Basic Continuation and Bifurcation Theory.

by Chris Deotte, MATH 275, Winter 2009

Motivation (Eigenfunctions)

In class, we learned that the solution of the heat equation:

$$u_t = \Delta u \;\text{ in } \Omega \times \mathbb{R}_+, \qquad u = 0 \;\text{ on }\; \partial\Omega \times \mathbb{R}_+$$

$$u(x,0) = v(x) \quad \text{is} \quad u(x,t) = \sum_{i=1}^{\infty} (v, \phi_i) e^{-\lambda_i t} \phi_i(x)$$

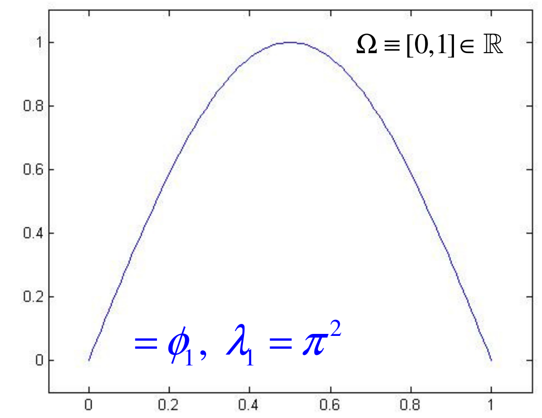
---

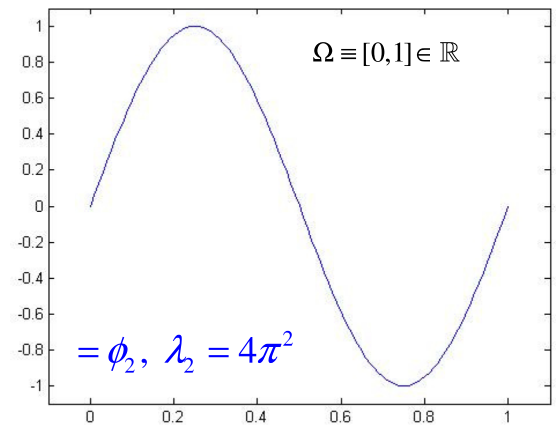
where $\phi_i,\ \lambda_i$ are solutions to the eigenvalue problem: $-\Delta u = \lambda u$ on Ω, $u = 0$ on $\partial\Omega$

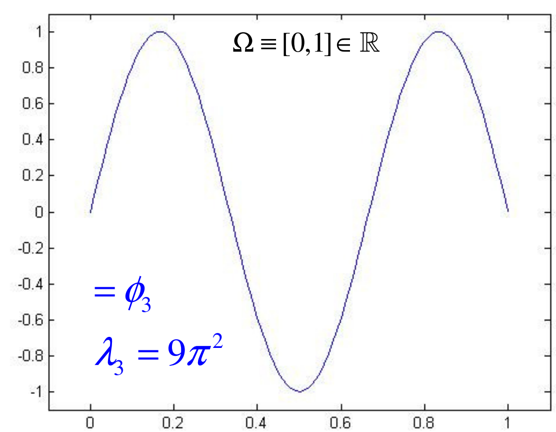
These are easy to find analytically for $\Omega \equiv [0,1]^n \in \mathbb{R}^n$

$$\phi_k(x) = \prod \sin(k_i \pi x_i), \quad \lambda_k = \pi^2 \Sigma k_i^2$$

But, how can we find the eigenfunctions for an arbitrary domain? i.e the unit circle?







Challenge Problem

How can we solve $-\Delta u = \lambda u$ in Ω , $u = 0$ on $\partial\Omega$

where Ω is unit circle, for values of $u \in B$ some Banach space and $\lambda \in \mathbb{R}$?

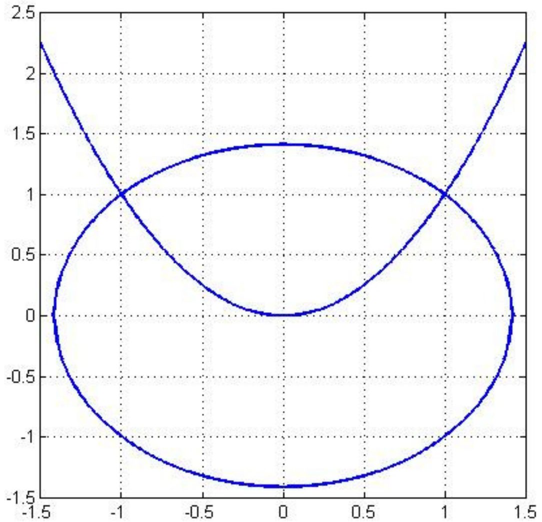
Standard Methods Fail

There's a problem! The operator $A(u,\lambda) = 0$ is not invertible!

That means that the solution, $x \in B \times \mathbb{R}$ may not exist or may not be unique. Our discrete approximate operator, $A_h(u_h, \lambda) = 0$ is also singular, therefore if we solved the discrete operator via an iterative method, convergence is not guaranteed nor can we find all the solutions at once. Even for a fixed λ , there may still be multiple solutions for *u*. Hence, we need a new method for finding the solutions.

Warm-Up Problem

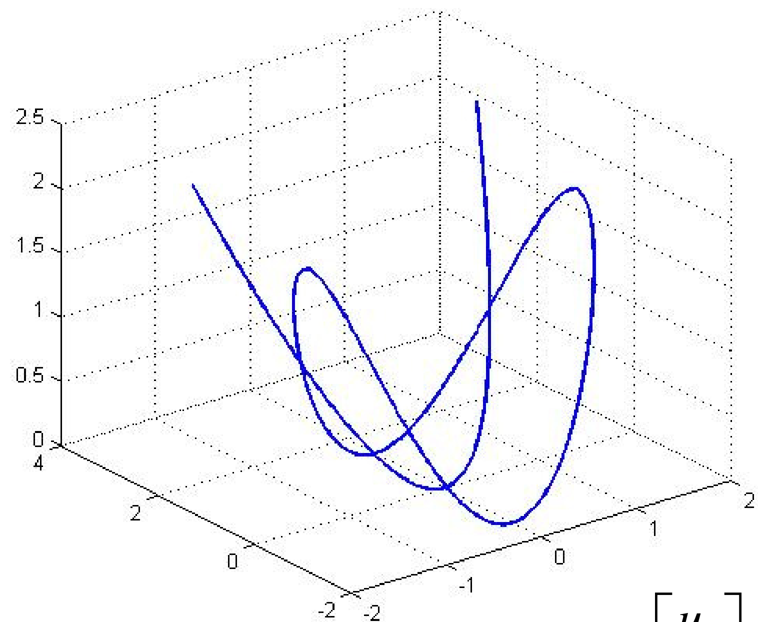
Find $u \in \mathbb{R}^3, \lambda \in \mathbb{R}$ such that $G(u,\lambda) = \begin{bmatrix} (u_1 - \lambda^2)(u_1{}^2 + \lambda^2 - 2) \\ u_2 - \lambda^2 \\ u_3 - \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$



λ vs. u_1

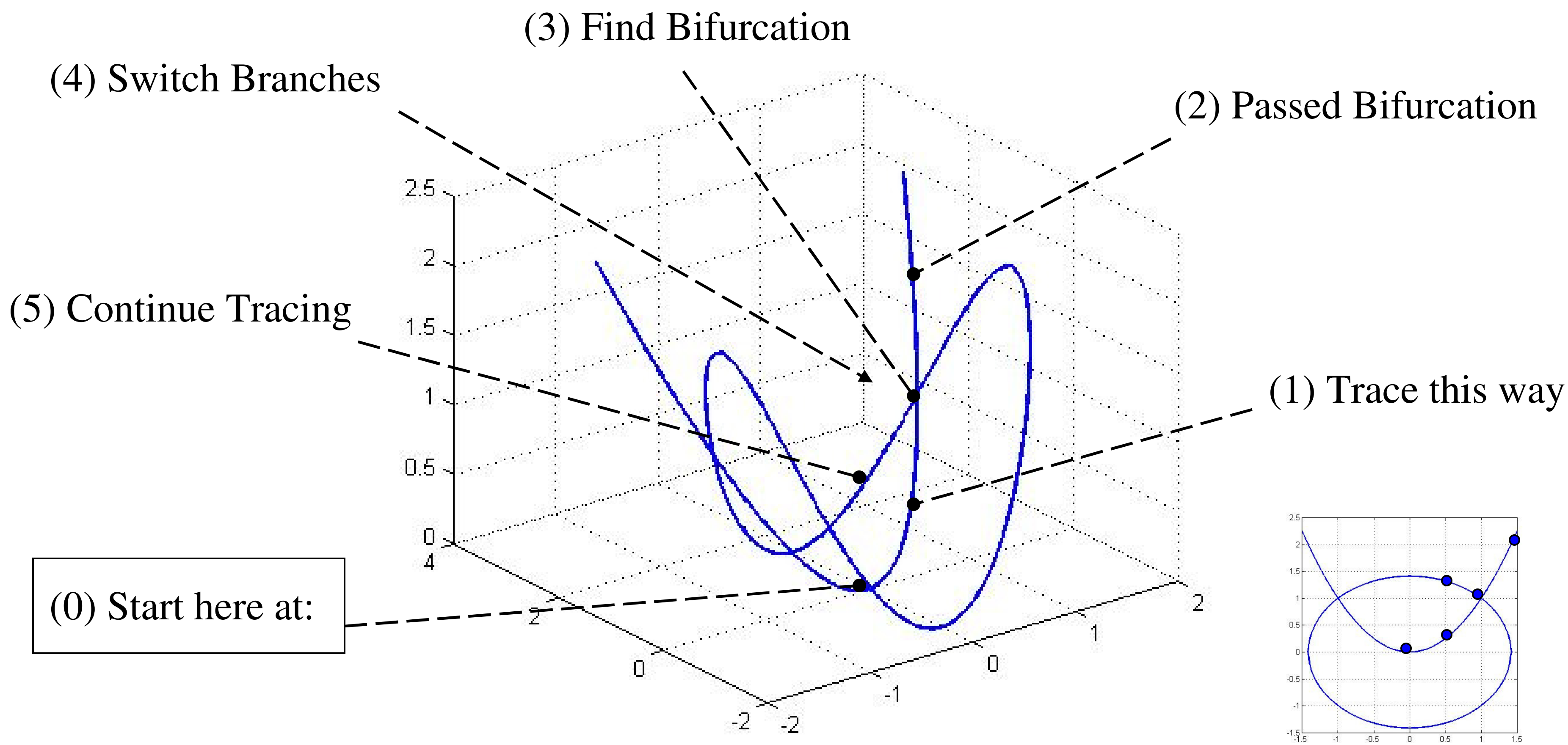
Solutions are 4 dimensional.

Here are projections of the solution space.



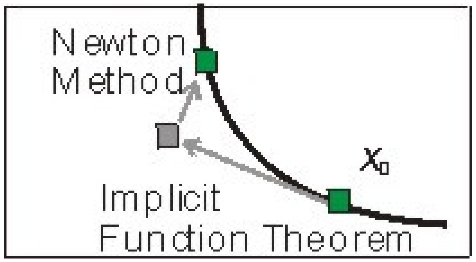
λ vs. $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Continuation and Bifurcation Method



How Do We Find the Solutions?

1) Start with a solution we know $x_0 \equiv [u_0, \lambda_0]^T \in \mathbb{R} \times B$

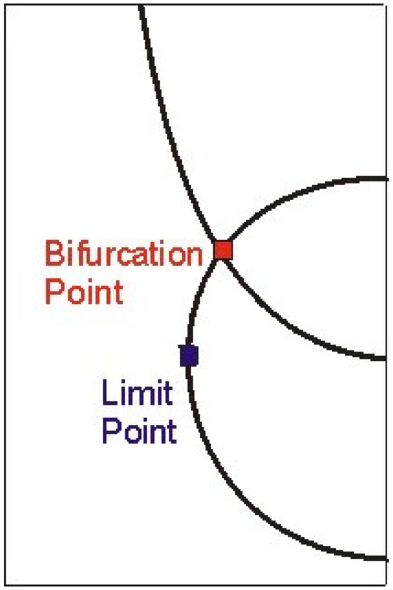


2) Calculate the tangent vector at x_0, $t = \left[\frac{\partial u}{\partial \lambda}(x_0), 1 \right]^T$
and step in that direction, $x_1 = x_0 + t\Delta\lambda$

Use the Implicit Function Theorem to calculate $\frac{\partial u}{\partial \lambda}(x_0) = -\left[G_u(x_0)\right]^{-1}\left[G_\lambda(x_0)\right]$

3) Then use Newton's Method to convergence back to the solution curve $x^{k+1} = x^k - \left[G_{x'}(x_k)\right]^{-1}\left[G(x_k)\right]$

and continue tracing out all the solutions.



This method will fail when G_u becomes singular at limit points and bifurcation points. Also, we would like better control of where the Newton Method will converge to.

$x' = x$ with one degree of freedom fixed

A Better Tracing Method

Instead of solving implicitly for *u* as a function of λ which can fail when $\det(G_u)=0$, implicitly solve for $x=[u,\lambda]^T$ as a function of arc length which won't fail at limit points.

Introduce a new equation, $N(u,\lambda,s)=0$ defining arc length *s* from solutions *u* and λ

$$\|\dot{x}(s)\|=1 \;\Rightarrow\; N(u,\lambda,s)=\|\dot{x}(s)\|^2-1=\|\dot{u}(s)\|^2+\left|\dot{\lambda}(s)\right|^2-1=0$$

More practical to approximate arc length at current solution, s_0 like this

$$(s-s_0)\|\dot{x}(s)\|^2=[\dot{x}(s)]\cdot[(s-s_0)\dot{x}(s)]\approx\begin{bmatrix}\dot{u}(s_0)\\ \dot{\lambda}(s_0)\end{bmatrix}\cdot\begin{bmatrix}u-u_0\\ \lambda-\lambda_0\end{bmatrix}=s-s_0 \;\Rightarrow$$

$$\boxed{N(u,\lambda,s)=[\dot{u}(s_0)]\cdot[(u-u_0)]+\dot{\lambda}(s_0)(\lambda-\lambda_0)-(s-s_0)=0}$$

Now we trace $P(u,\lambda,s)=\begin{bmatrix}G(u,\lambda)\\ N(u,\lambda,s)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=0$ using the tangent related to *s* not λ

without any problems at limit points and we can step past bifurcation points.

$$t=\dot{x}(s_0)=\begin{bmatrix}\dot{u}(s_0)\\ \dot{\lambda}(s_0)\end{bmatrix}=-\begin{bmatrix}G_u & G_\lambda\\ N_u & N_\lambda\end{bmatrix}^{-1}\begin{bmatrix}G_s\\ N_s\end{bmatrix}=-[P_x(s_0)]^{-1}[P_s(s_0)]$$

Locating Branches

Bifurcation points are points where $[P_x]$ is singular, i.e $\det[P_x]=0$

(which corresponds to $[G_u]$ being singular and $[G_\lambda]$ being in the range of $[G_u]$

since $P_x = \begin{bmatrix} G_u & G_\lambda \\ \dot{u}(s_0) & \dot{\lambda}(s_0) \end{bmatrix}$ and $G_u \dot{u}(s_0) + G_\lambda \dot{\lambda}(s_0) = 0$)

You know you've stepped past a bifurcation

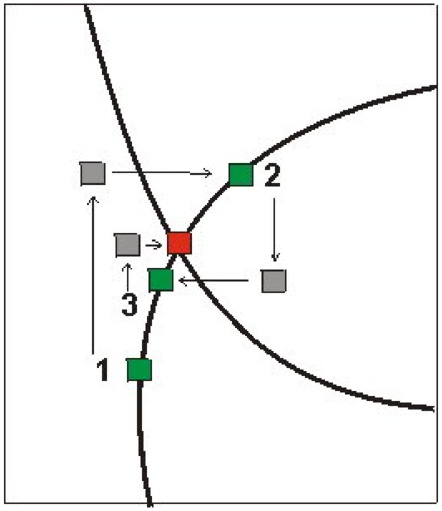
if $\det(P_x)$ changes signs (odd bifurcation).

(Better test: $\nu_{\min}[\psi_l]\cdot[\psi_r]$ changes signs

$\psi_r, \psi_l \equiv$ right/left singular vector of $\nu_{\min}$

$\nu_{\min} \equiv \min$ singular value of $[\mathrm{G_u}]$

Then locate bifurcation points using bisection



$\det[P_x(s_1)]>0$

$\det[P_x(s_2)]<0$

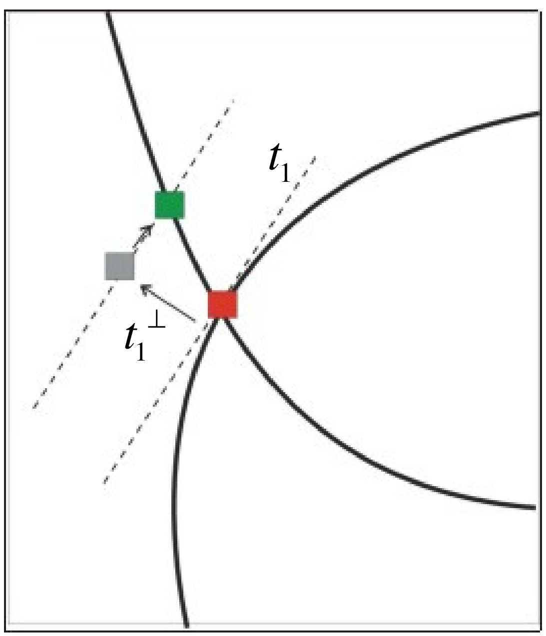
$\det[P_x(s_3)]>0$

$\det[P_x(s_4)]=0$

Switching Simple Branches

At a bifurcation point, step out perpendicularly from the tangent and iterate with Newton's Method constrained on a parallel line to the original tangent.

The problem is, which perpendicular do we choose? There are *n* degrees of freedom when choosing a perpendicular line where *n* is the dimension of *B*, our Banach space.



The plane on the left is $\text{span}\left\{[\phi_0,1]^T,[\phi_1,0]^T\right\}=\text{span}\left\{v_0,v_1\right\}$

If $t_1=\alpha v_0+\beta v_1$ then $t_1^{\perp}=-\beta v_0+\alpha v_1$, $\left(v_i\cdot v_j=\delta_{ij}\right)$

More Detail:

particular + homogeneous

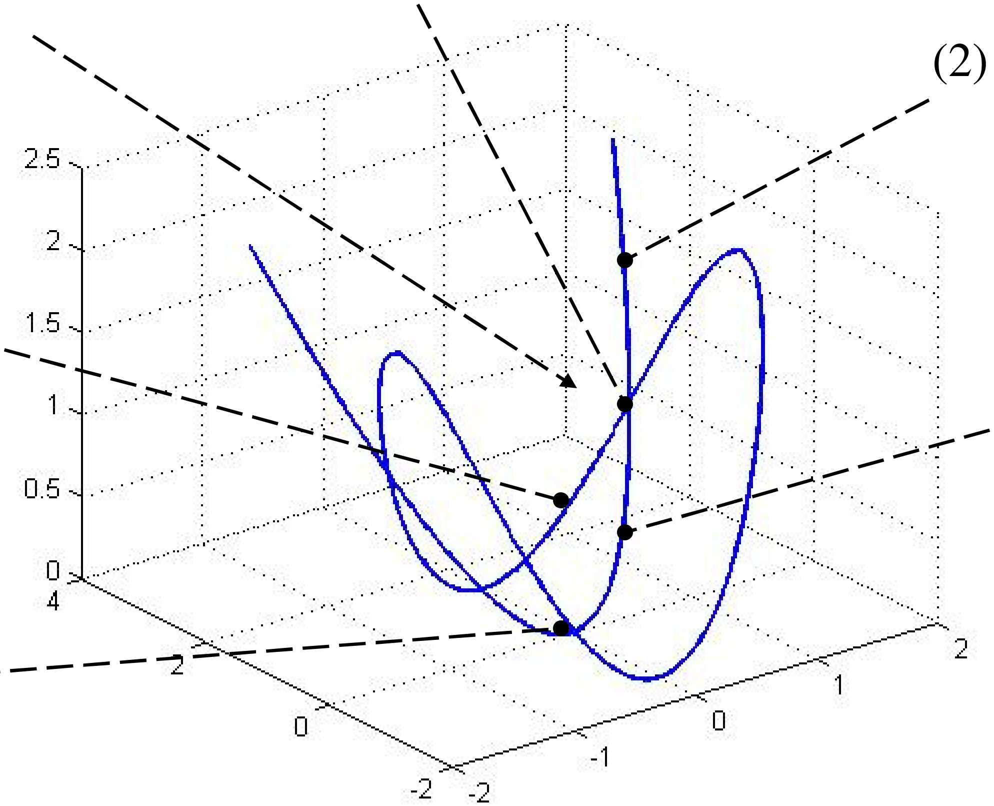
$$G_u\dot{u}(s_0)+G_\lambda\dot{\lambda}(s_0)=0 \;\Rightarrow\; \dot{u}\left(s_0\right)=\dot{\lambda}\left(s_0\right)\phi_0+\sum_{j=1}^{m}\alpha_j\phi_j \quad ,\phi_i\in B$$

$$G_u\phi_0+G_\lambda=0,\ \text{Null}[G_u]=span\left\{\phi_k\right\}_{\text{k=1}}^{\text{m}} \quad , \phi_i\cdot\phi_k=0,\ i\neq k$$

$$\alpha_k=[\phi_k]\cdot[\dot{u}(s_0)] \quad \text{Then if } t_1=[\alpha_0\phi_0+\alpha_1\phi_1\ ,\alpha_0]$$

$$t_1^{\perp}=\left[-\alpha_1\left\|\phi_1\right\|^2\phi_0+\alpha_0\left(1+\left\|\phi_0\right\|^2\phi_2\right)\ ,-\alpha_1\left\|\phi_1\right\|^2\right]$$

Review Basic Method



(0) Start here at:

$x_0 = [u, \lambda]^T = [0,0,0,0]$

$t = [0,0,\sqrt{2},\sqrt{2}]$

(1) Trace this way

$x_1 = [0.25, 0.25, 0.5, 0.5]^T$

$t = [0.5, 0.5, 0.5, 0.5]^T$

(2) Passed Bifurcation

$x_2 = [2.25, 2.25, 1.5, 1.5]^T$

$\det\left[P_x(x_1)\right] < 0$

$\det\left[P_x(x_3)\right] > 0$

(3) Find Bifurcation

$x_3 = [1.0, 1.0, 1.0, 1.0]^T$

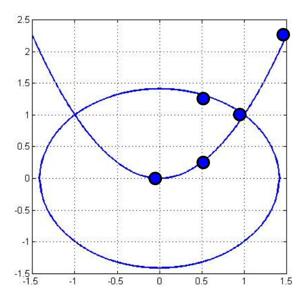
(4) Switch Branches

$t_1 = [-0.63, -0.63, -0.32, -0.32]^T$

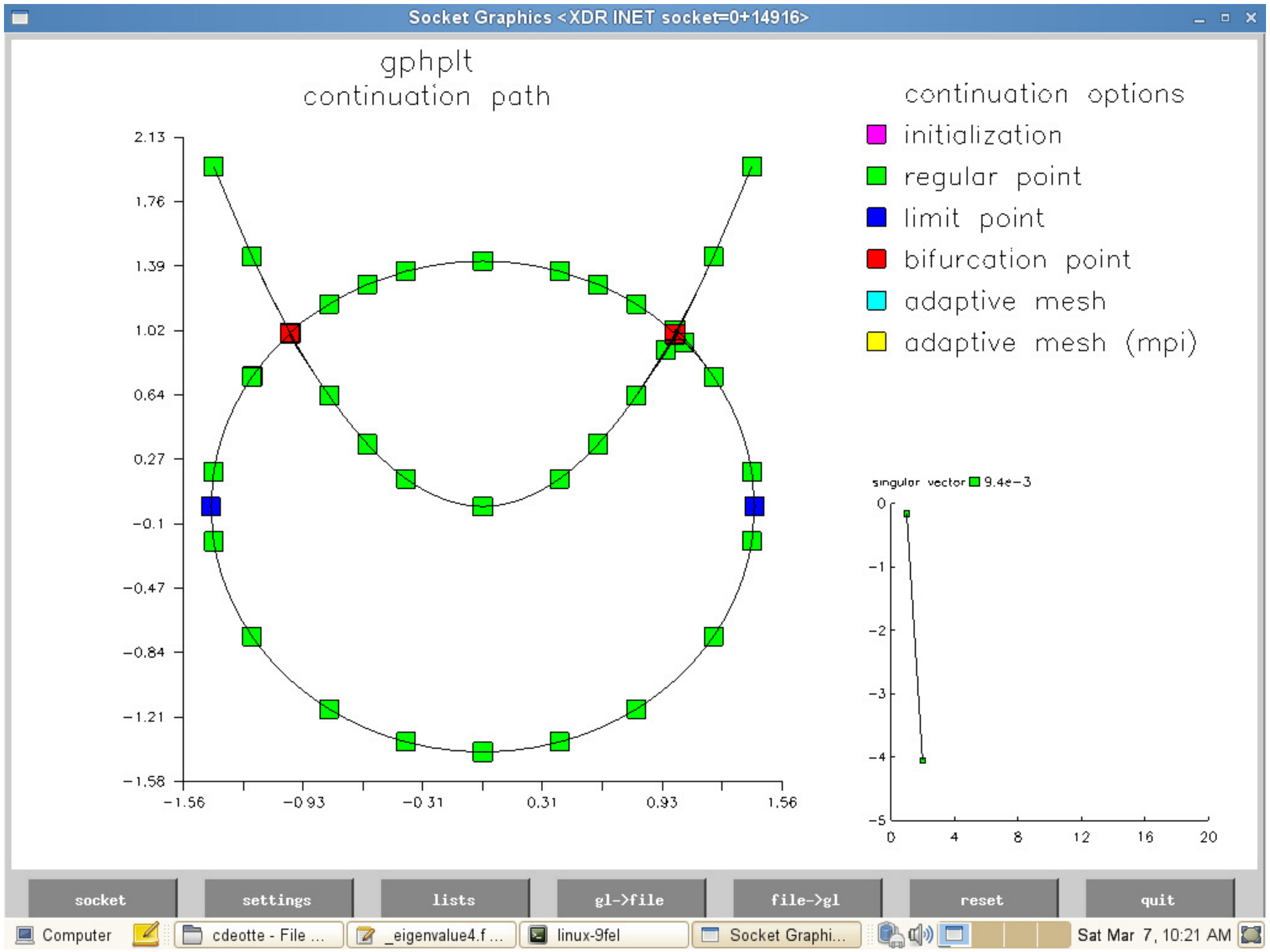
$t_2 = [0.38, -0.76, -0.38, -0.38]^T$

(5) Continue Tracing

$x_5 = [1.32, 0.25, 0.5, 0.5]^T$



Solving with PLTMG10.0



*PLTMG10 is designed to handle more complicated elliptic boundary value PDE's. You wouldn't really use it for this problem. $-\nabla \cdot a(x, y, u, \nabla u, \lambda) + f(x, y, u, \nabla u, \lambda) = 0$

Randy Bank's PLTMG10 software, allows one to manually apply the techniques presented here. His software displays the continuation path, solutions, and much more.

His normalization equation, N and branch switching algorithm is a bit more sophisticated.

He also utilizes a user defined functional on B to help with graphing and continuation.

PLTMG10 on Challenge Problem

Open PLTMG's Fortran file and define our equation, boundary conditions, and domain. Then compile the altered Fortran file and execute it.

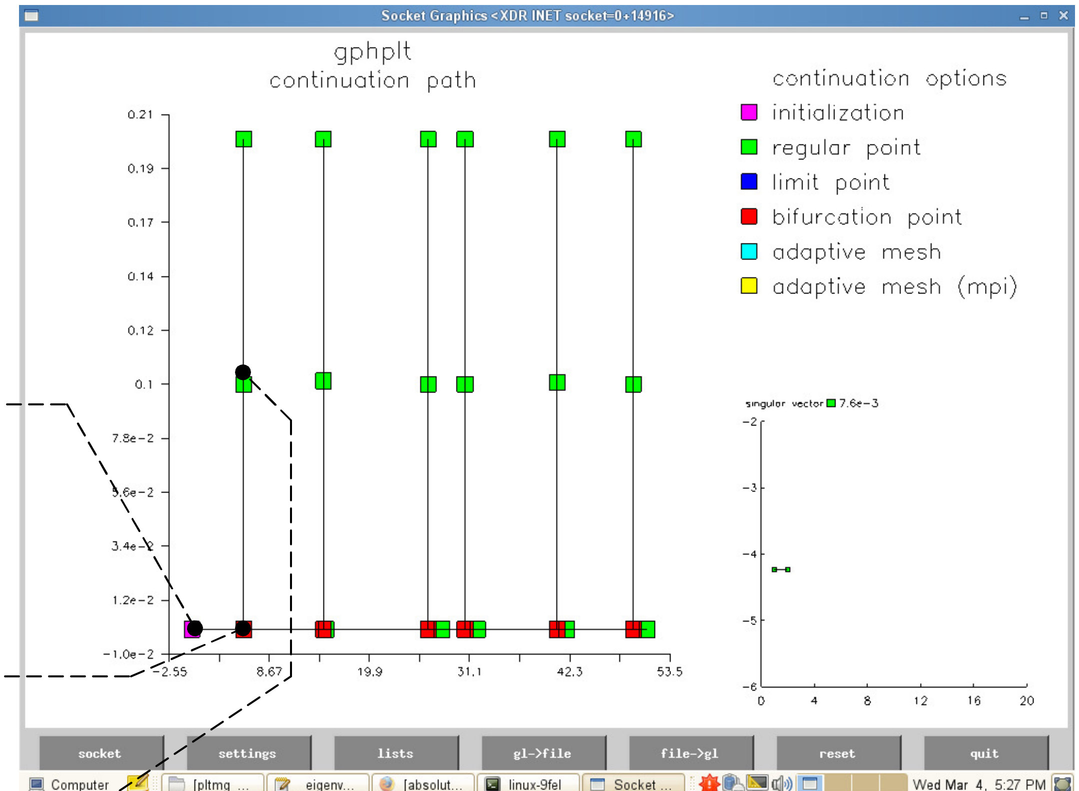
(1) First refine our mesh with TRIGEN
IADAPT=4 IREFN=8

(2) Next set our starting point with
PLTMG IPROB=3 ITASK=3
RTRGT=0 and RLTRGT=0

(3) Trace the curve with PLTMG
IPROB=3 ITASK=1 RLTRGT=10
(if it passes a bifurcation it will go back and find it)

(4) Switch branches with PLTMG
IPROB=3 ITASK=2
(changes tangent direction, nothing else)

(5) Trace new branch with PLTMG
ITASK=0 RTRGT=0.1
(will trace until functional equals 0.1)

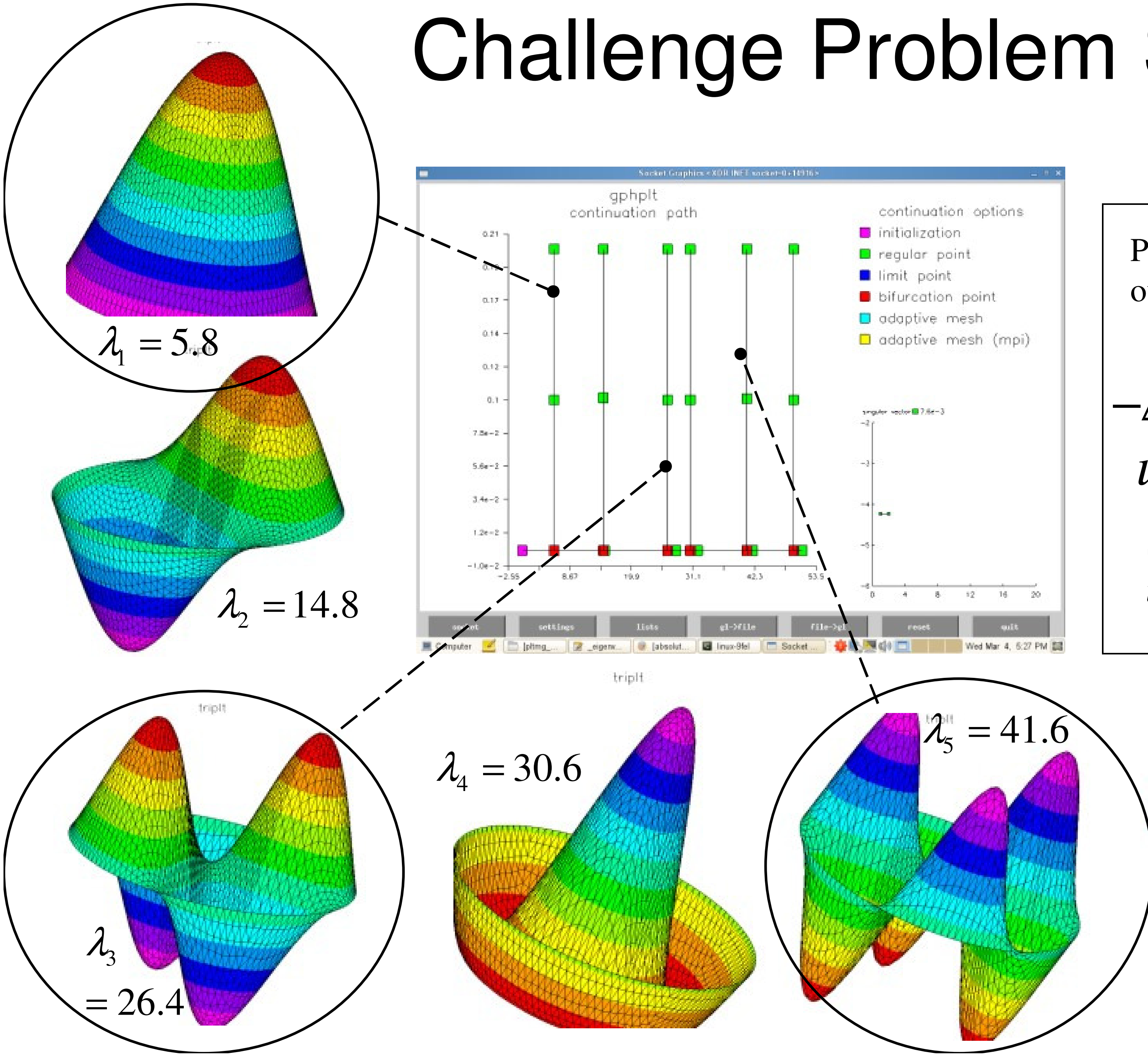


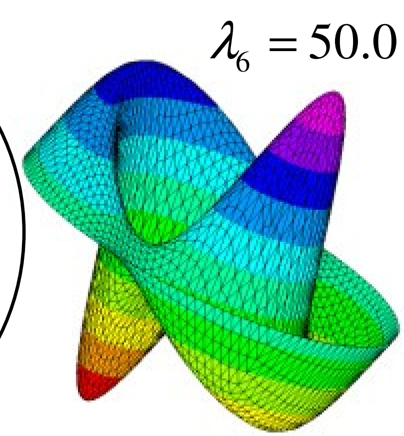
$$-\Delta u = \lambda u \text{ in } \Omega \text{ ,}$$
$$u = 0 \text{ on } \partial\Omega$$

$\Omega \equiv$ unit circle $\quad u \in B, \;\; \lambda \in \mathbb{R}$

Using the techniques presented, we can now solve our original problem:

Challenge Problem Solved.





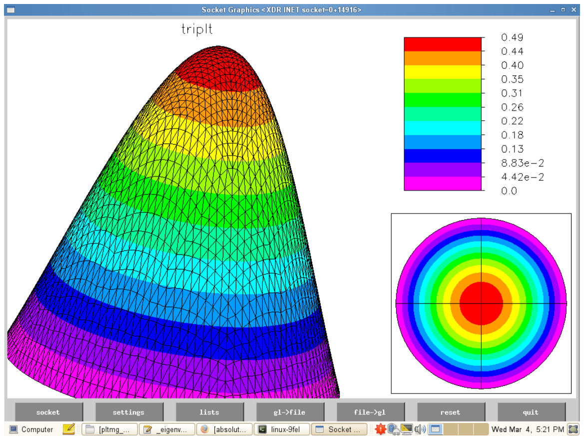
PLTMG10 has solved our original problem:

$$-\Delta u = \lambda u \text{ in } \Omega\ ,$$
$$u = 0 \text{ on } \partial\Omega$$

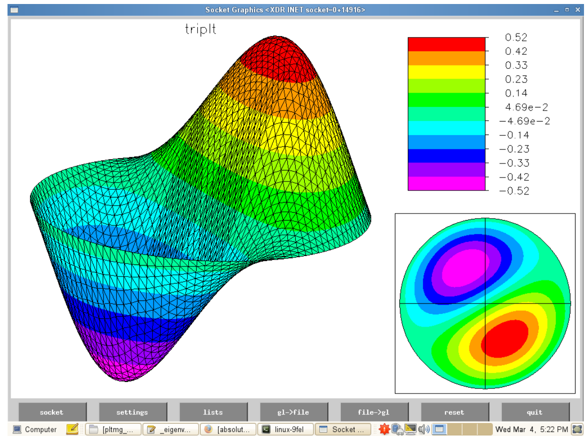
$\Omega \equiv$ unit circle

$u \in B, \quad \lambda \in \mathbb{R}$

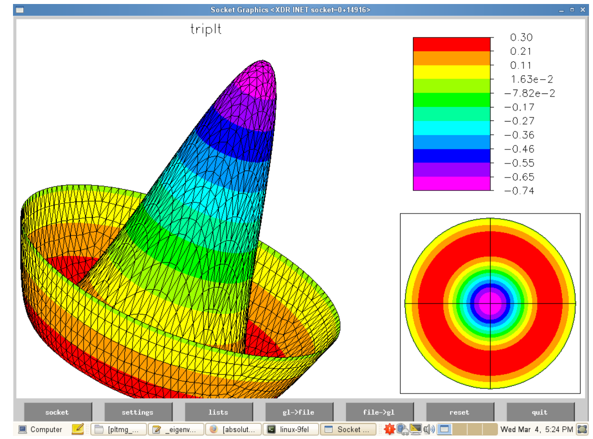
Eigenfunctions on Unit Circle



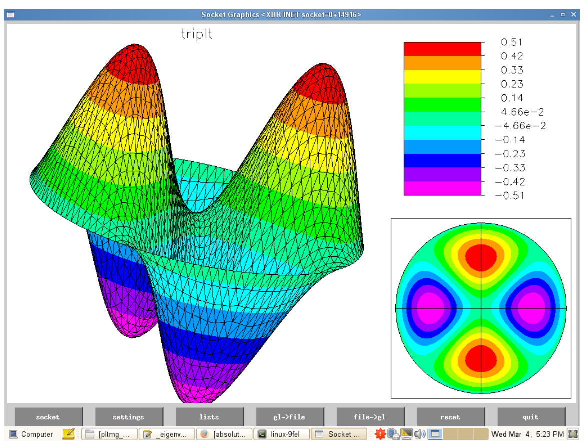
$\lambda_1 = 5.8$



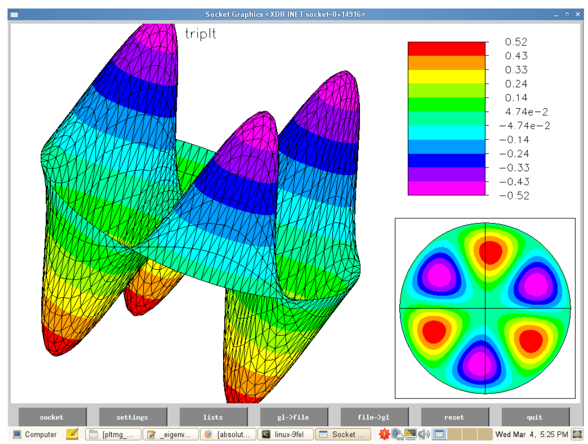
$\lambda_2 = 14.8$



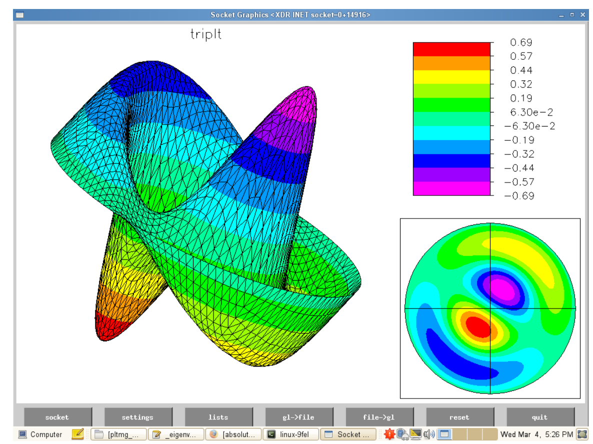
$\lambda_3 = 26.4$



$\lambda_4 = 30.6$

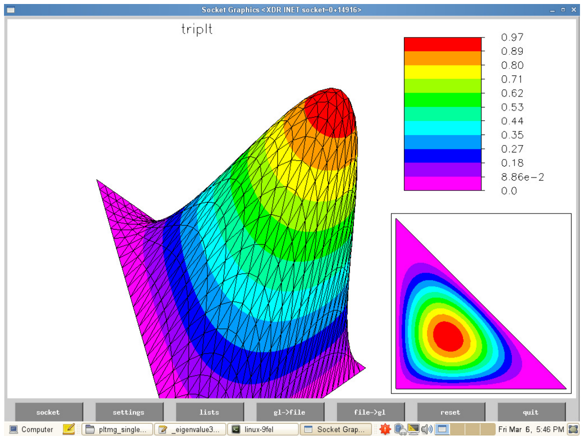


$\lambda_5 = 41.6$

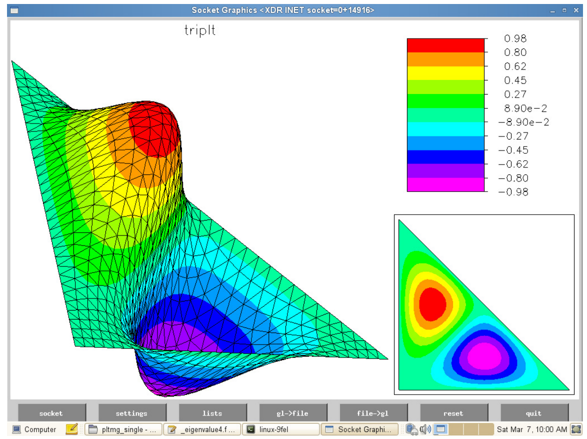


$\lambda_6 = 50.0$

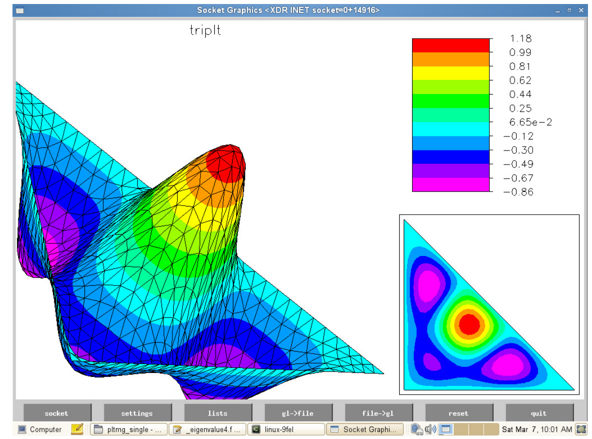
Eigenfunctions on Unit Triangle



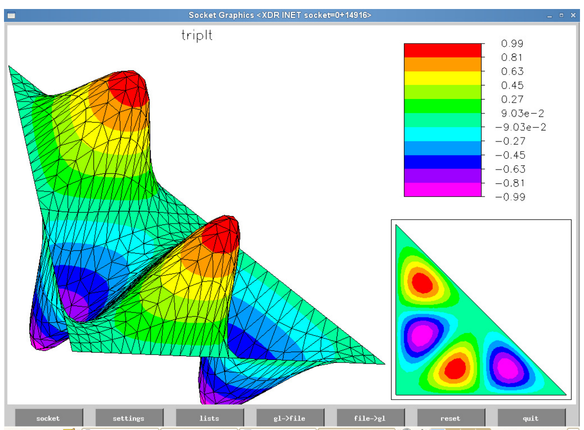
$\lambda_1 = 49.5$



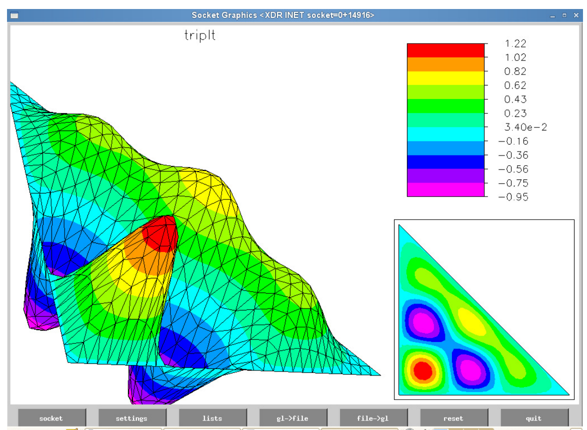
$\lambda_2 = 99.6$



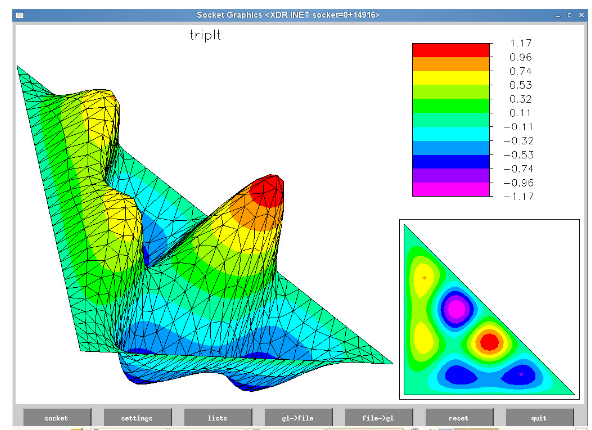
$\lambda_3 = 170.3$



$\lambda_4 = 201.6$



$\lambda_5 = 252$



$\lambda_6 = 262.4$

References

- Keller, Herbert B., “Numerical Solution of Bifurcation and Nonlinear Eigenvalue Problems”, *Applications of Bifurcation Theory* (1977) 347-369

- Bank, Randolph E., “PLTMG: A Software Package for Solving Elliptic Partial Differential Equations User’s Guide 10.0” (2007)

- For more info about or to download the software used in this presentation, visit: http://ccom.ucsd.edu/~reb/software.html