. (4.35); In the case that the Ricci scalar $R < 0$, then assume that $a^\vee_R > 0$, $a^\vee_\rho \geqslant 0$ and $a^\vee_\sigma \geqslant 0$;

(iii) Assume that the constants $a^\vee_\tau$, $K^\vee$, $\hat{\phi}^\vee_N$, and $\phi^\vee_D$ defined in §4.2 are all positive.

Then, there exists a function $\phi \in [\phi_-, \phi_{w+}] \cap A^{1,2} \subset W^{1,2}$ which is a solution of Eq. (4.9), where ϕ_{w+} is the super-solution found in Lemma 6, and ϕ_- is the sub-solution given in Lemma 8 for the case $R < 0$, and given in Lemma 9 for the case $R \geqslant 0$.

Remark. The proof of Theorem 9 begins by shifting the equation (4.9) in an appropriate way without changing its solutions. Then it is shown that a solution of the shifted equation exists iff there exists a fixed point of a certain map. It is then established that this map is compact, and thanks to the shifting it is also monotone increasing. These properties establish the existence of a fixed point. The last step in the proof is to show that this fixed point is a solution of the original Eq. (4.9).

Remark. We note that the weak boundary value problem for the Hamiltonian and momentum constraint equations introduced in §2.3 is well-defined for source functions satisfying Eq. (2.22), that is, a_τ, a_ρ, and a_w belong to $L^{6/5}$. However, Theorem 9 above requires that these coefficients belong to $L^{3/2}$. This extra regularity is needed to shift the Hamiltonian constraint equation. It is not clear if there exists a different shifting procedure that also works for for coefficients in L^p with $6/5 \leqslant p < 3/2$. This issue is also present in [38], where rough solutions are found in $W^{k,2}$ for $k > 3/2$; here we are basically asking for higher Lebesgue index p instead the higher Sobolev index k in [38] to make the shift possible.

Proof. *(Theorem 9.)* The assumption *(iii)* is required in Lemma 6 for the existence of the (local) super-solution ϕ_{w+} given in Eq. (4.14). The assumptions *(ii)-(iii)* are required to find a sub-solution: In the case of Ricci scalar $R < 0$ the global sub-solution is given in Lemma 8; In the case that the Ricci scalar $R \geqslant 0$ the global sub-solution is given in Lemma 9 in the case $a^\vee_\rho > 0$, and is given in Lemma 10 in the case $a^\vee_\sigma > \sigma_0$. Summarizing, in these cases there exists at least *local* sub- and super-solutions for the Hamiltonian constraint Eq. (4.9), which is sufficient for our needs here.

The condition $K^{\vee} > 0$ in assumption *(iii)* implies that the constant $\hat{\mathtt{k}}$ in Eq. (4.2) is positive in both cases where $\partial\mathcal{M}_D = \emptyset$ and $\partial\mathcal{M}_D \neq \emptyset$; so, the operator A_L is invertible.

We now use the sub- and super-solutions ϕ_-, ϕ_{w+} to restrict the domain of the functionals f_{wF} and f_w to the set $[\phi_-, \phi_{w+}] \subset L^2$. The interval itself and the functionals f_{wF}, f_w are well-defined, due to the property $0 < \phi_- \leqslant \phi_{w+}$. Let $\alpha \in L^{p/2}$ be the function given by

$$\alpha := 5\,\phi_{w+}^4 a_\tau + |a_R| + 3\,\frac{\phi_{w+}^2}{\phi_-^6}\,a_\rho + 7\,\frac{\phi_{w+}^6}{\phi_-^{14}}\,a_w, \tag{4.48}$$

and introduce a function $s \in L^{p/2}$ such that $(s-\alpha) \in L^{p/2}_+$. Then, define the shifted operators

$$A_L^s : W^{1,2} \to W_D^{-1,2}, \qquad A_L^s\phi(\underline{\varphi}) := A_L\phi(\underline{\varphi}) + (s\phi, \underline{\varphi}), \tag{4.49}$$

$$f_w^s : [\phi_-, \phi_{w+}] \subset L^2 \to W_D^{-1,2}, \qquad f_w^s(\phi)(\underline{\varphi}) := f_w(\phi)(\underline{\varphi}) - (s\phi, \underline{\varphi}). \tag{4.50}$$

First, note that the operator A_L^s is well-defined, since for all $s \in L^{3/2}$, and ϕ, $\underline{\varphi} \in W^{1,2}$ the generalized Hölder inequality ($p=1$, $p_1 = 3/2$, $p_2 = p_3 = 6$, in the notation given at the end of §1.1) implies

$$(s\phi, \underline{\varphi}) \leqslant \|s\|_{3/2}\,\|\phi\|_6\,\|\underline{\varphi}\|_6 \leqslant c^2\,\|s\|_{3/2}\,\|\phi\|_{1,2}\,\|\underline{\varphi}\|_{1,2},$$

where we used the fact that the imbedding $W^{1,2} \to L^6$ is continuous with imbedding constant $c > 0$. The functional f_w^s is also well-defined because the shift term can be bounded as follows

$$(s\phi, \underline{\varphi}) \leqslant \|s\|_{6/5}\,\|\phi\|_\infty\,\|\underline{\varphi}\|_6 \leqslant c\,\|s\|_{3/2}\,\|\phi\|_\infty\,\|\underline{\varphi}\|_{1,2}.$$

In fact, the first inequality shows that the shift on the functional f_w is well-defined for the shift function $s \in L^{6/5}$. However, we have just seen that the shift on the operator A_L is well-defined only for the shift function $s \in L^{3/2} \subset L^{6/5}$.

This operator A_L^s is invertible since A_L is invertible (due to the hypothesis $K^{\vee} > 0$) and since the function s is non-negative (see for example [25] for a proof). This shifted operator A_L^s satisfies the maximum principle, a result shown in Lemma 20 in the Appendix. Therefore, Lemma 21 in that Appendix shows that $(A_L^s)^{-1}$ is a monotone increasing operator.

Second, note that the function s satisfies $(s-\alpha) \in L^{p/2}_+$, which implies that the operator f_w^s is monotone decreasing. The latter means that given functions ϕ_2, $\phi_1 \in [\phi_-, \phi_{w+}]$ with $\phi_2 - \phi_1 \in L^\infty_+$, the functional f_w^s satisfies $-[f_w^s(\phi_2) - f_w^s(\phi_1)] \in W^{-1,2}_{D+}$. The proof of this property is the following: given such functions ϕ_2 and ϕ_1, compute

$$\begin{aligned}
\big(f_w^s(\phi_2) - f_w^s(\phi_1)\big)(\underline{\varphi}) &= \big(f_w(\phi_2) - f_w(\phi_1)\big)(\underline{\varphi}) - (s[\phi_2 - \phi_1], \underline{\varphi}) \\
&= \big([f_{wF}(\phi_2) - f_{wF}(\phi_1)], \underline{\varphi}\big) - (s[\phi_2 - \phi_1], \underline{\varphi}) \\
&= \big(a_\tau\big[(\phi_2)^5 - (\phi_1)^5\big], \underline{\varphi}\big) + \big(a_R[\phi_2 - \phi_1], \underline{\varphi}\big) \\
&\quad - (s[\phi_2 - \phi_1], \underline{\varphi}) - \big(a_\rho\big[(\phi_2)^{-3} - (\phi_1)^{-3}\big], \underline{\varphi}\big) \\
&\quad - \big(a_w\big[(\phi_2)^{-7} - (\phi_1)^{-7}\big], \underline{\varphi}\big).
\end{aligned} \tag{4.51}$$

Now, the conditions $0 < \phi_1 \leqslant \phi_2$ and $\phi_1,\ \phi_2 \in [\phi_-, \phi_{w+}]$ imply the following inequalities,

$$
\begin{aligned}
(\phi_2)^5 - (\phi_1)^5 &= \Big(\sum_{j=0}^{4} (\phi_2)^j (\phi_1)^{4-j}\Big)(\phi_2 - \phi_1) \\
&\leqslant 5\,(\phi_{w+})^4\,(\phi_2 - \phi_1), \qquad (4.52)\\
-\big[(\phi_2)^{-3} - (\phi_1)^{-3}\big] &= \frac{1}{(\phi_2\phi_1)^3}\,\Big(\sum_{j=0}^{2} (\phi_2)^j (\phi_1)^{2-j}\Big)(\phi_2 - \phi_1) \\
&\leqslant 3\,\frac{(\phi_{w+})^2}{(\phi_-)^6}\,(\phi_2 - \phi_1), \qquad (4.53)\\
-\big[(\phi_2)^{-7} - (\phi_1)^{-7}\big] &= \frac{1}{(\phi_2\phi_1)^7}\,\Big(\sum_{j=0}^{6} (\phi_2)^j (\phi_1)^{6-j}\Big)(\phi_2 - \phi_1) \\
&\leqslant 7\,\frac{(\phi_{w+})^6}{(\phi_-)^{14}}\,(\phi_2 - \phi_1). \qquad (4.54)
\end{aligned}
$$

These inequalities and Eq. (4.51) imply

$$
\big(f_w^s(\phi_2) - f_w^s(\phi_1)\big)(\underline{\varphi}) \leqslant \big([\alpha - s](\phi_2 - \phi_1), \underline{\varphi}\big),
$$

where α is given in Eq. (4.48). The choice $s \in L^{p/2}$ and $s \geqslant \alpha$ implies that

$$
\big(f_w^s(\phi_2) - f_w^s(\phi_1)\big)(\underline{\varphi}) \leqslant 0 \qquad \forall\, \underline{\varphi} \in W^{1,2}_{D+},
$$

which establishes that f_w^s is monotone decreasing.

Having introduced the shifted operators A_L^s and f_w^s, we now remark that a function $\phi \in [\phi_-, \phi_{w+}] \cap W^{1,2}$ is solution of $A_L\phi + f_w(\phi) = 0$ iff ϕ is solution of $A_L^s\phi + f_w^s(\phi) = 0$. So far we have the following structure:

$$
\begin{gathered}
f_w^s : [\phi_-, \phi_{w+}] \subset L^2 \to W_D^{-1,2}, \\
(A_L^s)^{-1} : W_D^{-1,2} \to A^{1,2} \subset W^{1,2}, \\
I : W^{1,2} \to L^2,
\end{gathered}
$$

where I is the identity imbedding, which is a compact map. Therefore, the operator

$$
T_w^s : [\phi_-, \phi_{w+}] \subset L^2 \to L^2, \qquad T_{\boldsymbol{w}}^s(\phi) := -I\,(A_w^s)^{-1} f_w^s(\phi), \tag{4.55}
$$

is well-defined. Both the operator $(A_L^s)^{-1}$ and the functional $-f_w^s$ are monotone increasing, therefore the operator T_w^s is also monotone increasing, a result that is proven in Lemma 22. Furthermore, this operator T_w^s is compact, because it is a composition of continuous maps and the compact imbedding $I : W^{1,2} \to L^2$ (for example see [20] page 486, Theorem 4, and also see the imbedding Theorems in [1] chapter VI). We established that the functions ϕ_- and ϕ_{w+} are sub- and super-solutions of Eq. (4.9), respectively. Therefore, Lemma 23 in the Appendix shows that these functions ϕ_- and ϕ_{w+} satisfy the inequalities in the order given by L^2_+,

$$
\phi_- \leqslant T_w^s(\phi_-), \qquad \phi_{w+} \geqslant T_w^s(\phi_{w+}).
$$

Since the order cone in L^2 is normal, all the hypotheses in Theorem 15 in the Appendix are satisfied. Thus, there exists $\phi \in [\phi_-, \phi_{w+}] \subset L^2$ a fixed point of T_w^s.

We now show that the fixed point ϕ satisfies that $\phi \in [\phi_-, \phi_{w+}] \cap W^{1,2}$. This result is a consequence of T^s_w being bounded in $W^{1,2}$. Indeed, given any function $\varphi \in [\phi_-, \phi_{w+}]$ we have that

$$\|T^s_w(\varphi)\|_{1,2} = \| - (A^s_L)^{-1} f^s_w(\varphi)\|_{1,2} \leqslant c_L \, \|f^s_w(\varphi)\|_{-1,2}. \tag{4.56}$$

Recalling the definition of the functional f^s_w, that is,

$$f^s_w(\varphi)(\underline{\varphi}) = (a_\tau \varphi^5, \underline{\varphi}) + (a_R \varphi, \underline{\varphi}) - (a_\rho \varphi^{-3}, \underline{\varphi}) - (a_w \varphi^{-7}, \underline{\varphi}) - \hat{\phi}^*_N(\mathsf{tr}_N \underline{\varphi}) - (s\varphi, \underline{\varphi}),$$

we have the following inequalities,

$$\begin{aligned} |f^s_w(\varphi)(\underline{\varphi})| &\leqslant (a_\tau, \underline{\varphi})\, \phi^5_{w+} + (|a_R|\varphi, \underline{\varphi}) + (a_\rho, \underline{\varphi})\, \phi^{-3}_- + (a_w, \underline{\varphi})\, \phi^{-7}_- \\ &\quad + |\hat{\phi}^*_N(\mathsf{tr}_N \underline{\varphi})| + (s, \underline{\varphi})\, \phi_{w+} \\ &\leqslant \Big[a^\wedge_\tau \, \phi^5_{w+} + a^\wedge_R \, \phi_{w+} + a^\wedge_\rho \, \phi^{-3}_- + a^\wedge_w \, \phi^{-7}_- \\ &\quad + c_t \|\hat{\phi}^*_N\|_{-\frac{1}{2},2,N} + s^\wedge \, \phi_{w+} \Big] \, \|\underline{\varphi}\|_{1,2}, \end{aligned}$$

where $s^\wedge$ is defined in an analogous way as $a^\wedge_\tau$ in Eq. (4.12), and c_t is a positive constant such that $\|\mathsf{tr}_N \underline{\varphi}\|_{\frac{1}{2},2,N} \leqslant c_t \|\underline{\varphi}\|_{1,2}$ for all $\underline{\varphi} \in W^{1,2}_D$. Therefore, introducing the constant

$$c_w := \big[a^\wedge_\tau \, \phi^5_{w+} + (a^\wedge_R + s^\wedge)\, \phi_{w+} + a^\wedge_\rho \, \phi^{-3}_- + a^\wedge_w \, \phi^{-7}_- + c_t \|\hat{\phi}^*_N\|_{-\frac{1}{2},2,N} \big],$$

we have the inequality

$$\sup_{0 \neq \underline{\varphi} \in W^{1,2}_D} \frac{|f^s_w(\varphi)(\underline{\varphi})|}{\|\underline{\varphi}\|_{1,2}} \leqslant c_w,$$

which yields to the desired inequality

$$\|T^s_w(\varphi)\|_{1,2} \leqslant k_w \qquad \forall \, \varphi \in [\phi_-, \phi_{w+}],$$

with $k_w = c_L c_w$. Therefore, the fixed point point $\phi \in [\phi_-, \phi_{w+}]$ satisfies

$$\|\phi\|_{1,2} = \|T^s_w(\phi)\|_{1,2} \leqslant k_w,$$

which establishes the property $\phi \in [\phi_-, \phi_{w+}] \cap W^{1,2}$. Therefore, we can apply the operator A^s_L on both sides of the equation $\phi = T^s_w(\phi)$, and then the fixed point function ϕ satisfies both the shifted and the non-shifted Hamiltonian constraint equations. The latter establishes that the function ϕ is a solution of the Hamiltonian constraint Eq. (4.9). □

4.7. **Regularity of solutions.** The following result states that when the regularity of the boundary data agrees with the equation coefficients regularity, the solution obtained by barrier methods is actually more regular than is stated in Theorem 9. Note that the Proposition 1 below does not apply in the case of the solutions found by variational methods. In this latter case the coefficients in the functional f_w belong to $W^{-1,2}_D$, so the bootstrap argument mentioned below does not apply.

Proposition 1. **(Regularity)** *Assume the hypotheses in Theorem 9, and in addition assume that the boundary $\partial\mathcal{M}$ is C^2. Assume that the boundary data satisfy $\hat{\phi}^*_N(\mathsf{tr}_N \underline{\varphi}) = (\hat{\phi}_N, \mathsf{tr}_N \underline{\varphi})_N$ for all $\underline{\varphi} \in W^{1,2}_D$ and the following condition holds*

$$\phi_D \in W^{2,(p/2)}, \quad \hat{\phi}_N \in W^{\frac{1}{(p/2)'},(p/2)}(\partial\mathcal{M}_N), \quad p = 3.$$

Then, the function $\phi \in [\phi_-, \phi_{w+}] \cap W^{1,2}$ solution of the Hamiltonian constraint Eq. (4.9) belongs to the space $W^{2,(p/2)}$.

Outline of the Proof. *(Proposition 1.)* A proof can be based on linear elliptic estimates (see for example Theorem 9.11 in [25]) and a standard bootstrap argument. □

5. Coupled system

Here we combine the results for the individual constraints derived earlier to establish a new non-CMC result for the coupled system. In §5.1 we use the global barriers found in §4.2 to establish existence of non-CMC solutions to the coupled constraints through fixed-point iteration and compactness arguments directly, rather than by using the Contraction Mapping Theorem as was done in the original work of Isenberg and Moncrief in [30].

It is interesting to note that for the main result on the non-CMC coupled system in [30], the near-CMC condition on the trace of the extrinsic curvature is actually used twice: once to obtain the global super-solution, and a second distinct time to construct a contraction for using the Contraction Mapping Theorem to get existence and uniqueness. Here, a weak version of the near-CMC condition must also be employed in §4.2 to drive a global super-solution for the Hamiltonian constraint in our weaker setting. However, by using a compactness argument for the coupled system in §5.1 rather than the Contraction Mapping Theorem, we avoid the second use of the near-CMC condition. If a global super-solution can be constructed without the near-CMC assumption, then our compactness argument would give existence of solutions to the coupled system in the fully general "far-from-CMC" case. What our proof technique gives up is uniqueness of solutions to the coupled system, which comes for free with existence when the contraction argument is used as in [30].

5.1. **Existence of weak solutions.** This section is dedicated to establishing existence of solutions to the weak Dirichlet-Robin boundary value problem (2.35)-(2.36). In the case that $R < 0$ there is no condition on the matter fields other than $\rho \geqslant 0$, but in the case that the Ricci scalar $R \geqslant 0$ it is required that either $a_\rho^\vee > 0$ or $a_\sigma^\vee > \sigma_0$, with the positive constant σ_0 defined in Eq. (4.35). The equation coefficients are required to have stronger regularity than those previously required in Secs. 4.1 and 4.6. Although our problem formulation is different (compact domains with boundary), our results can be viewed as extending the result in [30] to weaker solution spaces, and extending their result for $R = -1$ to scalar curvature having any sign. The work presented can similarly be viewed as extending the CMC results on rough solutions in [38] to the non-CMC case, for compact domains with boundary. As remarked earlier, the "near-CMC" assumption required for the Contraction Mapping Argument in [30] is not required for the compactness argument below.

Theorem 10. (Non-CMC) *Consider the weak formulation for the Hamiltonian and momentum constraints defined in §2.3. Assume the background metric $h \in C^2(\overline{\mathcal{M}}, 2)$ and that the following conditions hold:*

(i) Fix a number $p > 3$ and denote by $q := 6p/(3+p)$. Fix source and boundary functions

$$\tau \in L^q, \quad (\nabla\tau)^* \in \boldsymbol{W}_{\!I\!D}^{-1,q}, \quad \sigma \in L^q(\mathcal{M},2), \quad \rho \in L_+^{q/2}, \quad \boldsymbol{j} \in \boldsymbol{L}^{q/2},$$

$$\phi_D \in W^{1,p}, \quad \hat{\phi}_N^* \in W^{-\frac{1}{p},p}(\partial\mathcal{M}_N, 0), \quad \boldsymbol{w}_{\!I\!D} \in \boldsymbol{W}^{1,q}, \quad \hat{\boldsymbol{w}}_{\!I\!N}^* \in W^{-\frac{1}{q},q}(\partial\mathcal{M}_{\!I\!N}, 1),$$

with $(\rho^2 - \boldsymbol{j}\cdot\boldsymbol{j}) \in int(L_+^{q/4})$ in the case $\boldsymbol{j} \neq 0$;

(ii) *In the case that the Ricci scalar R of the background metric is non-negative, then assume that either the constant $a^{\vee}_{\rho} > 0$ or the constant $a^{\vee}_{\sigma} > \sigma_0$, with the positive constant σ_0 defined in Eq. (4.35); In the case that the Ricci scalar R is negative, then assume that $a^{\vee}_{R} > 0$, $a^{\vee}_{\rho} \geqslant 0$ and $a^{\vee}_{\sigma} \geqslant 0$;*

(iii) *Assume that the function a_τ and the constant $\mathtt{K}_1$ defined in Eq. (4.21) satisfy that $a^{\vee}_{\tau} - \mathtt{K}_1 > 0$; also assume that the constants $K^{\vee}$, $\hat{\phi}^{\vee}_{N}$, and $\phi^{\vee}_{D}$ are all positive.*

Then, there exists a solution

$$\phi \in [\phi_-, \phi_+] \cap A^{1,p} \subset W^{1,p}, \qquad \boldsymbol{w} \in \boldsymbol{A}^{1,q} \subset \boldsymbol{W}^{1,q}, \qquad p > 3, \quad q = \frac{6p}{3+p},$$

of the weak Dirichlet-Robin boundary value problem for the Hamiltonian and momentum constraint Eqs. (2.35)-(2.36), where ϕ_+ is the super-solution found in Lemma 7, and ϕ_- is the sub-solution given in Lemma 8 for the case $R < 0$, and given in Lemma 9 for the case $R \geqslant 0$.

Proof. *(Theorem 10.)* Notice that the definition of the numbers p and q satisfies that $3 < q < p$. The assumption $\tau \in L^q$ indicates that $a_\tau = \tau^2/12 \in L^{q/2}$, which implies that the linear functional a^*_τ given by as $a^*_\tau(\underline{\varphi}) = (a_\tau, \underline{\varphi})$ for all $\underline{\varphi} \in W^{1,p'}_D$ is a well-defined element $a^*_\tau \in W^{-1,p}_D$. The proof of the latter statement is based in the Hölder inequality, which implies

$$|a^*_\tau(\underline{\varphi})| \leqslant \|a_\tau\|_{(q/2)} \, \|\underline{\varphi}\|_{(q/2)'};$$

since $(q/2) = 3p/(3+p)$, the relations $\frac{1}{(q/2)} + \frac{1}{(q/2)'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$ imply that $(q/2)' = 3p'/[3-p']$. Now the coefficient $p > 3$ implies that $p' < 3/2$, so we conclude that the imbedding $W^{1,p'} \subset L^{(q/2)'}$ is continuous (see [25], Corollary 7.11 in §7.7), and then there exists a positive constant c_s such that

$$|a^*_\tau(\underline{\varphi})| \leqslant c_s \, \|a_\tau\|_{(q/2)} \, \|\underline{\varphi}\|_{1,p'},$$

which establishes that $a^*_\tau \in W^{-1,p}_D$. The coefficient functions $a_\rho = \kappa\rho/4$ and $a_\sigma = \sigma^2/8$ belong to $L^{q/2}$, and the same holds for the coefficient $a_{\mathcal{L}w} = (\mathcal{L}\boldsymbol{w})^2/8$ whenever the vector $\boldsymbol{w} \in \boldsymbol{W}^{1,q}$. A similar argument as above shows that the functionals

$$a^*_\rho(\underline{\varphi}) := (a_\rho, \underline{\varphi}), \quad a^*_\sigma(\underline{\varphi}) := (a_\sigma, \underline{\varphi}), \quad a^*_{\mathcal{L}w}(\underline{\varphi}) := (a_{\mathcal{L}w}, \underline{\varphi}), \qquad \forall\, \underline{\varphi} \in W^{1,p'}_D$$

are well-defined elements in $W^{-1,p}_D$. The choice of the Dirichlet and Robin boundary data and the Gelfand triple structure reviewed in the Appendix imply that the functional f defined in Eq. (2.27) is a well-defined map

$$f : [\phi_1, \phi_2] \subset L^\infty \times \boldsymbol{W}^{1,q} \to W^{-1,p}_D.$$

The assumption that the function $\boldsymbol{j} \in L^{q/2}$ implies that $\boldsymbol{j} \in L^{r/2}$ with $(r/2) = 3q/(3+q)$, since for $q > 3$ holds that $3 < r < q$. Hence, the functional $\boldsymbol{b}^*_j(\underline{\boldsymbol{\omega}}) := (\kappa \boldsymbol{j}, \underline{\boldsymbol{\omega}})$ for all $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}^{1,q'}_{I\!D}$ is a well-defined element $\boldsymbol{b}^*_j \in \boldsymbol{W}^{-1,q}_{I\!D}$. The proof is again based in the Hölder inequality

$$|\boldsymbol{b}^*_j(\underline{\boldsymbol{\omega}})| \leqslant \|\kappa \boldsymbol{j}\|_{(r/2)} \, \|\underline{\boldsymbol{\omega}}\|_{(r/2)'},$$

The condition $q > 3$ implies the inequality $q' < 3/2$ and the relations $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{(r/2)} + \frac{1}{(r/2)'} = 1$ imply that $(r/2)' = 3q'/[3-q']$. From the latter relation and the inequality $q' < 3/2$ we conclude that the imbedding $W^{1,q'} \subset L^{(r/2)'}$ is continuous, so there exists a positive constant c_s such that

$$|\boldsymbol{b}_j^*(\underline{\boldsymbol{\omega}})| \leqslant c_s \, \|\boldsymbol{j}\|_{(r/2)} \, \|\underline{\boldsymbol{\omega}}\|_{1,q'},$$

which establishes that $\boldsymbol{b}_j^* \in W_{I\!D}^{-1,q}$. The assumption $(\nabla\tau)^* \in \boldsymbol{W}_{I\!D}^{-1,q}$ implies that $\boldsymbol{b}_\tau(\underline{\boldsymbol{\omega}}) := (\frac{2}{3}\nabla\tau, \underline{\boldsymbol{\omega}})$ for all $\underline{\boldsymbol{\omega}} \in \boldsymbol{W}_{I\!D}^{1,q'}$ is a well-defined element $\boldsymbol{b}_\tau^* \in \boldsymbol{W}_{I\!D}^{-1,q}$. The choice of the Dirichlet and Robin boundary data and the Gelfand triple structure reviewed in the Appendix imply that the functional $\boldsymbol{f}$ defined in Eq. (2.28) is a well-defined map

$$\boldsymbol{f} : [\phi_1, \phi_2] \subset L^\infty \to \boldsymbol{W}_{I\!D}^{-1,q}.$$

The regularity assumptions on the equation coefficients and the assumptions *(ii)*-*(iii)* are sufficient conditions to establish the existence of a global sub-solution ϕ_- for the Hamiltonian constraint Eq. (2.35). In the case that the $R < 0$ this result is proved in Lemma 8, and in the case that $R \geqslant 0$ this result is proved either in Lemma 9 in the case $a_\rho^\vee > 0$, or in Lemma 10 in the case $a_\sigma^\vee > \sigma_0$. The regularity assumptions on the equation coefficients and the assumptions *(iii)* are the sufficient conditions to establish the existence of a global super-solution ϕ_+ for the Hamiltonian and momentum constraint Eqs. (2.35)-(2.36), which was established in Lemma 7.

We now use these global sub- and super-solutions of Eqs. (2.35)-(2.36) to define the domain $[\phi_-, \phi_+] \subset L^\infty$ of the operators f and $\boldsymbol{f}$ given in Eqs. (2.27)-(2.28). The inequality $K^\vee > 0$ in assumption *(iii)* implies that the operator $A_{I\!L} : W^{1,2} \to W_D^{-1,2}$ defined in Eq. (2.33) is invertible, which was established in Theorem 3. The regularity result in Proposition 1 implies that the operator $A_L : W^{1,p} \to W_D^{-1,p}$ with the same action as defined in Eq. (2.33) is also invertible. Regarding the Hamiltonian constraint equation, we introduce the same shifting done in the proof of Theorem 9, that is, fix a function $s \in L^{q/2}$, given by

$$s := 5\,\phi_+^4 a_\tau + |a_R| + 3\,\frac{\phi_+^2}{\phi_-^6}\,a_\rho + 7\,\frac{\phi_+^6}{\phi_-^{14}}\,a_w,$$

and introduce the shifting operators

$$\begin{aligned} &A_L^s : W^{1,p} \to W_D^{-1,p}, && A_L^s\phi(\underline{\varphi}) := A_L\phi(\underline{\varphi}) + (s\phi, \underline{\varphi}),\\ &f^s : [\phi_{w-}, \phi_{w+}] \subset L^\infty \times \boldsymbol{W}^{1,q} \to W_D^{-1,p}, && f^s(\phi, \boldsymbol{w})(\underline{\varphi}) := f(\phi, \boldsymbol{w})(\underline{\varphi}) - (s\phi, \underline{\varphi}). \end{aligned}$$

Then, we have the following structure,

$$\begin{gathered} f^s : [\phi_-, \phi_+] \subset L^\infty \times \boldsymbol{W}^{1,q} \to W_D^{-1,p},\\ (A_L^s)^{-1} : W_D^{-1,p} \to A^{1,p} \subset W^{1,p},\\ I : W^{1,p} \to L^\infty, \end{gathered} \qquad \begin{gathered} \boldsymbol{f} : [\phi_-, \phi_+] \subset L^\infty \to \boldsymbol{W}_{I\!D}^{-1,q},\\ (A_{I\!L})^{-1} : \boldsymbol{W}_{I\!D}^{-1,q} \to \boldsymbol{A}^{1,q}, \end{gathered}$$

where the map $I : W^{1,p} \to L^\infty$ is the identity imbedding, which is compact for $p > 3$. Therefore, the following operators are well-defined,

$$\begin{aligned} &S : [\phi_-, \phi_+] \subset L^\infty \to \boldsymbol{W}^{1,q}, && S(\phi) := -(A_{I\!L})^{-1}\boldsymbol{f}(\phi),\\ &T_{\boldsymbol{w}}^s : [\phi_-, \phi_+] \subset L^\infty \to L^\infty, && T_{\boldsymbol{w}}^s(\phi) := -I(A_L^s)^{-1} f(\phi, \boldsymbol{w}). \end{aligned}$$

Since we can choose ϕ_{w+} in Theorem 9 to be the constant ϕ_+ found in Lemma 7, then this Theorem 9 and the regularity results in Proposition 1 imply that exists a fixed point $\varphi \in [\phi_-, \phi_+] \cap W^{1,p}$ of the iteration

$$\varphi_{k+1} := T^s_{\boldsymbol{w}}(\varphi_k), \qquad \varphi_0 = \phi_-.$$

Therefore, the sequence $\{\phi^n, \boldsymbol{w}^n\}$ given by $\phi^0 = \phi_-$, $\boldsymbol{w}^0 = \boldsymbol{w}_{\mathbb{D}}$, and

$$\boldsymbol{w}^n = S\big(\phi^{n-1}\big), \qquad \phi^n = T^s_{\boldsymbol{w}^n}\big(\phi^n\big), \qquad n \in \mathbb{N},$$

is well-defined, where $\phi^n \in [\phi_-, \phi_+] \cap W^{1,p}$ is a fixed point of the operator $T_{\boldsymbol{w}^n}$, with $n \in \mathbb{N}$. By definition and by Theorem 9, each element in this sequence satisfies the equations

$$A_L \phi^n + f\big(\phi^n, \boldsymbol{w}^n\big) = 0, \qquad A_{\mathbb{L}} \boldsymbol{w}^n + \boldsymbol{f}\big(\phi^{n-1}\big) = 0.$$

We will show that the sequence $\{\phi^n, \boldsymbol{w}^n\} \subset [\phi_-, \phi_+] \cap W^{1,p} \times \boldsymbol{W}^{1,q}$ is bounded. The proof is as follows. First, the elliptic estimates for the momentum constraint given in §3.1, which imply, that there exists positive constants $\tilde{\mathtt{K}}_1$, $\tilde{\mathtt{K}}_2$ such that

$$\|\boldsymbol{w}^n\|_{1,q} \leqslant \hat{\mathtt{K}}_1\, \phi_+^6 + \hat{\mathtt{K}}_2, \qquad \forall\, n \in \mathbb{N}.$$

Second, a calculation similar to the one performed after Eq. (4.56) changing norms in $W^{-1,2}$ with norms in $W^{-1,p}$ for $p > 3$ implies that

$$\|T^s_{w^n}(\phi^n)\|_\infty \leqslant c_0\, \|T^s_{w^n}(\phi^n)\|_{1,p} \leqslant k_{w^n},$$

$$\begin{aligned} k_{w^n} := c_0 c_L \big[&\tilde{a}^\wedge_\tau\, \phi_+^5 + (\tilde{a}^\wedge_R + \tilde{s}^\wedge_n)\, \phi_+ + \tilde{a}^\wedge_\rho\, \phi_-^{-3} \\ &+ \tilde{a}^\wedge_{w^n}\, \phi_-^{-7} + c_0 \|\hat{\phi}^*_N\|_{-\frac{1}{p},p,N} \big], \end{aligned}$$

where we have introduced the constants $\tilde{a}^\wedge_\tau$, $\tilde{a}^\wedge_R$, $\tilde{a}^\wedge_\rho$, and $\tilde{a}^\wedge_{w^n}$, which are defined in a similar way as in Eqs. (4.12)-(4.13) changing the norms in $W^{1,2}_D$ by norms in $W^{1,p}_D$. We have also introduced the number $\tilde{s}^\wedge$ given by

$$\tilde{s}^\wedge_n := 5\, \phi_+^4\, \tilde{a}^\wedge_\tau + a^\wedge_R + 3\, \frac{\phi_+^2}{\phi_-^6}\, \tilde{a}^\wedge_\rho + 7\, \frac{\phi_+^6}{\phi_-^{14}}\, \tilde{a}^\wedge_{w^n}.$$

The dependence on n in k_{w^n} is due to the term $\tilde{a}^\wedge_{w^n}$. However, Eq. (4.20) implies that $a^\wedge_{w^n}$ can be bounded for all $n \in \mathbb{N}$, and we obtain the inequality

$$\|T^s_{w^n}(\phi^n)\|_\infty \leqslant k_0,$$

$$\begin{aligned} k_0 := c_0 c_L \big[&\tilde{a}^\wedge_\tau\, \phi_+^5 + (\tilde{a}^\wedge_R + \tilde{s}^\wedge_{\max})\, \phi_+ + \tilde{a}^\wedge_\rho\, \phi_-^{-3} \\ &+ (\tilde{\mathtt{K}}_1 \phi_+^{12} + \tilde{\mathtt{K}}_2)\, \phi_-^{-7} + c_0 \|\hat{\phi}^*_N\|_{-\frac{1}{p},p,N} \big], \\ \tilde{s}^\wedge_{\max} := 5\, &\phi_+^4 \tilde{a}^\wedge_\tau + \tilde{a}^\wedge_R + 3\, \frac{\phi_+^2}{\phi_-^6}\, \tilde{a}^\wedge_\rho + 7\, \frac{\phi_+^6}{\phi_-^{14}}\, (\tilde{\mathtt{K}}_1 \phi_+^{12} + \tilde{\mathtt{K}}_2). \end{aligned}$$

This establishes that the sequence $\{\phi^n, \boldsymbol{w}^n\}$ is bounded in $W^{1,p} \times \boldsymbol{W}^{1,q}$. The latter space is a reflexive Banach space, so the sequence $\{\phi^n, \boldsymbol{w}^n\}$ has a weakly convergent subsequence, that is, there exist elements $\phi \in W^{1,p}$ and $\boldsymbol{w} \in \boldsymbol{W}^{1,q}$ such that

$$\phi^{n_j} \rightharpoonup \phi \in W^{1,p}, \qquad \boldsymbol{w}^{n_j} \rightharpoonup \boldsymbol{w} \in \boldsymbol{W}^{1,q}.$$

The imbeddings $W^{1,p} \to L^\infty$ and $\boldsymbol{W}^{1,q} \to \boldsymbol{L}^\infty$ are compact since $3 < q < p$, which implies that

$$\phi^{n_j} \to \phi \text{ in } L^\infty, \qquad \boldsymbol{w}^{n_j} \to \boldsymbol{w} \text{ in } \boldsymbol{L}^\infty,$$

that is, the convergence is strong in these spaces. We now first note that $\phi \in [\phi_-, \phi_+]$, since the sequence $\phi^{n_j} \to \phi$ in the supremum norm and the interval $[\phi_-, \phi_+]$ is a closed set in L^∞. Second, we now show that indeed

$$\boldsymbol{w}^{n_j} \to \boldsymbol{w} \text{ in } \boldsymbol{W}^{1,q},$$

that is, the sequence $\{\boldsymbol{w}^{n_j}\}$ converges strongly to $\boldsymbol{w}$ in $\boldsymbol{W}^{1,q}$. The proof is to show that $\{\boldsymbol{w}^{n_j}\}$ is a Cauchy sequence in $\boldsymbol{W}^{1,q}$. This is shown by the following calculation, where we rename $\{\phi^{n_j}, \boldsymbol{w}^{n_j}\}$ simply as $\{\phi^n, \boldsymbol{w}^n\}$. Then, we obtain,

$$\begin{aligned}
\|\boldsymbol{w}^n - \boldsymbol{w}^m\|_{1,q} &\leqslant c_1 \,\|\boldsymbol{b}^*_\tau\|_{-1,q} \Big\| \big[(\phi^n)^6 - (\phi^m)^6\big] \Big\|_\infty \\
&= c_1 \,\|\boldsymbol{b}^*_\tau\|_{-1,q} \Big\| \Big[\sum_{j=0}^{5} (\phi^n)^j (\phi^m)^{5-j}\Big] \big(\phi^n - \phi^m\big) \Big\|_\infty \\
&\leqslant c_1 \,\|\boldsymbol{b}^*_\tau\|_{-1,q} \Big[\sum_{j=0}^{5} \|\phi^n\|_\infty^j \,\|\phi^m\|_\infty^{5-j}\Big] \big\|\phi^n - \phi^m\big\|_\infty,
\end{aligned}$$

which leads to

$$\|\boldsymbol{w}^n - \boldsymbol{w}^m\|_{1,q} \leqslant 6c_1\, \phi_+^6 \,\|\boldsymbol{b}^*_\tau\|_{-1,q} \big\|\phi^n - \phi^m\big\|_\infty.$$

Since ϕ^n is Cauchy in L^∞, we have established that $\boldsymbol{w}^n$ is a Cauchy sequence in $\boldsymbol{W}^{1,q}$.

The final step in the proof is to verify that ϕ and $\boldsymbol{w}$ satisfy the constraint equations (2.35)-(2.36). Since $\phi \in [\phi_-, \phi_+]$, the function $T^s_w(\phi) \in W^{1,p} \subset L^\infty$ is well-defined. What we have to show is that

$$T^s_w(\phi) = \phi, \qquad S(\phi) = \boldsymbol{w}. \tag{5.1}$$

The first equation in (5.1) can be written conveniently as follows:

$$T^s_w(\phi) - \phi = \Big(T^s_w(\phi) - T^s_w(\phi^n)\Big) + \Big(T^s_w(\phi^n) - T^s_{w^n}(\phi^n)\Big) + (\phi^n - \phi),$$

therefore,

$$\|T^s_w(\phi) - \phi\|_\infty \leqslant \|T^s_w(\phi) - T^s_w(\phi^n)\|_\infty + \|T^s_w(\phi^n) - T^s_{w^n}(\phi^n)\|_\infty + \|\phi^n - \phi\|_\infty. \tag{5.2}$$

The first term on the right hand side above satisfies the inequalities

$$\begin{aligned}
\|T^s_w(\phi) - T^s_w(\phi^n)\|_\infty &\leqslant c_0 \,\|T^s_w(\phi) - T^s_w(\phi^n)\|_{1,p} \\
&= c_0 \,\|(A^s_L)^{-1}\big(f^s(\phi, \boldsymbol{w}) - f^s(\phi^n, \boldsymbol{w})\big)\|_{1,p} \\
&\leqslant c_0 c_L \,\|f^s(\phi, \boldsymbol{w}) - f^s(\phi^n, \boldsymbol{w})\|_{-1,p},
\end{aligned}$$

and the last line above can be bounded as follows,

$$\begin{aligned}
\|f^s(\phi, \boldsymbol{w}) - f^s(\phi^n, \boldsymbol{w})\|_{-1,p} \leqslant\; & \|a^*_\tau\|_{-1,p} \,\|\phi^5 - (\phi^n)^5\|_\infty \\
&+ \big(\|a^*_R\|_{-1,p} + \|s^*\|_{-1,p}\big) \,\|\phi - \phi^n\|_\infty \\
&+ \|a^*_\rho\|_{-1,p} \,\|\phi^{-3} - (\phi^n)^{-3}\|_\infty \\
&+ \|a^*_w\|_{-1,p} \,\|\phi^{-7} - (\phi^n)^{-7}\|_\infty.
\end{aligned}$$

Notice that $\|a^*_\tau\|_{-1,p} = \tilde{a}^\wedge_\tau$, and the same holds for a^*_R, s^*, a^*_ρ and a^*_w. These relations together with Eqs. (4.52)-(4.54) imply that

$$\begin{aligned}\|f^s(\phi,\boldsymbol{w}) - f^s(\phi^n,\boldsymbol{w})\|_{-1,p} &\\ &\leqslant \Big[5\phi_+^4\,\tilde{a}^\wedge_\tau + (\tilde{a}^\wedge_R + \tilde{s}^\wedge) + 3\frac{\phi_+^2}{\phi_-^6}\,\tilde{a}^\wedge_\rho + 7\frac{\phi_+^6}{\phi_-^{14}}\,\tilde{a}^\wedge_w\Big]\,\|\phi - \phi^n\|_\infty \\ &= \big(\tilde{\alpha}^\wedge + \tilde{s}^\wedge\big)\,\|\phi-\phi^n\|_\infty.\end{aligned} \tag{5.3}$$

Theorem 9 requires $s \geqslant \alpha$, with α given in Eq. (4.48). Choosing $s = \alpha$ we obtain,

$$\|f(\phi,\boldsymbol{w}) - f(\phi^n,\boldsymbol{w})\|_{-1,p} \leqslant 2\tilde{\alpha}^\wedge\,\|\phi-\phi^n\|_\infty,$$

with

$$\tilde{\alpha}^\wedge := 5\,\phi_+^4\,\tilde{a}^\wedge_\tau + \tilde{a}^\wedge_R + 3\,\frac{\phi_+^2}{\phi_-^6}\,\tilde{a}^\wedge_\rho + 7\,\frac{\phi_+^6}{\phi_-^{14}}\,\tilde{a}^\wedge_w.$$

The number $\tilde{\alpha}^\wedge$ can be bounded independently of $\boldsymbol{w}$ since we have the inequality $\tilde{a}^\wedge_w \leqslant \tilde{\mathtt{K}}_1\,\phi_+^{12} + \tilde{\mathtt{K}}_2$, which is obtained in a similar way as the inequality in Eq. (4.20), just changing the norms used in that result to the appropriate norms needed here. Therefore, the following bound holds,

$$\|T^s_w(\phi) - T^s_w(\phi^n)\|_\infty \leqslant k_T\,\|\phi-\phi^n\|_\infty,$$

with the constant k_T given by

$$k_T := 2c_0c_L\Big[5\,\phi_+^4\,\tilde{a}^\wedge_\tau + \tilde{a}^\wedge_R + 3\,\frac{\phi_+^2}{\phi_-^6}\,\tilde{a}^\wedge_\rho + 7\,\frac{\phi_+^6}{\phi_-^{14}}\,\big(\tilde{\mathtt{K}}_1\,\phi_+^{12} + \tilde{\mathtt{K}}_2\big)\Big]. \tag{5.4}$$

(Although we do not need to exploit this fact here, note that from the definition of k_T it can be seen that there always exists source functions and boundary data small enough such that $0 \leqslant k_T < 1$, a condition that implies that this map T^s_w is a k_T-contraction, as is it defined in [57] page 17.)

The second term on the right hand side in Eq. (5.2) satisfies the following bounds

$$\begin{aligned}\|T^s_w(\phi^n) - T^s_{w^n}(\phi^n)\|_\infty &\leqslant c_0\,\|T^s_w(\phi^n) - T^s_{w^n}(\phi^n)\|_{1,p} \\ &\leqslant c_0\,\big\|(A^s_L)^{-1}\big[f^s(\phi^n,\boldsymbol{w}) - f^s(\phi^n,\boldsymbol{w}^n)\big]\big\|_{1,p} \\ &\leqslant c_0c_L\,\|f^s(\phi^n,\boldsymbol{w}) - f^s(\phi^n,\boldsymbol{w}^n)\|_{-1,p} \\ &\leqslant c_0c_L\,\|a^*_{\boldsymbol{w}} - a^*_{\boldsymbol{w}^n}\|_{-1,p}\,\|(\phi^n)^{-7}\|_\infty \\ &\leqslant c_0c_L\,\phi_-^{-7}\,\|a^*_{\boldsymbol{w}} - a^*_{\boldsymbol{w}^n}\|_{-1,p}.\end{aligned}$$

Recall that the functional a^*_w can be expressed as $a^*_w(\underline{\varphi}) = (a_w,\underline{\varphi})$ for all $\underline{\varphi} \in W^{1,p'}_D$, with $a_w = (\sigma + \boldsymbol{\mathcal{L}}\boldsymbol{w})^2/8$. We then conclude that $a_w \in L^{q/2}$, with $q = 6p/(3+p)$, which implies that for $p > 3$ we have the inequality $3 < q < p$. Hence, there exists a positive constant c_s such that $\|a^*_w\|_{-1,p} \leqslant c_s\|a_w\|_{(q/2)}$. So we have the inequality

$$\|T^s_w(\phi^n) - T^s_{w^n}(\phi^n)\|_\infty \leqslant c_0c_Lc_s\,\phi_-^{-7}\,\|a_{\boldsymbol{w}} - a_{\boldsymbol{w}^n}\|_{(q/2)}.$$

Now, the function $a_w - a_{w^n}$ can be written as

$$\begin{aligned}a_{\boldsymbol{w}} - a_{\boldsymbol{w}^n} &= \frac{1}{8}\big[(\sigma + \boldsymbol{\mathcal{L}}\boldsymbol{w})^2 - (\sigma + \boldsymbol{\mathcal{L}}\boldsymbol{w}^n)^2\big] \\ &= \frac{1}{8}\big[2\sigma + \mathcal{L}(\boldsymbol{w}+\boldsymbol{w}^n)\big]\,\mathcal{L}(\boldsymbol{w}-\boldsymbol{w}^n),\end{aligned}$$

with each factor in L^q, so a simple case of the generalized Hölder inequality (see the last part of §1.1) implies that

$$\|a_w - a_{w^n}\|_{(q/2)} \leqslant \frac{1}{8}\,\|2\sigma + \mathcal{L}(\boldsymbol{w}+\boldsymbol{w}^n)\|_q\,\|\mathcal{L}(\boldsymbol{w}-\boldsymbol{w}^n)\|_q.$$

Then, we have the further inequalities

$$\begin{aligned}
\|a_w - a_{w^n}\|_{(q/2)} &\leqslant \frac{1}{8}\,\big(2\|\sigma\|_q + \|\mathcal{L}(\boldsymbol{w}+\boldsymbol{w}^n)\|_q\big)\,\|\mathcal{L}(\boldsymbol{w}-\boldsymbol{w}^n)\|_q\\
&\leqslant \frac{c_{\mathcal{L}}}{8}\,\big(2\|\sigma\|_q + c_{\mathcal{L}}\,\|\boldsymbol{w}+\boldsymbol{w}^n\|_{1,q}\big)\,\|\boldsymbol{w}-\boldsymbol{w}^n\|_{1,q}\\
&\leqslant \frac{c_{\mathcal{L}}}{4}\,\big[\|\sigma\|_q + c_{\mathcal{L}}\,(\tilde{\mathtt{K}}_1\,\phi_+^6 + \tilde{\mathtt{K}}_2)\big]\,\|\boldsymbol{w}-\boldsymbol{w}^n\|_{1,q}.
\end{aligned}$$

Denote $k_2 = c_0 c_L c_s c_{\mathcal{L}} \phi_-^{-7}\big[\|\sigma\|_q + c_{\mathcal{L}}\,(\tilde{\mathtt{K}}_1\,\phi_+^6 + \tilde{\mathtt{K}}_2)\big]/4$, then

$$\|T^s_{w^n}(\phi^n) - T^s_{\boldsymbol{w}}(\phi^n)\|_\infty \leqslant k_2\,\|\boldsymbol{w}^n - \boldsymbol{w}\|_{1,q}.$$

Therefore, the inequality in Eq. (5.2) implies

$$\|T^s_w(\phi) - \phi\|_\infty \leqslant k_T\,\|\phi-\phi^n\|_\infty + k_2\,\|\boldsymbol{w}-\boldsymbol{w}^n\|_{1,q} + \|\phi^n - \phi\|_\infty,$$

and all the terms in the right hand side approaches zero when n approaches infinity, so we conclude that $T^s_w(\phi) = \phi$.

We now show that the second equation in (5.1) also holds using the following argument. We begin with the convenient representation

$$S(\phi) - \boldsymbol{w} = \big[S(\phi) - S(\phi^n)\big] + (\boldsymbol{w}^{n+1} - \boldsymbol{w}).$$

This gives

$$\|S(\phi) - \boldsymbol{w}\|_{1,q} \leqslant \|S(\phi) - S(\phi^n)\|_{1,q} + \|\boldsymbol{w}^{n+1} - \boldsymbol{w}\|_{1,q}. \tag{5.5}$$

The first term on the right hand side can be bounded as follows,

$$\begin{aligned}
\|S(\phi) - S(\phi^n)\|_{1,q} &= \big\|-(A_{\mathbb{L}})^{-1}\big(\boldsymbol{f}(\phi) - \boldsymbol{f}(\phi^n)\big)\big\|_{1,q}\\
&\leqslant c_{\mathbb{L}}\,\|\boldsymbol{f}(\phi) - \boldsymbol{f}(\phi^n)\big\|_{-1,q}\\
&\leqslant c_{\mathbb{L}}\,\big\|\boldsymbol{b}^*_\tau\|_{-1,q}\,\|\phi^6 - (\phi^n)^6\|_\infty,
\end{aligned}$$

where to get the last line we used the product property elements in L^∞ and elements in $W^{-1,q}_{\mathbb{D}}$, which is discussed in the Gelfand triple part of the Appendix. Recalling the identity

$$\phi^6 - (\phi^n)^6 = (\phi - \phi^n)\sum_{j=0}^{5}(\phi)^j(\phi^n)^{5-j},$$

and that ϕ, and $\phi^n \in [\phi_-, \phi_+]$, one finds

$$\|S(\phi) - S(\phi^n)\|_{1,q} \leqslant k_3\,\|\phi - \phi^n\|_\infty,$$

with $k_3 = 6c_{\mathbb{L}}\,\|\boldsymbol{b}^*_\tau\|_{-1,q}\,\phi_+^5$. Finally, inequality (5.5) and the inequalities above imply

$$\|S(\phi) - \boldsymbol{w}\|_{1,q} \leqslant k_3\,\|\phi-\phi^n\|_\infty + \|\boldsymbol{w}^{n+1} - \boldsymbol{w}\|_{1,q}.$$

The right hand side in equation above approaches zero as n approaches infinity. Therefore we conclude that $S(\phi) = \boldsymbol{w}$. This result establishes the Theorem. $\square$

5.2. **Regularity of solutions.** A bootstrap type argument shows that the regularity of weak solutions is actually related to the minimum regularity of the equation coefficients and of the boundary data.

Proposition 2. (Non-CMC Regularity) *Assume the hypotheses in Theorem 10, assume that the boundary set $\partial\mathcal{M}$ is C^2, and recall the parameters $q = 6p/(3+p)$ and $p > 3$. If the extension ϕ_D of the Dirichlet boundary data and the Robin data $\hat{\phi}_N$ for the Hamiltonian constraint equation (2.31) satisfy*

$$\phi_D \in W^{2,(q/2)}, \qquad \hat{\phi}_N \in W^{\frac{1}{(q/2)'},(q/2)}(\partial\mathcal{M}_N, 0),$$

then the solution $\phi \in [\phi_-, \phi_+] \cap W^{1,p}$ and $\boldsymbol{w} \in \boldsymbol{W}^{1,q}$ of the weak Dirichlet-Robin boundary value formulation for the Hamiltonian and momentum constraint Eqs. (2.35)-(2.36) found in Theorem 10 also satisfies $\phi \in [\phi_-, \phi_+] \cap W^{2,(q/2)}$ and $\boldsymbol{w} \in \boldsymbol{W}^{1,q}$.

Outline of the Proof. *(Proposition 2.)* The proof can again be based on linear elliptic estimates (see Theorem 9.11 in [25], and [14], Vol. II, page 296) and a standard bootstrap argument. □

6. Summary

In this article, we considered the conformal decomposition of Einstein's constraint equations introduced by Lichnerowicz and York, on a compact manifold with boundary. We began by developing some basic technical results for the momentum constraint operator, and then established existence and uniqueness of $W^{1,2}$-solutions to the momentum constraint (with conformal factor as fixed data) using variational methods. Among the technical results we established were generalized Korn inequalities for the conformal Killing operator on a compact manifold with boundary, Lemma 4, which does not appear to be in the literature. An alternative invertibility argument for the divergence of the conformal Killing operator is given using Riesz-Schauder theory, which yielded similar results in the case where the Dirichlet part of the boundary is non-empty. In both cases, the assumptions on the data were quite weak so that standard techniques cannot be used to establish additional regularity.

We then considered the Hamiltonian constraint (with momentum vector as fixed data); using order cones in Banach spaces, we derived weak sub- and super-solutions to the Hamiltonian constraint. These can be viewed as non-trivial generalizations of the barriers constructed previously in the literature to a setting with much weaker assumptions on the data. We also establish some related *a priori* L^∞-bounds on any $W^{1,2}$-solution to the Hamiltonian constraint (Theorem 6). Although such results are standard for semi-linear scalar problems with monotone nonlinearities (for example, see [32]), our results hold for a class of non-monotone nonlinearities that includes the Hamiltonian constraint nonlinearity and appear to be new. The generalized sub- and super-solutions are subsequently used together with variational methods to establish existence (and uniqueness when scalar curvature $R \geqslant 0$) of solutions to the Hamiltonian constraint in $L^\infty \cap W^{1,2}$. Our arguments allowed the scalar curvature R to have any sign; the case of non-negative R required the additional assumption that either the matter energy density or the trace-free, divergence-free part of the extrinsic curvature be positive. Again, we made very weak assumptions on the data so that standard techniques cannot be used to establish additional regularity of the solutions. Due to the lack of Gâteaux-differentiability of the nonlinearity in the space $W^{1,2}$, the connection between the energy used in the variational argument and the Hamiltonian constraint as its Euler condition for stationarity was non-trivial, and was established through several Lemmas. Although our problem formulation is slightly different (bounded domains with matter), the final result for weak solutions of the Hamiltonian constraint could be viewed as lowering the regularity of the recent result of Maxwell [38] on "rough" CMC solutions in $W^{k,2}$ for $k > 3/2$ down to $L^\infty \cap W^{1,2}$. We also gave an alternative non-variational argument using the more standard barrier methods, which requires more assumptions on the data, and yields essentially the Maxwell result for our problem formulation.

We then combined the weak solution results for the individual Hamiltonian and momentum constraints to establish an existence result for the coupled system in the case of nonconstant mean curvature, through fixed-point iteration and compactness arguments rather than through the Contraction Mapping Theorem as used in the original 1996 work of Isenberg and Moncrief. This result requires more regularity than that needed for the results established for the individual constraints, with solutions for the conformal factor in $W^{1,p}$ for $p > 3$, and momentum vector in $W^{1,q}$ for $q = 6p/(3+p)$, but still extends the existing theory for the system in two ways. First, although our problem formulation is somewhat different (bounded domains with matter), the results could be viewed as extending the 1996 result of Isenberg and Moncrief on nonconstant mean curvature with Ricci scalar $R = -1$, to weaker solution spaces, and to cases where the Ricci scalar R can have sign. Second, again although the problem formulation is different, the result could be viewed as extending the recent rough solution work of Maxwell from the CMC case to the non-CMC case.

It is interesting to note that for the main result on the non-CMC coupled system in [30], the near-CMC condition on the trace of the extrinsic curvature is actually used twice: once to obtain the global super-solution, and a second distinct time to construct a contraction for using the Contraction Mapping Theorem to get existence and uniqueness. By using a compactness argument directly rather than the Contraction Mapping Theorem, we avoid the second use of the near-CMC condition. If a global super-solution can be constructed without the near-CMC assumption, then our compactness argument would give existence of solutions to the coupled system in the fully general "far-from-CMC" case. What our proof technique gives up is uniqueness of solutions to the coupled system, which comes for free with existence when the contraction argument used as in [30].

The variational approach used for the Hamiltonian constraint in the article should allow for the treatment of the case where the coefficient of the leading nonlinear term ϕ^5 becomes slightly negative, by using the Mountain Pass approach as in the recent work of Hebey, Pacard, and Pollack.[3] The variational approach presented here also for the momentum constraint might make possible the combined variational treatment of systems involving the Hamiltonian and/or momentum constraints as part of a large variational system. Finally, if the existing non-constant sub- and super-solutions in the literature for the Hamiltonian constraint can be

[3] E. Hebey, F. Pacard, and D. Pollack. *A variational analysis of Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds.* Available as gr-qc/0702031v1, 2007.

extended to our less regular setting, it would allow for weakening some of the assumptions on the signs of the coefficients used for our results above.[4]

Although our presentation was for 3-manifolds, the results in the paper remain valid for higher spatial dimensions with minor adjustments, and the techniques we employed should extend to other cases such as closed and (fully or partially) open manifolds through use of techniques such as weighted Sobolev spaces.

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Appendix A. Some tools from nonlinear functional analysis

A.1. **Gelfand triples.** A readable reference for Gelfand triples is §17.1 in [51]. See also §23.4 in [58], where they are called evolution triples. The vector spaces (X, H, X^*) form a **Gelfand triple** iff the space H is a Hilbert space, X is a reflexive Banach space with dual space X^*, and there exists a continuous imbedding $I : X \to H$ such that $I(X)$ is dense in H. It can be shown that: If $I : X \to H$ is continuous and $I(X)$ dense in H, then the dual map $I^* : H^* \to X^*$ is continuous; in addition, since X is reflexive, $I^*(H^*)$ is dense in X^*. For the proof see [58], page 417. Denote by $R : H^* \to H$ the **Riesz map** defined as follows: given an element $h^* \in H^*$ the element $Rh^* \in H$ is given by $h^*(\underline{h}) = (Rh^*, \underline{h})_H$ for all $\underline{h} \in H$. It can be shown that this map is a bijection, therefore it is invertible, and its inverse satisfies that for all $h \in H$ the element $R^{-1}h \in H^*$ and the following equation holds, $R^{-1}h(\underline{h}) = (h, \underline{h})_H$ for all $\underline{h} \in H$. A Gelfand triple is usually denoted as

$$X \xrightarrow{I} H \equiv H^* \xrightarrow{I^*} X^*.$$

Gelfand triples are useful to study weak formulations of elliptic PDE. An example of a Gelfand triple is given by the Sobolev spaces $(W^{1,p}, L^2, W^{-1,p'})$, with $p' = p/(p-1)$.

The property of a Gelfand triple we are most interested in is that $I^*(H^*)$ is dense in X^*. This property together with the existence of the Riesz map imply that for

[4] J. Isenberg. *Private communication*, 2007.

all $x^* \in X^*$ there exists a sequence $\{h_n\} \subset H$ such that the elements $x_n^* \in X^*$, defined as $x_n^*(\underline{x}) := (h_n, I\underline{x})_H$ for all $\underline{x} \in X^*$, satisfy

$$x^* = \lim_{n\to\infty} x_n^*, \qquad \text{in } X^*. \tag{A.1}$$

The elements x_n^* can be written in terms of the Riesz map R and the imbedding I as follows, $x_n^* = I^* R^{-1} h_n$. The definition of the map I^* implies that for all $h^* \in H^*$ holds

$$I^* h^* \in X^*, \qquad I^* h^*(\underline{x}) = h^*(I\underline{x}) \qquad \forall\, \underline{x} \in X.$$

Therefore, Eq. (A.1) can be expressed as follows: for all $x^* \in X^*$ there exists a sequence $\{h_n\} \subset H$ such that

$$x^*(\underline{x}) = \lim_{n\to\infty} (h_n, I\underline{x})_H \qquad \forall\, \underline{x} \in X.$$

Definition 1. (Product) *Let $X \xrightarrow{I} H \equiv H^* \xrightarrow{I^*} X^*$ be a Gelfand triple, and let V be a vector space such that there exists an imbedding $\mathcal{I} : V \to H$. Furthermore, assume that for all $v \in V$ and $h \in H$ there exists a map $v, h \mapsto (\mathcal{I}v)h \in H$, where the element $(\mathcal{I}v)h$ in H satisfies that for every $v \in V$ there exists a positive constant c_v such that*

$$\big((\mathcal{I}v)h, \underline{h}\big)_H \leqslant c_v |(h, \underline{h})_H| \qquad \forall\, h, \underline{h} \in H.$$

Then, given any $x^* \in X^*$ define the map $v, x^* \mapsto (vx)^* \in X^*$ as follows:

$$(vx)^*(\underline{x}) := \lim_{n\to\infty} x_n^*\big((\mathcal{I}v)(I\underline{x})\big) \qquad \forall \underline{x} \in X.$$

An example of the situation above is the following: Let the Gelfand triple be given by the Sobolev spaces of scalar valued functions $(W^{1,p}, L^2, W^{-1,p'})$, let the subspace $V = L^\infty$, and let the map $v, h \mapsto vh$ be defined as pointwise multiplication a.e. in the domains of v and h. Then, all the properties in Def. 1 are satisfied. Indeed, the name "product" for the definition above originates in this example. This product given in Def. 1 satisfies the following property:

$$\|(vx)^*\|_{X^*} \leqslant c_v\, \|x^*\|_{X^*} \qquad \forall\, x^* \in X^* \text{ and } \forall\, v \in V.$$

The proof is the following calculation:

$$\|(vx)^*\|_{X^*} = \lim_{n\to\infty} \|(vx_n)^*\|_{X^*} = \lim_{n\to\infty} \sup_{0\neq\underline{x}\in X} \frac{|\big((\mathcal{I}v)h_n, (I\underline{x})\big)_H|}{\|\underline{x}\|_X};$$

noticing that $|\big(\mathcal{I}v)h_n, (I\underline{x})\big)_H| \leqslant c_v |(h_n, (I\underline{x}))_H|$ holds for all $\underline{x} \in X$, then it also holds for the supremum in $\underline{x}$, which leads us to the following inequalities,

$$\|(vx)^*\|_{X^*} \leqslant c_v \lim_{n\to\infty} \sup_{0\neq\underline{x}\in X} \frac{|\big(h_n, (I\underline{x})\big)_H|}{\|\underline{x}\|_X} = c_v \lim_{n\to\infty} \|x_n^*\|_{X^*} = c_v\, \|x^*\|_{X^*}.$$

We now show that the map $v, x^* \mapsto (vx)^*$ is well-defined in the sense that it is independent of the sequence x_n^* that approximates x^*. The proof is the following: Let $\{x_{h_n}^*\}$ and $\{\tilde{x}_{\tilde{h}_n}^*\}$ be sequences in X^* such that as $n \to \infty$ holds

$$x_{h_n}^* \to x^*, \quad \tilde{x}_{\tilde{h}_n}^* \to x^* \quad \text{in} \quad X^*.$$

Introduce the sequences $\{(vx)_{\tilde{h}_n}^*\}$ and $\{(v\tilde{x}_{\tilde{h}_n})^*\}$ in X^* such that

$$(vx_{h_n})^* \to (vx)^*, \quad (v\tilde{x}_{\tilde{h}_n})^* \to (\widetilde{vx})^* \quad \text{in} \quad X^*.$$

Then, the following argument shows that

$$\begin{aligned}\|(vx)^* - (\widetilde{vx})^*\|_{X^*} &\leqslant \|(vx)^* - (vx_{h_n})^*\|_{X^*} + \|(vx_{h_n})^* - (v\tilde{x}_{\tilde{h}_n})^*\|_{X^*} \\ &\quad + \|(v\tilde{x}_{\tilde{h}_n})^* - (\widetilde{vx})^*\|_{X^*} \\ &\leqslant \epsilon + c_v \|x^*_{h_n} - \tilde{x}^*_{\tilde{h}_n}\|_{X^*} + \epsilon \\ &\leqslant 2\epsilon + c_v\big(\|x^*_{h_n} - x^*\|_{X^*} + \|x^* - \tilde{x}^*_{\tilde{h}_n}\|_{X^*}\big) \\ &\leqslant 2(1+c_v)\epsilon, \end{aligned} \tag{A.2}$$

where the second inequality comes from the following one,

$$\|(vx_{h_n})^* - (v\tilde{x}_{\tilde{h}_n})^*\|_{X^*} = \sup_{0\neq \underline{x}\in X} \frac{\big|\big(v[h_n - \tilde{h}_n], I\underline{x}\big)_H\big|}{\|\underline{x}\|_X} \leqslant c_v \|x^*_{h_n} - \tilde{x}^*_{\tilde{h}_n}\|_{X^*}.$$

Since $\epsilon \to 0$ as $n \to \infty$, we then conclude from Eq. (A.2) that $(vx)^* = (\widetilde{vx})^*$.

A.2. **Gårding inequality and Riesz-Schauder theory.** We recall now (without giving a proof) the well-known result of Riesz and Schauder. Standard references for this result are in [25] page 76, in [51] page 166, and in [57] page 372.

Theorem 11. (Riesz-Schauder) *Let X be a Banach space, $K : X \to X$ be a linear and compact map, and $\mathcal{I}_X : X \to X$ be the identity map. Then, the following statements hold:*

(i) $\dim N_{(\mathcal{I}_X - K)} = \dim N_{(\mathcal{I}^*_X - K^*)}$*;*

(ii) Given $f \in X$ the equation

$$(\mathcal{I}_X - K)x = f \tag{A.3}$$

has a solution iff $x^*_n(f) = 0$ for all $x^*_n \in N_{(\mathcal{I}^*_X - K^*)}$;

(iii) If $\dim N_{(\mathcal{I}^*_X - K^*)} = 0$, then the condition $x^*_n(f) = 0$ is trivially satisfied for all elements $f \in X$, hence for every $f \in X$ there exist a unique element $x \in X$ solution of Eq. (A.3). Furthermore, the operator $(\mathcal{I}_X - K)^{-1}$, whose existence is asserted here, is linear and bounded.

(iv) If $\dim N_{(\mathcal{I}^*_X - K^*)} > 0$ and the condition $x^*_n(f) = 0$ is satisfied, then the solutions x of Eq. (A.3) are not unique, and given any solution x then $\hat{x} = x + x_n$ is also a solution, with $x_n \in N_{(\mathcal{I}_X - K)}$;

We now use the Riesz-Schauder Theorem above to show whether a linear equation involving a bounded bilinear form satisfying Gårding's inequality has solutions. Let (X, H, X^*) be a Gelfand triple, as it is defined in the previous subsection of this Appendix. Introduce a bilinear form $a : X \times X \to \mathbb{R}$ and consider the following problem: Given an element $f^* \in X^*$ find an element $x \in X$ solution of the equation

$$a(x, \underline{x}) = f^*(\underline{x}) \qquad \forall\, \underline{x} \in X. \tag{A.4}$$

It is convenient to reformulate this problem in terms of operators instead of bilinear forms. Introduce the operator $A : X \to X^*$, with action $Ax(\underline{x}) := a(x, \underline{x})$ for all $x, \underline{x} \in X$. Then, the problem above has the following form: Given an element $f^* \in X^*$ find an element $x \in X$ solution of the equation

$$Ax = f^*. \tag{A.5}$$

It is also convenient to introduce the Banach adjoint operator $A^* : X \to X^*$ defined as $A^*x(\underline{x}) := A\underline{x}(x)$ for all $x, \underline{x} \in X$. We are identifying X with its double dual space X^{**}. Let N_A, and N_{A^*} be the null spaces of the operators A and A^*, respectively.

Theorem 12. *Let (X, H, X^*) be a Gelfand triple, and in addition assume that the imbedding $I : X \to H$ is compact. Let $A : X \to X^*$ be a linear, bounded operator satisfying Gårding's inequality, that is, there exist positive constants k_0 and K_0 such that*

$$\|Ax\|_{X^*} \leqslant K_0 \|x\|_X, \qquad k_0 \|x\|_X^2 \leqslant \|Ix\|_H^2 + Ax(x), \qquad \forall\, x \in X.$$

Then, $\dim N_A = \dim N_{A^*}$ and Eq. (A.5) has a solution $x \in X$ iff $f^*(\tilde{x}_n) = 0$ for all $\tilde{x}_n \in N_{A^*}$. Furthermore, the following statements hold:

(i) If $\dim N_{A^*} = 0$, then there exists a unique $x \in X$ solution of Eq. (A.5) for all $f^* \in X^*$; Furthermore, there exists a positive constant c_0 such that the following estimate holds,

$$\|x\|_X \leqslant c_0 \|Ax\|_{X^*} \qquad \forall\, x \in X; \tag{A.6}$$

(ii) If $\dim N_{A^*} > 0$ and the condition $f^*(\tilde{x}_n) = 0$ for all $\tilde{x}_n \in N_{A^*}$ holds, then the solution x is not unique, since $x' := x + x_n$ is also a solution, with $x_n \in N_A$.

Proof. *(Theorem 12.)* Given the operator A, introduce the operator $A_X : X \to X^*$ with action

$$A_X x(\underline{x}) := Ax(\underline{x}) + (Ix, I\underline{x})_H.$$

The assumptions that the operator A is bounded and satisfies Gårding's inequality imply that the operator A_X is bounded and coercive, respectively, hence, invertible. Notice that A_X can be written in terms of operators as follows: $A_X = A + J$, where $J : X \to X^*$ is given by $J := I^* R^{-1} I$, since

$$Jx(\underline{x}) = I^* R^{-1} I x(\underline{x}) = R^{-1} I x(I\underline{x}) = (Ix, I\underline{x})_H.$$

The Eq. (A.5) can be re-expressed as follows:

$$Ax = f^* \quad \Leftrightarrow \quad (A_X - J)x = f^* \quad \Leftrightarrow \quad (\mathcal{I}_X - A_X^{-1} J)x = A_X^{-1} f^*,$$

where $\mathcal{I}_X : X \to X$ is the identity map. Introduce the notation $f_X := A_X^{-1} f^* \in X$ and the operator $K : X \to X$ given by $K := A_X^{-1} J$. So, x is solution of Eq. (A.5) iff it solves the equation

$$(\mathcal{I}_X - K)x = f_X. \tag{A.7}$$

Since the imbedding $I : X \to H$ is compact, and the remaining maps that define K are continuous, we conclude that K is compact (for example see [20] page 486, Theorem 4). Then, the operator $\mathcal{I}_X - K$ is a Fredholm operator of index zero, and Theorem 11 implies that $\dim N_{\mathcal{I}_X - K} = \dim N_{\mathcal{I}_X^* - K^*}$. By construction we have that $N_{\mathcal{I}_X - K} = N_A$. One can also show that $x_n^* \in N_{\mathcal{I}_X^* - K^*}$ iff $\tilde{x}_n := (A_X^{-1})^* x_n^* \in N_{A^*}$. Due to A_X is a bijection, this shows that

$$\dim N_A = \dim N_{A^*}.$$

Theorem 11 implies that Eq. (A.7) has solution iff $x_n^*(f_X) = 0$ for all $x_n^* \in N_{\mathcal{I}_X^* - K^*}$. This condition can be rewritten as follows:

$$0 = x_n^*(f_X) = x_n^*(A_X^{-1} f^*) = f^*\big((A_X^{-1})^* x_n^*\big) = f^*(\tilde{x}_n) \qquad \forall\, \tilde{x}_n \in N_{A^*},$$

which is the condition appearing in Theorem 12. In the case that $\dim N_A = 0$, then $\dim N_{A^*} = 0$, and so the condition $x_n^*(f_X) = 0$ is trivially satisfied for all $f_X \in X$. Therefore, Theorem 11 implies that for every element $f^* \in X^*$ there always exists a unique solution $x \in X$ of Eq. (A.5). This statement defines the operator $A^{-1} : X^* \to X$, and Theorem 11 asserts that this operator is linear and bounded, the latter property implies that there exists a positive constant c_0 such that

$$\|x\|_X \leqslant c_0 \, \|Ax\|_{X^*} \qquad \forall\, x \in X.$$

This establishes part *(i)* in Theorem 12. In the case that $\dim N_A > 0$ and the condition $x_n^*(f_X) = 0$ is satisfied, then Theorem 11 says that a solution $x \in X$ exists, and $x' = x + x_n$ is also a solution, where $x_n \in N_A$. This establish part *(ii)* in Theorem 12. □

A.3. **Variational methods.** These notes follow the main ideas in Chapter 4 of Part Two in [33], and §1, §2 in Chapter 1 in [48]. An introduction into this subject is §7.1 in [39]. The main result of this Section is Theorem 13. We could not find in the literature this result precisely in this form, needed for the Hamiltonian constraint problem, so for completeness we included the proof of the Theorem.

Given a Banach space X, a subset $U \subset X$ is called **closed under weak convergence** (closed$_w$) iff for all sequence $\{x_n\} \subset U$ such that $x_n \rightharpoonup x_0$ in X holds that $x_0 \in U$. Every closed$_w$ set in a Banach space is closed, but the converse statement is not true. A particular class of closed sets that are also closed$_w$ are closed and convex sets. Given a vector space V, a subset $U \subset V$ is called **convex** iff for all $\tilde{x}$, $\hat{x} \in U$ the elements $[t\tilde{x} + (1-t)\hat{x}] \in U$ for $t \in [0,1]$. The proof of the above statement is based in a result by Mazur (see Theorem 2.2.4 on page 142 in [33]) that says: In a Banach space, for every sequence $\{x_n\}$ such that $x_n \rightharpoonup x_0$ there exists a sequence $\{y_m\}$ such that $y_m \to x_0$, where the element y_m are constructed as a convex combinations of the x_n, that is,

$$y_m := \sum_{n=1}^{m} \lambda_n x_n, \qquad \Big(\lambda_n \geqslant 0, \quad \sum_{n=1}^{m} \lambda_n = 1\Big). \tag{A.8}$$

Using this result is not difficult to show that every closed and convex set U in a Banach space X is also closed$_w$, as the following argument shows: given $\{x_n\} \subset U$ such that $x_n \rightharpoonup x_0$ in X, use Mazur's idea to construct the sequence $\{y_m\}$ as a convex combination of the x_n such that $y_m \to x_0$ in X. However, U is convex, so $\{y_m\} \subset U$, and it is also closed, so $x_0 \in U$. This establishes that U is also closed$_w$.

Let X be a Banach space, and introduce the functional $J : X \to \overline{\mathbb{R}}$. The symbol $\overline{\mathbb{R}}$ means that there might exist points $x_0 \in U$ such that there exists a sequence $x_n \to x_0$ with $\lim_{n\to\infty} J(x_n) = \infty$ or equal $-\infty$. If such points exist, then J is an unbounded operator with domain D_J strictly included in X. This idea is summarized with the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. The notation $J : X \to \overline{\mathbb{R}}$ is more convenient than the notation $J : D_J \subset X \to \mathbb{R}$ in cases where the continuity or the differentiability of the functional is not important in the situation under study. An example is the problem of finding the local or global minimum of a functional using direct methods, which do not include computing the Euler equations for the functional. Given any subset $U \subset X$ of a Banach space X, the functional $J : U \subset X \to \overline{\mathbb{R}}$ is called **proper** on U iff for all $\{x_n\} \subset U$ such that $\|x_n\|_X \to \infty$ holds $J(x_n) \to +\infty$. A particular case of proper functionals are coercive functionals, where J is called **coercive** on U iff there exist positive constants c_0, c_1, such that for all $x \in U$ holds $J(x) \geqslant c_0 \, \|x\|_X^2 - c_1$. The functional J is **bounded below** on U iff there exists $\alpha_0 \in \mathbb{R}$ such that $J(x) \geqslant \alpha_0$ for all $x \in U$. All coercive functionals are bounded below by $-c_1$.

The functional $J : X \to \overline{\mathbb{R}}$ is **lower semi-continuous** (lsc) at the element $x_0 \in X$ iff for all sequence $\{x_n\} \subset X$ with $x_n \to x_0$ in X holds $J(x_0) \leqslant \liminf_{n\to\infty} J(x_n)$. Any continuous functional is lsc. The functional $J : X \to \overline{\mathbb{R}}$ is called **lower semi-continuous under weak convergence** (lsc_w) at the element $x_0 \in X$ iff for all sequence $\{x_n\} \subset X$ with $x_n \rightharpoonup x_0$ in X holds

$$J(x_0) \leqslant \liminf_{n\to\infty} J(x_n).$$

Given any set $U \subset X$ a functional $J : U \subset X \to \overline{\mathbb{R}}$ is lsc (respectively lsc_w) on U if it is lsc (respectively lsc_w) on all points in U. Not every lsc functional is lsc_w. The later property is a stronger condition on the functional than the former property, due to the set of all sequences that converge weakly in a Banach space is bigger than the set of all sequences that converge strongly. An example of a lsc_w functional, mentioned in [48], is the norm in an arbitrary Banach space, as the following argument shows: Let X be a Banach space, $\{x_n\} \subset X$ be any sequence such that $x_n \rightharpoonup x_0$, then there always exists an element $x^*_{x_0} \in X^*$ such that $x^*_{x_0}(x_0) = \|x^*_{x_0}\|_{X^*}\, \|x_0\|_X$; then we obtain

$$\|x^*_{x_0}\|_{X^*}\, \|x_0\|_X = x^*_{x_0}(x_0) = \liminf_{n\to\infty} x^*_{x_0}(x_n) \leqslant \|x^*_{x_0}\|_{X^*}\ \liminf_{n\to\infty} \|x_n\|_X,$$

and this implies that $\|x_0\|_X \leqslant \liminf_{n\to\infty} \|x_n\|_X$, establishing our assertion. The well-known case of the norm in a Hilbert space H being lsc_w follows from the previous argument choosing $x^*_{x_0}(\underline{x}) = (x_0, \underline{x})_H$, where $\underline{x}$ is any element in H.

Theorem 13. (Existence of a minimizer) *Let X be a reflexive Banach space, and $U \subset X$ be a closed_w subset. Let $J : U \subset X \to \mathbb{R}$ be a proper, bounded below, and lsc_w functional. Then, there exists an element $x_0 \in U$ minimizer of the functional J in the set U, that is,*

$$J(x_0) = \inf_{x\in U} J(x).$$

Proof. *(Theorem 13.)* The functional J is bounded below in U, therefore there exists a positive constant α_0 such that for all $x \in U$ holds $J(x) \geqslant \alpha_0$. Then, there exists a minimizing sequence, that is, a sequence $\{x_n\} \subset U$ such that $J(x_n) \to \alpha_0$ as $n \to \infty$. The functional J is proper on U, therefore the sequence $\{x_n\}$ is bounded. The Banach space X is reflexive, which implies that there exists $x_0 \in X$ such that $x_n \rightharpoonup x_0$ as $n \to \infty$. The set U is closed_w, therefore $x_0 \in U$. Finally, the functional J is lsc_w and the sequence $\{x_n\}$ is a minimizing sequence, which imply that

$$J(x_0) \leqslant \liminf_{n\to\infty} J(x_n) = \alpha_0 \leqslant \inf_{x\in U} J(x).$$

The definition of infimum implies $J(x_0) = \inf_{x\in U} J(x)$, which establishes the Theorem. □

Let $U \subset X$ be a convex set in a Banach space X. A functional $J : U \subset X \to \overline{\mathbb{R}}$ is called **convex** iff for all x, $\underline{x} \in U$ and $t \in [0,1]$ holds

$$J(tx + (1-t)\underline{x}) \leqslant tJ(x) + (1-t)J(\underline{x}).$$

A convex functional J is called **strictly convex** iff for all x, $\underline{x} \in U$, with $x \neq \underline{x}$, and $t \in (0,1)$ holds

$$J(tx + (1-t)\underline{x}) < tJ(x) + (1-t)J(\underline{x}).$$

Besides these main Theorems above, the following Lemma is also needed in the proof of Theorem 7.

Lemma 12. *Let X be a Banach space, $U \subset X$ be a closed, convex set, and $J : U \subset X \to \overline{\mathbb{R}}$ be a convex and lsc functional. Then, the functional J is lsc$_w$.*

Proof. *(Lemma 12.)* Let $\{x_n\} \subset U$ be any sequence such that $x_n \rightharpoonup x_0$ in X, and denote $\alpha_0 := \liminf_{n\to\infty} J(x_n)$. Earlier in this Section it was shown, using a Mazur's sequence, that a closed and convex set in a Banach space is also closed$_w$, therefore $x_0 \in U$. Using once again Mazur's result, let $\{y_m\} \subset U$ be a convex combination of the elements x_n such that $y_m \to x_0$ in U. Furthermore, let the convex combination of the elements x_n start at $n = N$ for some number $N \in \mathbb{N}$ instead of $n = 1$, that is,

$$y_m = \sum_{n=N}^{m} \lambda_n x_n, \qquad \Big(\lambda_n \geqslant 0, \quad \sum_{n=N}^{m} \lambda_n = 1\Big).$$

The functional J is convex, therefore,

$$J(y_m) \leqslant \sum_{n=N}^{m} \lambda_n J(x_n), \tag{A.9}$$

By definition of the constant α_0 and after selecting a subsequence if necessary, given any positive number ϵ there exists a number $N(\epsilon) \in \mathbb{N}$ such that

$$J(x_n) < \alpha_0 + \epsilon, \qquad \forall\, n \geqslant N(\epsilon).$$

Then, Eq. (A.9) implies

$$J(y_m) < \Big(\sum_{n=N(\epsilon)}^{m} \lambda_n\Big)\,(\alpha_0 + \epsilon) = \alpha_0 + \epsilon.$$

so by choosing ϵ small enough we conclude that $\liminf_{m\to\infty} J(y_m) \leqslant \alpha_0$, or alternatively,

$$\liminf_{m\to\infty} J(y_m) \leqslant \liminf_{n\to\infty} J(x_n).$$

Finally, recalling that the sequence $y_m \to x_0$ and that the functional J is lsc, we have that $J(x_0) \leqslant \liminf_{m\to\infty} J(y_m)$, which together with equation above says,

$$J(x_0) \leqslant \liminf_{n\to\infty} J(x_n).$$

This equation establishes the Lemma. □

For completeness we now state the following result, which establishes the existence and uniqueness of minimizers for certain type of convex functionals.

Theorem 14. (Minimizers of convex functionals) *Let X be a Banach space and $U \subset X$ be a closed, convex set. Let $J : U \subset X \to \overline{\mathbb{R}}$ be a convex, lsc, and coercive functional. Then, there exists an element $x \in U$ minimizer of the functional J in the set U. Furthermore, if the functional J is strictly convex, then the minimizer x is unique.*

Proof. *(Theorem 14.)* We know that a closed, convex set U in a reflexive Banach space X is a closed$_w$ set. And a convex and lsc functional J on a closed$_w$ set U is lsc$_w$, result proved in Lemma 12. Since the functional J is also coercive, then J is proper and bounded below. Therefore, Theorem 13 implies that the functional J has a minimizer $x \in U$.

Assume now that the functional J is strictly convex, and assume that there exist two minimizers x_1, $x_2 \in U$ of the functional J, that is,

$$J_0 := J(x_1) = J(x_2) = \inf_{x \in X} J(x).$$

We will now construct a contradiction. Assume that the minimizers are different, $x_1 \neq x_2$, and introduce the elements $x_t := tx_1 + (1-t)x_2$, for $t \in (0,1)$. The strict convexity of the functional J implies

$$J(x_t) = J\big(tx_1 + (1-t)x_2\big) < tJ(x_1) + (1-t)J(x_2) = tJ_0 + (1-t)J_0 = J_0.$$

We then conclude that $J(x_t) < J(x_1) = J(x_2)$, contradicting the assumption that the elements x_1, x_2 are minimizers of J. Therefore the minimizer must be unique. This establishes the Theorem. □

We finish this Section with a calculation that is useful to verify whether a Gâteaux differentiable functional is convex or strictly convex. Let $B_\epsilon(x_0) \subset X$ be an open ball of radius ϵ centered at the element $x_0 \in X$. If a convex functional $J : U \subset X \to X$ is Gâteaux differentiable in the convex set U with Gâteaux derivative DJ, then, there exists a positive and small enough number ϵ such that the following inequality holds

$$DJ(x)v \leqslant J(x+v) - J(x) \qquad \forall\, x \in \mathrm{int}(U), \quad \forall\, v \in B_\epsilon(0). \tag{A.10}$$

The proof is the following calculation: Fix a positive number ϵ, then given both $x \in \mathrm{int}(U)$ and $v \in B_\epsilon(0)$ there exists a small enough number ϵ such that $x+v \in U$. The convexity of the set U and of the functional J imply that

$$J\big(t(x+v) + (1-t)x\big) \leqslant tJ(x+v) + (1-t)J(x),$$

which in turn implies

$$\frac{1}{t}\big[J(x+tv) - J(x)\big] \leqslant J(x+v) - J(x).$$

Then, Eq. (A.10) follows by taking the limit $t \to 0^+$ in the inequality above. A similar proof establishes the following result: If the functional J is a strictly convex and Gâteaux differentiable in a convex set U, then there exists a positive and small enough number ϵ such that the following inequality holds

$$DJ(x)v < J(x+v) - J(x) \qquad \forall\, x \in \mathrm{int}(U), \quad \forall\, v \in B_\epsilon(0), \quad v \neq 0. \tag{A.11}$$

A.4. **Ordered Banach spaces.** These notes follow the main ideas and definitions given Chapter 7.1, page 275, in [57], while some examples were taken from [3] and [19]. Let X be a Banach space, $\mathbb{R}_+$ be the non-negative real numbers. A subset $C \subset X$ is a **cone** iff given any $x \in C$ and $a \in \mathbb{R}_+$ the element $ax \in C$. A subset $X_+ \subset X$ is an **order cone** iff the following properties hold:

(i) The set X_+ is non-empty, closed, and $X_+ \neq \{0\}$;
(ii) Given any a, $b \in \mathbb{R}_+$ and x, $\underline{x} \in X_+$ then $ax + b\underline{x} \in X_+$;
(iii) If $x \in X_+$ and $-x \in X_+$, then $x = 0$.

The second property above says that every order cone is in fact a cone, and that the set X_+ is convex. The space $X = \mathbb{R}^2$ is a convenient Banach space to picture non-trivial examples of cones and order cones, as can be seen in Fig. 1. A pair X, X_+ is called an **ordered Banach space** iff X is a Banach space and $X_+ \subset X$ is an order cone. The reason for this name is that the order cone X_+ defines several relations on elements in X, called order relations, as follows:

$$u \geqslant v \text{ iff } u - v \in X_+, \qquad u > v \text{ iff } u \geqslant v \text{ and } u \neq v,$$
$$u \gg v \text{ iff } u - v \in \operatorname{int}(X_+), \qquad u \not\geqslant v \text{ iff } u \geqslant v \text{ is false;}$$

finally it is also used the notation $u \leqslant v$, $u < v$, and $u \ll v$ to mean $v \geqslant u$, $v > u$, $v \gg u$, respectively. A simple example of an ordered Banach space is $\mathbb{R}$ with the usual order. Another example can be constructed when this order on $\mathbb{R}$ is transported into $C^0(\overline{\mathcal{M}}, 0)$, the set of scalar-valued functions on a set $\mathcal{M} \subset \mathbb{R}^n$, with $n \geqslant 1$. An order on $C^0(\overline{\mathcal{M}}, 0)$ is the following: the functions $u, v \in C^0(\overline{\mathcal{M}}, 0)$ satisfy $u \geqslant v$ iff $u(x) \geqslant v(x)$ for all $x \in \mathcal{M}$. The following Lemmas summarize the main properties of order relations in Banach spaces.

Lemma 13. *Let X, X_+ be an ordered Banach space. Then, for all elements u, v, $w \in X$, hold: (i) $u \geqslant u$; (ii) If $u \geqslant v$ and $v \geqslant u$, then $u = v$; (iii) If $u \geqslant v$ and $v \geqslant w$, then $u \geqslant w$.*

Proof. *(Lemma 13.)* The property that $u - u = 0 \in X_+$ implies that $u \geqslant u$. If $u \geqslant v$ and $v \geqslant u$ then $u - v \in X_+$ and $-(u - v) \in X_+$, therefore $u - v = 0$. Finally, if $u \geqslant v$ and $v \geqslant w$, then $u - v \in X_+$ and $v - w \in X_+$, which means that $u - w = (u - v) + (v - w) \in X_+$. □

Furthermore, the order relation is compatible with the vector space structure and with the limits of sequences.

Lemma 14. *Let X, X_+ be an ordered Banach space. Then, for all u, $\hat{u}$, v, $\hat{v}$, $w \in X$, and a, $b \in \mathbb{R}$, hold*

- *(i) If $u \geqslant v$ and $a \geqslant b \geqslant 0$, then $au \geqslant bv$;*
- *(ii) If $u \geqslant v$ and $\hat{u} \geqslant \hat{v}$, then $u + \hat{u} \geqslant v + \hat{v}$;*
- *(iii) If $u_n \geqslant v_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} u_n \geqslant \lim_{n\to\infty} v_n$.*

Proof. *(Lemma 14.)* The first two properties are straightforward to prove, and we do not do it here. The third property holds because the order cone is a closed set. Indeed, $u_n \geqslant v_n$ means that $u_n - v_n \in X_+$ for all $n \in \mathbb{N}$, and then $\lim_{n\to\infty}(u_n - v_n) \in X_+$ because X_+ is closed, then Property *(iii)* follows. □

The remaining order relations have some other interesting properties.

Lemma 15. *Let X, X_+ be an ordered Banach space. Then, for all u, v, $w \in X$, and $a \in \mathbb{R}$, hold: (i) If $u \gg v$ and $v \gg w$, then $u \gg w$; (ii) If $u \gg v$ and $v \geqslant w$, then $u \gg w$; (iii) If $u \geqslant v$ and $v \gg w$, then $u \gg w$; (iv) If $u \gg v$ and $a > 0$, then $au \gg av$.*

The Proof of Lemma 15 is similar to the previous Lemma, and is not reproduced here. Given an ordered Banach space X, X_+, and two elements $u \geqslant v$, introduce the intervals

$$[v, u] := \{w \in X : v \leqslant w \leqslant u\}, \qquad (v, u) := \{w \in X : v \ll w \ll u\}.$$

Analogously, introduce the intervals $[v, u)$ and $(v, u]$. See Fig. 1 for an example in $X = \mathbb{R}^2$. Useful order cones for solving PDE are those that define an order structure in the Banach space which is related with the norm and the notion of boundness. These type of order cones are called normal. More precisely, an order cone X_+ in a Banach space X is called **normal order cone** iff there exists $0 < a \in \mathbb{R}$ such that for all u, $v \in X$ with $0 \leqslant v \leqslant u$ holds $\|v\| \leqslant a\,\|u\|$.

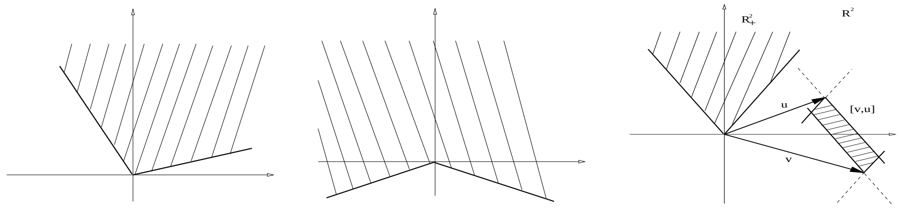

FIGURE 1. The shaded regions in the first picture represents an order cone, while the second picture represents a cone that is not an order cone. The shaded region between u and v in the third picture represents the closed interval $[v, u]$, constructed with the order cone $\mathbb{R}^2_+$, which is also represented by a shaded region.

Lemma 16. *If X, X_+ is an ordered Banach space with normal order cone X_+, then every closed interval in X is bounded.*

Proof. *(Lemma 16.)* Let $w \in [v, u]$, then $v \leqslant w \leqslant u$, and so $0 \leqslant w - v \leqslant u - v$. Since the cone X_+ is normal, this implies that there exists $a > 0$ such that $\|w - v\| \leqslant a\,\|u - v\|$. Then, the inequalities $\|w\| \leqslant \|w - v\| + \|v\| \leqslant a\,\|u - v\| + \|v\|$, which hold for all $w \in [v, u]$, establish the Lemma. $\square$

Not every order cone is normal. For example, consider the Sobolev spaces $W^{k,p}$ of scalar-valued functions on an n-dimensional, compact manifold $\mathcal{M}$, with Lipschitz continuous boundary, where k is a non-negative integer, and $p > 1$ is a real number. An order cone in $W^{k,p}$ is defined translating the order on the real numbers, almost everywhere in $\mathcal{M}$, that is,

$$W^{k,p}_+ := \{u \in W^{k,p} : u \geqslant 0 \text{ a.e. in } \mathcal{M}\}.$$

In the case $k = 0$, that is, we have $W^{0,p} = L^p$, the order cone above is a normal cone [3, 57]. However, in the case $k \geqslant 1$ the cone above cannot be normal, since on the one hand, the cone definition involves information only of the values of $u(x)$ and not of its derivatives; on the other hand, the norm in $W^{k,p}$ contains information of both the values of $u(x)$ and its derivatives. Since there is no boundary conditions on $\partial\mathcal{M}$ in the definition of $W^{k,p}$, there is no way to relate the values of a function in $\mathcal{M}$ with the values of its derivatives. (In other words, there is no Poincaré inequality for elements in $W^{k,p}$, with $k \geqslant 1$.)

An order cone $X_+ \subset X$ is **generating** iff $\mathrm{Span}(X_+) = X$. An order cone $X_+ \subset X$ is called **total** iff $\mathrm{Span}(X_+)$ is dense in X. Total order cones are important because the order structure associated with them can be translated from the space X into its dual space X^*.

Lemma 17. *Let X, X_+ be an ordered Banach space. If X_+ is a total order cone, then an order cone in X^* is given by the set $X^*_+ \subset X^*$ defined as*

$$X^*_+ := \{u^* \in X^* : u^*(v) \geqslant 0 \quad \forall\, v \in X_+\}.$$

Proof. *(Lemma 17.)* We check the three properties in the definition of the order cone. The first property is satisfied because X_+ is an order cone, so there exists $v \neq 0$ in X_+, and then there exists $u^* \neq 0$ in X^* such that $u^*(v) = 1 \geqslant 0$, so X^*_+ is non-empty. Trivially, $0 \in X^*_+$. Finally, X^*_+ is closed because the order relation

$\geqslant$ for real numbers is used in its definition. The second property of an order cone is satisfied, because given any u^*, $v^* \in X_+^*$ and any non-negative a, $b \in \mathbb{R}$, then for all $\underline{u} \in X_+$ holds

$$(au^* + bv^*)(\underline{u}) = au^*(\underline{u}) + bv^*(\underline{u}) \geqslant 0$$

since each term is non-negative. This implies that $(au^* + bv^*) \in X_+^*$. The third property is satisfied because the order cone X_+ is total. Suppose that the element $u^* \in X_+^*$ and $-u^* \in X_+^*$, then for all $\underline{u} \in X_+$ holds that $u^*(\underline{u}) \geqslant 0$ and $-u^*(\underline{u}) \geqslant 0$, which implies that $u^*(\underline{u}) = 0$ for all $\underline{u} \in X_+$. Therefore, $u^* \in X_+^{\perp\!\!\!\flat} \subset X^*$, where the super-script $\perp\!\!\!\flat$ in $X_+^{\perp\!\!\!\flat}$ means the Banach annihilator of the set X_+, which is a subset of the space X^*. Therefore, we conclude that $u^* \in \left[\mathrm{Span}(X_+)\right]^{\perp\!\!\!\flat}$. Since the order cone is total, $\overline{\mathrm{Span}(X_+)} = X$, that implies $\left[\mathrm{Span}(X_+)\right]^{\perp\!\!\!\flat} = \{0\}$, so $u^* = 0$. This establishes the Lemma. $\square$

An order cone X_+ in a Banach space X is called a **solid cone** iff X_+ has non-empty interior. The following result asserts that solid order are generating. We remark that the converse is not true. In the examples below we present function spaces frequently used in solving PDE with order cones having empty interior which are indeed generating.

Lemma 18. *Let X, X_+ be an order Banach space. If X_+ is a solid cone, then X_+ is generating.*

Proof. *(Lemma 18.)* The cone X_+ has a non-empty interior, so there exists $x_0 \in \mathrm{int}(X_+)$ and $x_0 \neq 0$. This means that given any $x \in X$ there exists $0 < a \in \mathbb{R}$ small enough such that both $x_+ := x_0 + ax$ and $x_- := x_0 - ax$ belong to $\mathrm{int}(X_+)$. But then, $x = (x_+ - x_-)/(2a)$, so $x \in \mathrm{Span}(X_+)$. This establishes the Lemma. $\square$

Here is a list of examples of several order cones used in function spaces. All these examples use order cones obtained from the usual order in $\mathbb{R}$. In particular, they refer to scalar-valued functions on an n- dimensional, compact manifold $\mathcal{M}$ with Lipschitz boundary.

- Introduce on C^k the cone $C_+^k := \{u \in C^k : u(x) \geqslant 0 \quad \forall x \in \mathcal{M}\}$. This is an order cone for all non-negative integer k. The cone is a normal cone in the particular case $k = 0$. The cone is solid for all $k \geqslant 0$, therefore it is a generating cone.
- Introduce on L^∞ the cone $L_+^\infty := \{u \in L^\infty : u \geqslant 0 \quad \text{a.e. in } \mathcal{M}\}$. This is a normal, order cone. It is a solid cone, therefore is generating.
- Introduce on $W^{k,\infty}$ the cone $W_+^{k,\infty} := \{u \in W^{k,\infty} : u \geqslant 0 \quad \text{a.e. in } \mathcal{M}\}$. This is an order cone. It is not normal for $k \geqslant 1$. The cone is solid, therefore it is generating.
- Introduce on L^p the cone $L_+^p := \{u \in L^p : u \geqslant 0 \quad \text{a.e. in } \mathcal{M}\}$. This is a normal, order cone every real numbers $p \geqslant 1$. The cone is not solid, however it is a generating cone.
- Introduce on $W^{k,p}$ the cone $W_+^{k,p} := \{u \in W^{k,p} : u \geqslant 0 \quad \text{a.e. in } \mathcal{M}\}$. This is an order cone every real numbers $p \geqslant 1$. The cone is not normal for $k \geqslant 1$. The cone is not solid for $kp \leqslant n$, and it is solid for $kp > n$. In both cases, the cone is generating.

A.5. **Maximum principles.** We have not seen in the literature an approach to maximum principles on ordered Banach spaces in the generality we present it in this Section. Let X, X_+ and Y, Y_+ be ordered Banach spaces. An operator

$A : D_A \subset X \to Y$ satisfies the **maximum principle** iff for every u, $v \in D_A$ such that $Au - Av \in Y_+$ holds that $u - v \in X_+$. In the particular case that the operator A is linear, then it satisfies the maximum principle iff for all $u \in X$ such that $Au \in Y_+$ holds that $u \in X_+$. The main example is the Laplace operator acting on scalar-valued functions defined on different domains. It is shown later on in this Appendix that the inverse of an operator that satisfies the maximum principle is monotone increasing. The following result gives a simple sufficient condition for an operator to satisfy the maximum principle. This result is useful on weak formulations of PDE.

Lemma 19. *Let X, X_+ be an ordered Banach space, and $A : X \to X^*$ be a linear and coercive map. Assume that X_+ is a generating order cone, and that for all $u \in X$ such that $Au \in X^*_+$ there exists a decomposition $u = u^+ - u^-$ with u^+, $u^- \in X_+$ that also satisfies $Au^+(u^-) = 0$. Then, the operator A satisfies the maximum principle.*

Proof. *(Lemma 19.)* Since the order cone X_+ is generating, the space X^* is also an ordered Banach space. Denote its order cone by X^*_+. The assumption that the order cone X_+ is generating also implies that for any element $u \in X$ there exists a decomposition $u = u^+ - u^-$ with u^+, $u^- \in X_+$. By hypothesis, there exists at least one decomposition with the extra property that $Au^+(u^-) = 0$. Now, by definition of the order in the space X^* we have that

$$Au \in X^*_+ \quad \Leftrightarrow \quad Au(\underline{u}) \geqslant 0 \quad \forall\, \underline{u} \in X_+.$$

Pick as test function $\underline{u} = u^-$. Then,

$$0 \leqslant Au(u^-) = A(u^+ - u^-)(u^-) = Au^+(u^-) - Au^-(u^-) = -Au^-(u^-),$$

where the last equality comes from the condition $Au^+(u^-) = 0$. Therefore, we have

$$Au^-(u^-) \leqslant 0 \quad \Rightarrow \quad u^- = 0,$$

because A is coercive. So we showed that $u = u^+ \in X_+$. This establish the Lemma. $\square$

An example is the weak form of the Laplace operator on scalar functions in the homogeneous Dirichlet problem on a compact manifold $\mathcal{M}$ with Lipschitz boundary. Consider the case $X = W^{1,2}_0$, with $Y = X^* = W^{-1,2}$, and $X_+ = W^{1,2}_+$, while $Y_+ = W^{-1,2}_+$. The Laplace operator in this case is given by $A : X \to X^*$ with action $Au(v) := (\nabla u, \nabla v)$. It is not difficult to check that this operator satisfies the hypothesis in Lemma 19. Therefore, this operator satisfies the maximum principle, that is, $Au \in W^{-1,2}_+$ implies $u \in W^{1,2}_+$, that is, $u \geqslant 0$ a.e. in the manifold $\mathcal{M}$. This result is in agreement with Theorem 8.1 in [25], where it is stated that: "If $Au \geqslant 0$, then $\inf_{\mathcal{M}} u \geqslant -\inf_{\partial\mathcal{M}} u^-$." Here we introduced the cut-off function $u^- := -\min(u, 0) \geqslant 0$. Recalling that in our case the domain of A contains only functions that vanish at the boundary, then $\inf_{\mathcal{M}} u \geqslant 0$, that is, $u \geqslant 0$ in $\mathcal{M}$.

The following example is again the Laplace operator that appears in equations when they are written in weak form, but this time using more complicated operator domains due to more complicated boundary conditions in the PDE equation. Let $(\mathcal{M}, h)$ be a 3-dimensional Riemannian manifold, where $\mathcal{M}$ is a smooth, compact manifold with a Lipschitz boundary $\partial\mathcal{M}$, and $h \in C^2(\overline{\mathcal{M}}, 2)$ is a positive definite metric. Assume that the boundary set can be decomposed as follows, $\partial\mathcal{M} =$

$\partial\mathcal{M}_D \cup \partial\mathcal{M}_N$ and $\overline{\partial\mathcal{M}_D} \cap \overline{\partial\mathcal{M}_N} = \emptyset$. Recall the definition of the Sobolev spaces

$$W^{1,2}_D := \{u \in W^{1,2}(\mathcal{M},\mathbb{R}) : \mathsf{tr}_D u = 0\}, \qquad W^{-1,2}_D := \big[W^{1,2}_D\big]^*.$$

Then, define the operator

$$A^s_L : W^{1,2}_D \to W^{-1,2}_D, \qquad A^s_L\phi(\underline{\phi}) := a_L(\phi,\underline{\phi}) + (s\phi,\underline{\phi}); \tag{A.12}$$

where a_L is the bilinear form

$$a_L : W^{1,2}_D \times W^{1,2}_D \to \mathbb{R}, \qquad a_L(\phi,\underline{\phi}) := (\nabla\phi,\nabla\underline{\phi}) + (K\,\mathsf{tr}_N\phi,\mathsf{tr}_N\underline{\phi})_N,$$

and the Robin coefficient $K \in L^\infty(\partial\mathcal{M}_N,0)$ satisfies the bounds

$$\hat{\mathtt{k}}\,\|\mathsf{tr}_N\phi\|^2_N \leqslant (K\mathsf{tr}_N\phi,\mathsf{tr}_N\phi)_N, \qquad \forall\phi \in W^{1,2}, \tag{A.13}$$

with $\hat{\mathtt{k}}$ a positive constant. Assume that the function $s \in L^{3/2}_+$, so the second term in the definition of the operator A^s_L is well defined.

Lemma 20. *The operator A^s_L defined in Eq. (A.12) satisfies the maximum principle.*

Proof. *(Lemma 20.)* We now verify all the hypothesis in Lemma 19. The cone $W^{1,2}_{D+} = W^{1,2}_+ \cap W^{1,2}_D$ is generating in $W^{1,2}_D$ therefore, $W^{-1,2}_D$ is also an ordered space. The constant $\hat{\mathtt{k}}$ is positive and the function s is non-negative, which implies that the operator A^s_L is coercive. Using the usual decomposition of a function u into $u^+(x) = \max_{\mathcal{M}}(u(x),0)$ and $u^-(x) = -\min_{\mathcal{M}}(u(x),0)$, then it is not difficult to show that $A^s_L u^+(u^-) = 0$, because the two parts of the decomposition of u are defined on non-intersecting parts of $\mathcal{M}$. Therefore, Lemma 20 follows from Lemma 19. $\square$

A.6. **Monotone operators.** Let X, X_+ and Y, Y_+ be two ordered Banach spaces. An operator $F : X \to Y$ is **monotone increasing** iff for all x, $\underline{x} \in X$ such that $x - \underline{x} \in X_+$ holds that $F(x) - F(\underline{x}) \in Y_+$. An operator $F : X \to Y$ is **monotone decreasing** iff for all x, $\underline{x} \in X$ such that $x - \underline{x} \in X_+$ holds that $-\big[F(x) - F(\underline{x})\big] \in Y_+$. The following result is a useful relation between linear, invertible operators that satisfy the maximum principle and monotone increasing operators.

Lemma 21. *Let X, X_+ and Y, Y_+ be two ordered Banach spaces. Let $A : X \to Y$ be a linear, invertible operator satisfying the maximum principle. Then, the inverse operator $A^{-1} : Y \to X$ is monotone increasing.*

Proof. *(Lemma 21.)* Let y, $\underline{y} \in Y$ be such that $y - \underline{y} \in Y_+$. Then,

$$A\big(A^{-1}(y-\underline{y})\big) \in Y_+ \quad \Rightarrow \quad A^{-1}(y-\underline{y}) \in X_+ \quad \Leftrightarrow \quad A^{-1}y - A^{-1}\underline{y} \in X_+.$$

This establishes that the operator A^{-1} is monotone increasing. $\square$

We are interested in a class of nonlinear problems where the principal part involves a linear operator $A : X \to Y$ satisfying the maximum principle, and the non-principal part involves a nonlinear operator $F : X \to Y$ which has monotonicity properties; problems of this type can be written as follows: Find an element $x \in X$ solution of the equation

$$Ax + F(x) = 0. \tag{A.14}$$

We now establish some results for this class of problems.

Lemma 22. *Let X, X_+ and Y, Y_+ be two ordered Banach spaces. Let $A : X \to Y$ be a linear, invertible operator satisfying the maximum principle. Let $F : X \to Y$ be a monotone decreasing (increasing) operator. Then, the operator $T : X \to X$ given by $T := -A^{-1}F$ is monotone increasing (decreasing).*

Proof. *(Lemma 22.)* Assume first that the operator F is monotone decreasing. So, given any x, $\underline{x} \in X$ such that $x - \underline{x} \in X_+$, the following inequalities hold,

$$
\begin{aligned}
x - \underline{x} \in X_+ \quad &\Rightarrow \quad -\big[F(x) - F(\underline{x})\big] \in Y_+,\\
&\Leftrightarrow \quad A\big(-A^{-1}\big[F(x) - F(\underline{x})\big]\big) \in Y_+,\\
&\Rightarrow \quad -A^{-1}\big[F(x) - F(\underline{x})\big] \in X_+,\\
&\Leftrightarrow \quad -\big[A^{-1}F(x) - A^{-1}F(\underline{x})\big] \in X_+,\\
&\Leftrightarrow \quad T(x) - T(\underline{x}) \in X_+,
\end{aligned}
$$

which establishes that the operator T is monotone increasing. In the case that the operator F is monotone increasing, then the first line in the proof above changes into $x - \underline{x} \in X_+$ implies that $F(x) - F(\underline{x}) \in Y_+$, and then all the remaining inequalities in the proof above are reverted. This establishes the Lemma. □

The next result translates the inequalities that satisfy sub- and super-solutions to the equation $Ax + F(x) = 0$, into inequalities for the operator $T = -A^{-1}F$.

Lemma 23. *Assume the hypothesis in Lemma 22.*

If there exists an element $x_+ \in X$ such that $Ax_+ + F(x_+) \in Y_+$, then this element satisfies that $x_+ - T(x_+) \in X_+$.

If there exists an element $x_- \in X$ such that $-\big[Ax_- + F(x_-)\big] \in Y_+$, then this element satisfies that $-\big[x_- - T(x_-)\big] \in X_+$.

Proof. *(Lemma 23.)* The first statement in the Lemma can be shown as follows,

$$
\begin{aligned}
Ax_+ + F(x_+) \in Y_+ &\Leftrightarrow A\big(x_+ + A^{-1}F(x_+)\big) \in Y_+\\
&\Rightarrow x_+ + A^{-1}F(x_+) \in X_+,
\end{aligned}
$$

which then establishes that $x_+ - T(x_+) \in X_+$. In a similar way, the second statement in the Lemma can be shown as follows,

$$
\begin{aligned}
-\big[Ax_- + F(x_-)\big] \in Y_+ &\Leftrightarrow A\big(-x_- - A^{-1}F(x_-)\big) \in Y_+\\
&\Rightarrow -x_- - A^{-1}F(x_-) \in X_+,
\end{aligned}
$$

which then establishes that $-\big[x_- - T(x_-)\big] \in X_+$. This establishes the Lemma. □

The last result can be found as Theorem 7.A in [57], page 283, and Corollary 7.18 on page 284. We reproduce it here for completeness, without the proof.

Theorem 15. (Fixed point for increasing operators) *Let X be an ordered Banach space, with a normal order cone X_+. Let $T : [x_-, x_+] \subset X \to X$ be a monotone increasing, compact map. If $-\big[x_- - T(x_-)\big] \in X_+$ and $x_+ - T(x_+) \in X_+$, then the iterations*

$$
\begin{aligned}
x_{n+1} &:= T(x_n), \qquad x_0 = x_-,\\
\hat{x}_{n+1} &:= T(\hat{x}_n), \qquad \hat{x}_0 = x_+,
\end{aligned}
$$

converge to x and $\hat{x} \in [x_-, x_+]$, respectively, and the following estimate holds,

$$
x_- \leqslant x_n \leqslant x \leqslant \hat{x} \leqslant \hat{x}_n \leqslant x_+, \qquad \forall n = \mathbb{N}. \tag{A.15}
$$

For nonlinear problems of the form (A.14), one can use Theorem 15 for monotone nonlinearities to conclude the following.

Corollary 3. **(Semi-linear equations with sub-/super-solutions)** *Let X, X_+ and Y, Y_+ be two ordered Banach spaces where X_+ is a normal order cone. Let $A : X \to Y$ be a linear, invertible operator satisfying the maximum principle. Let x_+, $x_- \in X$ be elements such that $(x_+ - x_-) \in X_+$, and then assume that the operator $F : [x_-, x_+] \subset X \to Y$ is monotone decreasing and compact. If the elements x_- and x_+ satisfy the relations*

$$- \big[Ax_- + F(x_-)\big] \in Y_+, \qquad Ax_+ + F(x_+) \in Y_+, \tag{A.16}$$

then there exists a solution $x \in [x_-, x_+] \subset X$ of the equation $Ax + F(x) = 0$.

Proof. *(Corollary 3.)* The operator A is invertible, then rewrite the equation $Ax + F(x) = 0$ as a fixed-point equation,

$$x = -A^{-1}F(x) =: T(x). \tag{A.17}$$

By Lemma 22, we know that the map $T : X \to X$ is monotone increasing. Moreover, this operator T it is compact, since is the composition of the continuous mapping $-A^{-1}$ and the compact map F. The elements x_- and x_+ satisfy Eq. (A.16), therefore, by Lemma 23, they are also sub- and super-solutions for the fixed-point equation involving the map T. It follows from Theorem 15 that there exists an element $x \in X$ solution to the fixed-point equation (A.17), and this solution satisfies the bounds $x_- \leqslant x \leqslant x_+$. □

A.7. ***A priori* estimates in ordered Banach spaces.** Many problems of the form in Eq. (A.14) do not have monotone nonlinearities. However, in the case that there exist sub- and super-solutions to Eq. (A.14) it is possible to introduce a "shift" into the equation. This shift transforms a problem that does not have a monotone nonlinearity into one that does, without destroying the maximum principle property required of the linear part. However, the disadvantage of the shift technique is that it requires additional regularity in the equation coefficients than the regularity needed for the original equation to be well-defined. On the other hand, it is possible to construct arguments leading to *a priori* order cone estimates on any possible solution (whether or not it exists) with very weak assumptions on the nonlinearity. Although such results are standard for semi-linear scalar problems with monotone nonlinearities (for example, see [32]), our result below holds for a class of semi-linear problems with non-monotone nonlinearities and appears to be new. Problems with monotone nonlinearities fit into this class, but it also includes a much larger set of nonlinearities. (See the second assumption in i in Lemma 24 below.)

The following result (Lemma 24 below) gives sufficient conditions for establishing *a priori* order cone estimates on solutions to certain PDE-like operator equations in ordered Banach spaces. These order estimates can be translated into norm estimates in the case that the order cone is normal. (See Corollary 4 following Lemma 24 below.) Note that the bounds established in Lemma 24 below are not necessarily sub- and super-solutions; establishing the bounds by first showing they are sub- and super-solutions and then using Corollary 3 would require a monotone nonlinearity, or use of the shifting technique requiring additional regularity assumptions.

Lemma 24. **(*A priori* order estimates)** *Let Y, Y_+ be an ordered Banach space with a generating order cone Y_+. Let $F : Y \to Y^*$ be a continuous map. Let $A : Y \to Y^*$ be a linear, continuous operator with $\dim N_A \geqslant 1$. Assume that there exists a subspace $X \subset Y$, with an induced order cone $X_+ = X \cap Y_+$, such that $A_X : X \to X^*$, the restriction of the operator A to the space X, is coercive. Let $u \in Y$ be a solution of the equation $Au + F(u) = 0$.*

(i) If there exists an element $y_\wedge \in N_A$ such that $(u - y_\wedge)^+ \in X_+$, and for all $y \in Y$ such that $(y - y_\wedge) \in Y_+$ holds that $F(y)\big((y - y_\wedge)^+\big) \geqslant 0$; Then, the solution $u \in Y$ satisfies $y_\wedge - u \in Y_+$.

(ii) If there exists an element $y_\vee \in N_A$ such that $(u - y_\vee)^- \in X_+$, and for all $y \in Y$ such that $-(y - y_\vee) \in Y_+$ holds that $F(y)\big((y - y_\vee)^-\big) \leqslant 0$. Then, the solution $u \in Y$ satisfies $u - y_\vee \in Y_+$.

Proof. *(Lemma 24.)* We first show part *(i)*. Given the solution $u \in Y$, introduce and element $u_D \in Y$ be an element such that $u - u_D \in X$. Second, notice that the element $(u - y_\wedge)$ belongs to the space Y, which has a generating order cone Y_+, so we know that there exists a decomposition

$$(u - y_\wedge) = (u - y_\wedge)^+ - (u - y_\wedge)^-,$$

with both elements $(u - y_\wedge)^+$, $(u - y_\wedge)^- \in Y_+$. The first assumption in *(i)* says that $(u - y_\wedge)^+ \in X_+$ and so the element $(u - y_\wedge)^+$ is a valid test function for the functional

$$\big[A_X(u - u_D) + F(u) + Au_D\big] \in X^*,$$

so we have the following,

$$\begin{aligned} Au(u - y_\wedge)^+ &= A(u - y_\wedge)(u - y_\wedge)^+ \\ &= A(u - y_\wedge)^+(u - y_\wedge)^+ \\ &= A_X(u - y_\wedge)^+(u - y_\wedge)^+. \end{aligned}$$

Therefore, we have the following inequalities,

$$\begin{aligned} 0 &= Au(u - y_\wedge)^+ + F(u)\big((u - y_\wedge)^+\big) \\ &= A_X(u - y_\wedge)^+(u - y_\wedge)^+ + F(u)\big((u - y_\wedge)^+\big) \\ &\geqslant a_0 \,\|(u - y_\wedge)^+\|_X^2 + F(u)\big((u - y_\wedge)^+\big), \qquad a_0 > 0 \\ &\geqslant a_0 \,\|(u - y_\wedge)^+\|_X^2. \end{aligned}$$

The last inequality implies that $(u - y_\wedge)^+ = 0$, which then says that $u - y_\wedge = -(u - y_\wedge)^-$, and we then conclude that $-(u - y_\wedge) \in Y_+$. This condition can be written using inequalities as $u \leqslant y_\wedge$.

We now prove part *(ii)*. The element $(u - y_\vee)$ also belongs to the space Y, which has a generating order cone Y_+, so we know that there exists a decomposition

$$(u - y_\vee) = (u - y_\vee)^+ - (u - y_\vee)^-,$$

with both elements $(u - y_\vee)^+$, $(u - y_\vee)^- \in Y_+$. The first assumption in part *(ii)* says that $(u - y_\vee)^- \in X_+$ and so the element $(u - y_\vee)^-$ is a valid test function for the functional

$$\big[A_X(u - u_D) + F(u) + Au_D\big] \in X^*,$$

so we have the following,

$$\begin{aligned} Au(u-y_\vee)^- &= A(u-y_\vee)(u-y_\vee)^- \\ &= -A(u-y_\vee)^-(u-y_\vee)^- \\ &= -A_X(u-y_\vee)^-(u-y_\vee)^-. \end{aligned}$$

Therefore, we have the following inequalities,

$$\begin{aligned} 0 &= Au(u-y_\vee)^- + F(u)\big((u-y_\vee)^-\big) \\ &= -A_X(u-y_\vee)^-(u-y_\vee)^- + F(u)\big((u-y_\vee)^-\big) \\ &\leqslant -a_0\,\|(u-y_\vee)^-\|_X^2 + F(u)\big((u-y_\vee)^-\big), \qquad a_0>0 \\ &\leqslant -a_0\,\|(u-y_\vee)^-\|_X^2. \end{aligned}$$

The last inequality implies that $(u-y_\vee)^- = 0$, which then says that $u - y_\vee = (u-y_\vee)^+$, and we then conclude that $(u-y_\vee) \in Y_+$. This condition can be written using inequalities as $u \geqslant y_\vee$. This inequality establishes the Lemma. □

Corollary 4. *Let Y, Y_+ be an ordered Banach space with a normal order cone Y_+. Let Z be a Banach space, and consider the space $W = Y \cap Z$, with order cone $W_+ := Y_+ \cap Z$. If there exist elements u, $y_\vee$, and $y_\wedge \in W$ such that $0 \leqslant y_\vee \leqslant u \leqslant y_\wedge$ in the order given by W_+, then there exists a positive constant c such that the following inequalities hold*

$$c\,\|y_\vee\|_Z \leqslant \|u\|_Z \leqslant \frac{1}{c}\,\|y_\wedge\|_Z.$$

Proof. *(Corollary 4.)* It follows directly from the definition of a normal order cone. □

As an example, Lemma 24 holds with the spaces taken to be $X = W_D^{1,2}$, $Y = W^{1,2}$, the linear operator taken to be $Au(v) = (\nabla u, \nabla v)$, and the nonlinear operator taken to be a monotone operator such as $F(u) = u^5$. Lemma 24 also holds for a non-monotone nonlinear operator satisfying the assumptions for the Lemma, such as $F(u) = u^5 - 2u^3$. An example of the space Z where Corollary 4 holds is $Z = L^\infty$.

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