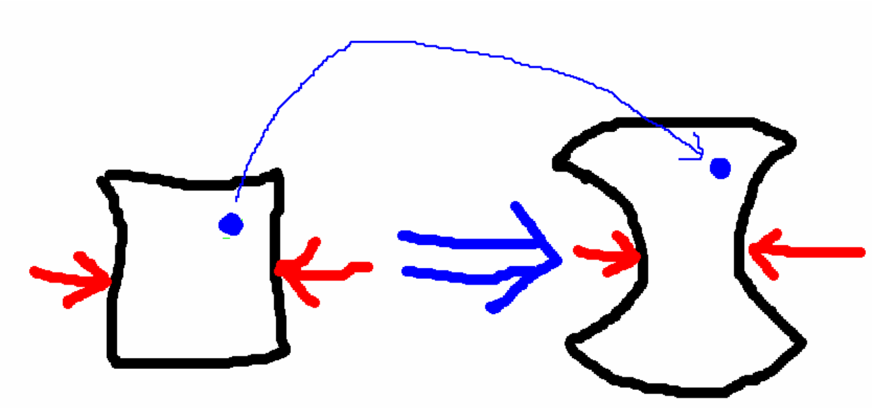
Investigating Elasticity

in 2D

Reasons

- I thought it would be fun to see how objects deform under external forces.
- I wanted to practice using the techniques learned in this class.



$$\bar{\Omega} \in \mathbb{R}^2$$

$$\phi : \bar{\Omega} \to \mathbb{R}^2$$

$$\phi(x, y) = [\phi_1, \phi_2] = [x, y] + [u_1, u_2] = id + u$$

where *u* is displacement and *id* is initial location.

What's The Equation? Object is at equilibrium thus forces add to zero

$$\sum(body\ forces)+\sum(surface\ forces)=\int_{\Omega} f_R(x_R)dx+\int_{\partial_N\Omega}\Sigma_R(x_R)\cdot n ds=\int_{\Omega}\left[f_R(x_R)+\mathrm{div}\Sigma_R(x_R)\right]dx=0$$

Boundary Conditions $\Sigma_R(x_R)\cdot n=g(x_R)$ on $\partial_N\Omega$ $\quad u(x_R)=0$ on $\partial_D\Omega$

How do we incorporate deformation? Well, how does the object balance external forces?

deformation $\Rightarrow C=\nabla\phi(x_R)^T\nabla\phi(x_R)\neq I \Rightarrow E\neq 0 \Rightarrow \Sigma_R(x_R)$ changes

Material reacts by deforming and creating internal stress to balance external forces

$$E(x_R)_{i,j}=\frac{1}{2}\left(\nabla\phi(x_R)^T\nabla\phi(x_R)-I\right)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\sum\left(\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)=\varepsilon(x)_{i,j}$$

Strain (a function of the deformation gradient, $\nabla\phi$)

$$\Sigma_R(x_R)=\lambda\mathrm{trace}(E)I+2\mu E+O(E^2)=\sigma(x_R)$$

(for St. Venant Kirchhoff material)

Second Piola-Kirchhoff Stress Tensor.
(a function of strain which in turn is a function of the deformation gradient.)

I will be using linear approximations.

The Three Problems (using linearization approximation):

P1: Energy Minimization Problem

Find $u \in B$ s.t. $J(u) \le J(v)\ \forall \mathrm{v} \in \mathrm{B}$

$$\text{Energy } = J(u) = \int_{\Omega} \left[\tfrac{1}{2} \varepsilon(u) : \sigma(u) - f(x) \cdot u \right] dx + \int_{\partial_N \Omega} g(x) \cdot u dx$$

P2: Weak form for Stationary of energy

Find $u \in B$ s.t. $< F(u), v >= 0\ \forall v \in B$

$$< J'(u), v >= \int_{\Omega} \left[2\mu \nabla^{(s)} u : \nabla^{(s)} v + \lambda (\nabla \cdot u)(\nabla \cdot v) - f(x) \cdot v \right] dx - \int_{\partial_N \Omega} g(x) \cdot v dx = 0$$

P3: Strong form with Differential Equations

$$-2\mu(\nabla \cdot \varepsilon(u)) - \lambda \nabla^2 u = f(x) \text{ in } \Omega$$

$$\sigma(u) \cdot n = g(x) \text{ on } \partial_N \Omega$$

$$u = 0 \text{ on } \partial_D \Omega$$

We will solve P2.

$$\text{Find } u \in B \text{ s.t. } A(u,v) = F(v) \ \forall v \in B$$

$$\text{where } A(u,v) = \int_{\Omega} \left[2\mu \nabla^{(s)} u : \nabla^{(s)} v + \lambda (\nabla \cdot u)(\nabla \cdot v) \right] dx$$

$$\text{and } F(v) = \int_{\Omega} f(x) \cdot v dx + \int_{\partial_N \Omega} g(x) \cdot v dx$$

$f(x) = (\textit{body force})$ and $g(x) = (\textit{surface traction})$

μ and λ are lame constants.

$\nabla^{(s)} u$ then $u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

$\mu = \frac{E}{2(1+\nu)} \approx 8.2031$ $\qquad$ $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \approx 10.4403$

$C = A : B$ then $c_{ij} = a_{ij} \cdot b_{ij}$

$E = 21.0$ is Young's Modulus and $\nu = 0.28$ is Poisson ratio

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Since A is bilinear, bounded, and coercive and F is linear and bounded then by the Lax Milgram Theorem there exists a unique solution. Also the solution can be shown to be continuously dependent on the data. Therefore this problem is well posed.

We will use the Galerkin Method to find an approximate solution within a known bounded error.

$$\text{Find } u_h \in V_h \subset B \text{ s.t. } A(u_h, v_h) = F(v_h) \;\; \forall v_h \in V_h$$

$$\text{where } A(u_h, v_h) = \int_\Omega \left[2\mu \nabla^{(s)} u_h : \nabla^{(s)} v_h + \lambda (\nabla \cdot u_h)(\nabla \cdot v_h) \right] dx$$

$$\text{and } F(v_h) = \int_\Omega f(x) \cdot v_h dx + \int_{\partial_N \Omega} g(x) \cdot v_h dx$$

$$\text{Error}= \|u - u_h\|_B \le \left(\frac{M}{m} \right) \inf_{v_h \in V_h} \|u - v_h\|_B \quad \text{where } M = \sup_{u,v \in B} \frac{A(u,v)}{\|u\|_B \, \|v\|_B} \text{ and } m = \inf_{u \in B} \frac{A(u,u)}{\|u\|_B^{\;2}}$$

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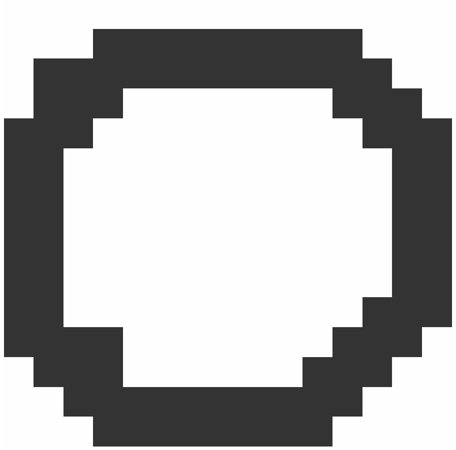
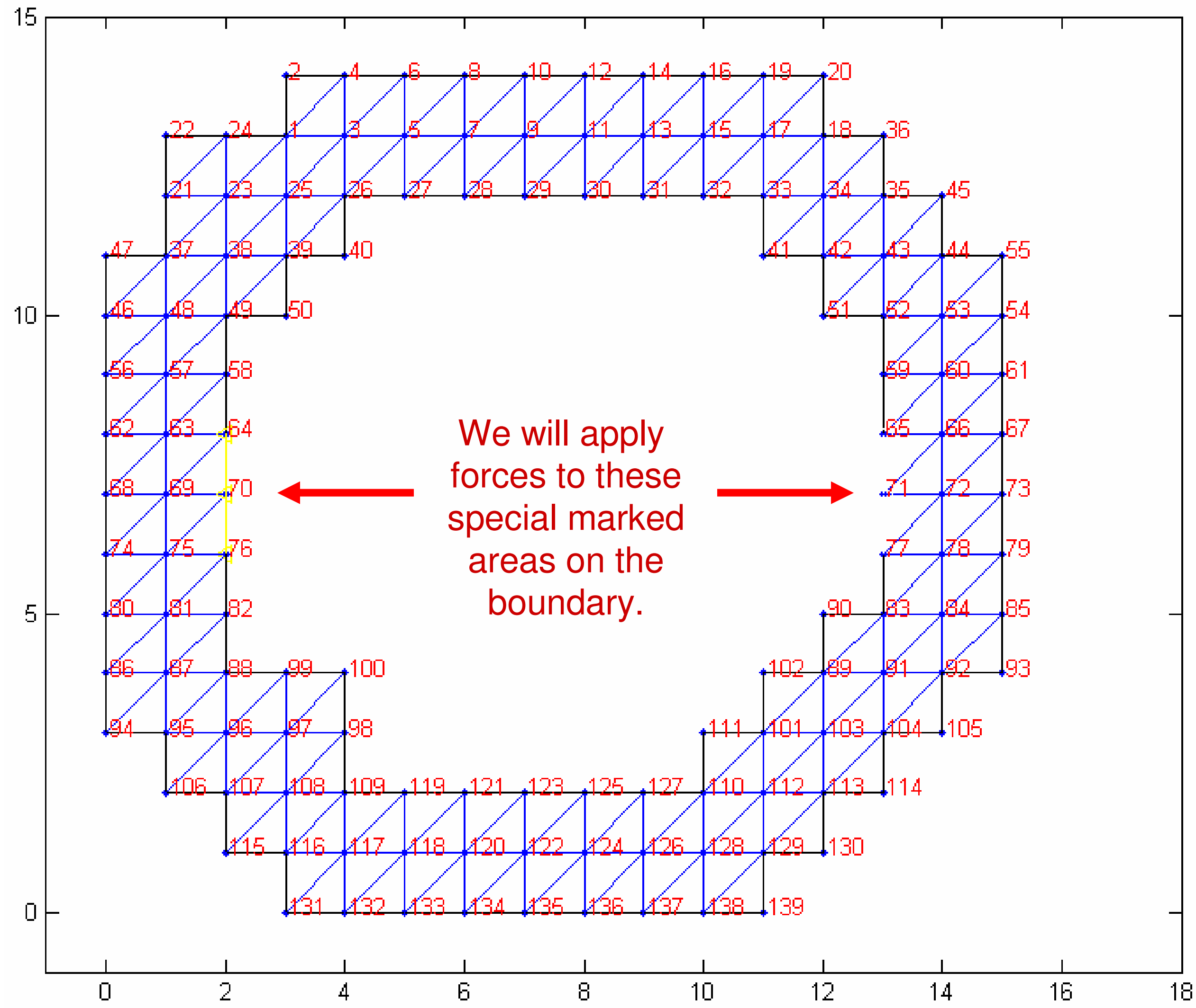
First we have to discretize our domain. Given a region in $\mathbb{R}^2$,we could subdivide it into regularly sized triangles as arranged below. That will simplify our calculations later.



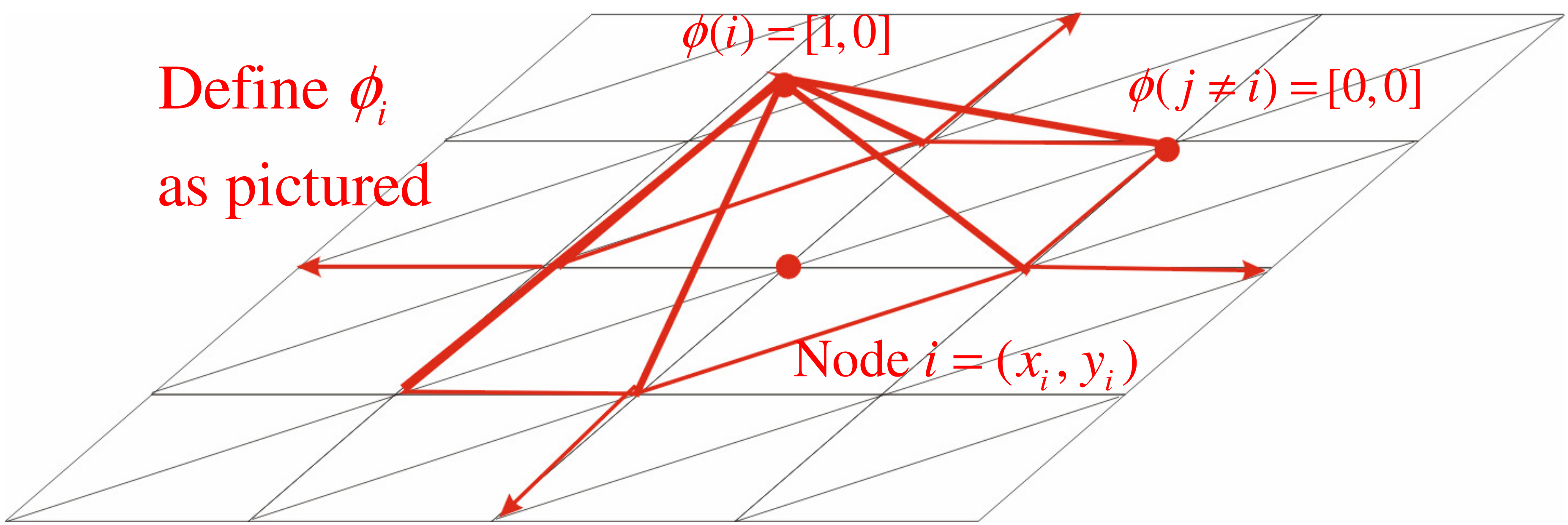
Here I broke a ring shaped domain into 140 nodes and 178 triangles following this regular pattern.



Next we must pick our basis functions. (Notice I've chosen 2 at each node)

$$\text{Let } V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n, \psi_1, \psi_2, \ldots, \psi_n\} \quad \phi_i, \psi_i \in \mathbb{R}^2$$

where n = number of nodes in $\bar{\Omega} \backslash \partial_D \Omega$ (for the ring domain, n=140)



and $\psi(i) = [0,1]$ and $\psi(j \neq i) = [0,0]$

Next we must write down our 2*n* linear equations in matrix form.

After choosing basis functions for V_h, all we have left to do is solve 2*n* linear equations for 2*n* unknown scalars. (where 2*n* = our number of basis functions).

$$A(u_h, v_h) = A\left(\sum_{i=1}^{n} \alpha_i \phi_i + \sum_{i=1}^{n} \beta_i \psi_i \ , \ v_h\right) = F(v_h) \ \forall \ basis\ function,\ v_h$$

$$\underset{2n\times 2n}{A} \cdot \underset{2n\times 1}{x} = \underset{2n\times 1}{F}$$

approximate solution:

$$u_h = \sum_{i=1}^{n} \alpha_i \phi_i + \sum_{i=1}^{n} \beta_i \psi_i$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad x = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$a_{11_{ij}} = A(\phi_i, \phi_j)$$

$$a_{22_{ij}} = A(\psi_i, \psi_j)$$

$$a_{12_{ij}} = A(\phi_i, \psi_j)$$

$$a_{21_{ij}} = A(\psi_i, \phi_j)$$

$$\alpha \in \mathbb{R}^n \qquad \beta \in \mathbb{R}^n$$

$$F_{1_i} = F(\phi_i) \qquad F_{2_i} = F(\psi_i)$$

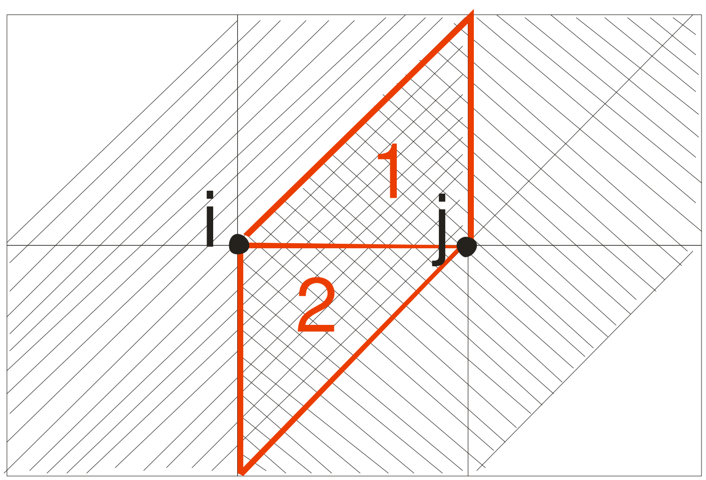
In order to write down our matrix equation, we need to find the matrix *A* and vector *F*

$$A(u_h, v_h) = \int_\Omega \left[2\mu \nabla^{(s)} u_h : \nabla^{(s)} v_h + \lambda (\nabla \cdot u_h)(\nabla \cdot v_h) \right] dx$$

Let $\phi_i = \begin{bmatrix} s \\ 0 \end{bmatrix}$, and $\phi_j = \begin{bmatrix} t \\ 0 \end{bmatrix}$ where $t = \begin{cases} 1 \text{ at node } k = j \\ 0 \text{ at node } k \neq j \end{cases}$ then $\nabla^{(s)} \phi_j = \frac{1}{2} \begin{bmatrix} 2\frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \\ \frac{\partial t}{\partial y} & 0 \end{bmatrix}$ and $\nabla \cdot \phi_j = \frac{\partial t}{\partial x}$

$$\text{Thus } A(\phi_i, \phi_j) = \int_\Omega \left[(2\mu + \lambda) \frac{\partial s}{\partial x} \frac{\partial t}{\partial x} + \mu \frac{\partial s}{\partial y} \frac{\partial t}{\partial y} \right] dx$$

For *i* next to *j*, you have:



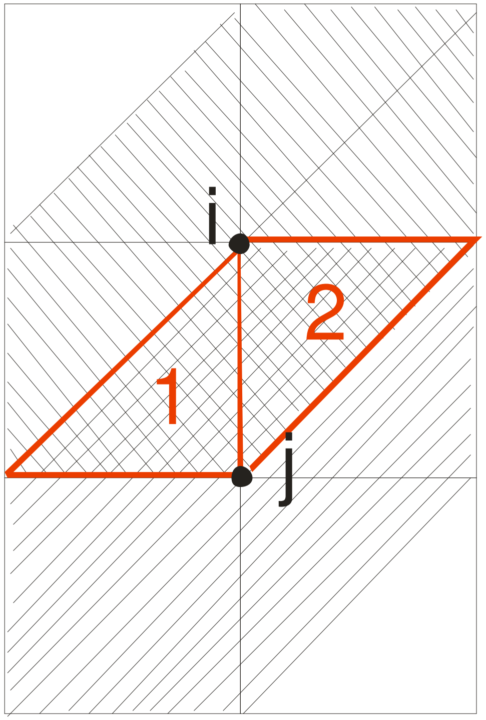
In 1: $\frac{\partial s}{\partial x} = \frac{-1}{h}, \quad \frac{\partial s}{\partial y} = 0, \quad \frac{\partial t}{\partial x} = \frac{1}{h}, \quad \frac{\partial t}{\partial y} = -\frac{1}{h}$

$$\int_1 \left[(2\mu + \lambda) \cdot \frac{-1}{h^2} + \mu \cdot 0 \cdot -\frac{1}{h} \right] dx = -\frac{1}{2}(2\mu + \lambda)$$

In 2: $\frac{\partial s}{\partial x} = \frac{-1}{h}, \quad \frac{\partial s}{\partial y} = \frac{1}{h}, \quad \frac{\partial t}{\partial x} = \frac{1}{h}, \quad \frac{\partial t}{\partial y} = 0$

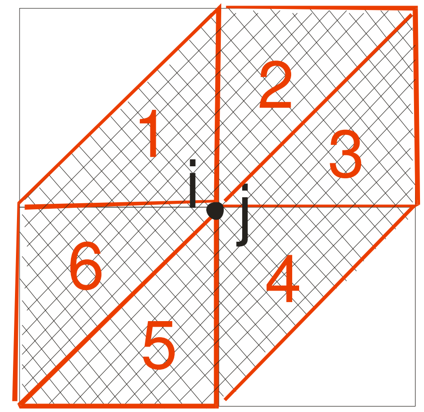
$$\int_2 \left[(2\mu + \lambda) \cdot \frac{-1}{h^2} + \mu \cdot \frac{1}{h} \cdot 0 \right] dx = -\frac{1}{2}(2\mu + \lambda)$$

$$A(\phi_i, \phi_j) = \int_1 + \int_2 = -2\mu - \lambda \approx -26.8$$



For *i* above *j, you have:*

$$A(\phi_i,\phi_j)=\int_1 + \int_2 = -\mu \approx -8.20$$

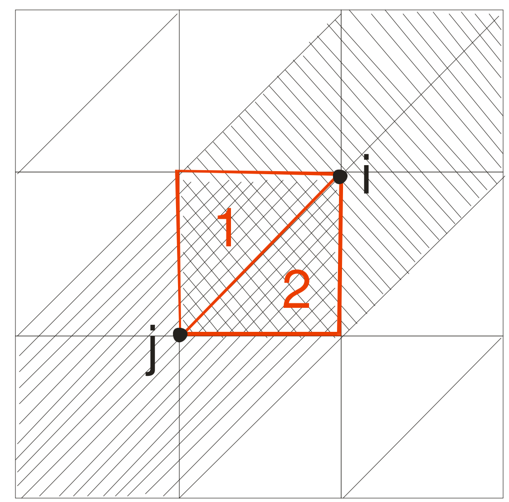


For *i* equal to *j, you have:*

$$A(\phi_i,\phi_j)=\int_1 + \int_2 + \int_3 + \int_4 + \int_5 + \int_6 = 6\mu + 2\lambda \approx 70.10$$

For *i* on diagonal from *j, you have:*

$$A(\phi_i,\phi_j)=\int_1 + \int_2 = 0$$



Then you have to calculate $A(\psi_i,\psi_j)$ and $A(\phi_i,\psi_j)$ for the different cases. And then finally you calculate $F(\phi_i)$ and $F(\psi_i)$ at the interior points and border points using the given boundary and initial conditions.

With our nice choice of basis functions and uniform mesh, our matrix *A* and vector *F* are defined by the following:

$$A(\phi_i,\phi_j) = A(\psi_i,\psi_j) = \begin{Bmatrix} 0 \text{ if } d(i,j) \geq 2 \\ 0 \text{ if } d(i,j) = \sqrt{2} \\ 70.1 \approx 6\mu + 2\lambda \text{ if } i = j \\ -26.8 \approx -2\mu - \lambda \text{ if } \left|\text{i-j}\right| = [1,0] \\ \text{-}8.20 \approx \text{-}\mu \text{ if } \left|\text{i-j}\right| = [0,1] \end{Bmatrix}$$

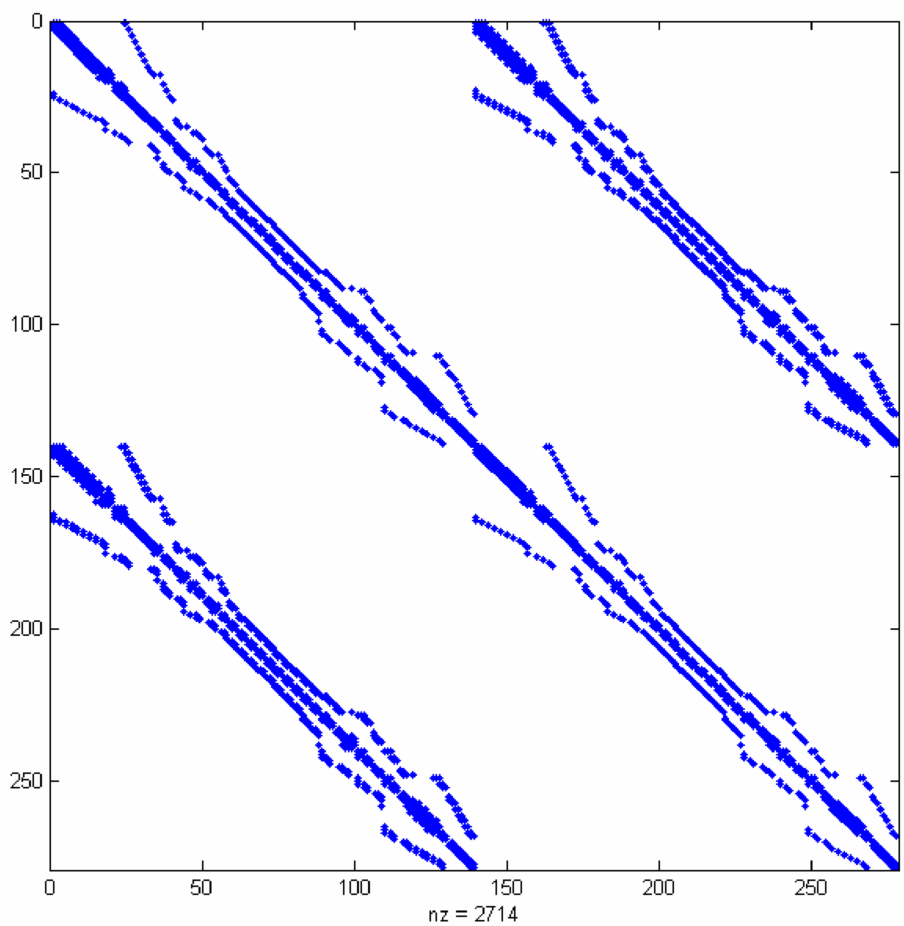
$$A(\phi_i,\psi_j) = A(\psi_i,\phi_j) = \begin{Bmatrix} 0 \text{ if } d(i,j) \geq 2 \\ \text{-}8.20 \approx \text{-}\mu \text{ if } d(i,j) = \sqrt{2} \\ -16.4 \approx -2\mu \text{ if } i = j \\ 8.20 \approx \mu \text{ if } d(i,j) = 1 \end{Bmatrix}$$

$$F(\phi_i) = f(i) \cdot \begin{bmatrix} h^2 \\ 0 \end{bmatrix} \text{ for } i \in \Omega^o \text{ and } F(\psi_i) = f(i) \cdot \begin{bmatrix} 0 \\ h^2 \end{bmatrix} \text{ for } i \in \Omega^o \quad , h = 1$$

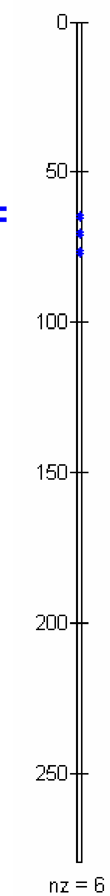
$$F(\phi_i) = g(i) \cdot \begin{bmatrix} h \\ 0 \end{bmatrix} \text{ for } i \in \partial_N \Omega \text{ and } F(\psi_i) = g(i) \cdot \begin{bmatrix} 0 \\ h \end{bmatrix} \text{ for } i \in \partial_N \Omega \quad , h = 1$$

Here is matrix *A* for the sample ring domain. *A* is a 280x280 matrix with 2714 non-zero entries indicated below by blue dots. Matrix A only depends on the basis functions and mesh layout. It is independent of the initial and boundary conditions (the given body and surface forces). Vector *F* depends on everything. It is a 280x1 vector with 6 non-zero entries.

A=



F=



Finally, we solve our matrix equation and get our approximate solution

If $A\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = F$, then $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = A^{-1}F$ and $\phi(x, y) = id + u = [x, y] + \sum_{i=1}^{n} \alpha_i \phi_i(x, y) + \sum_{i=n+1}^{2n} \beta_i \psi_i(x, y)$