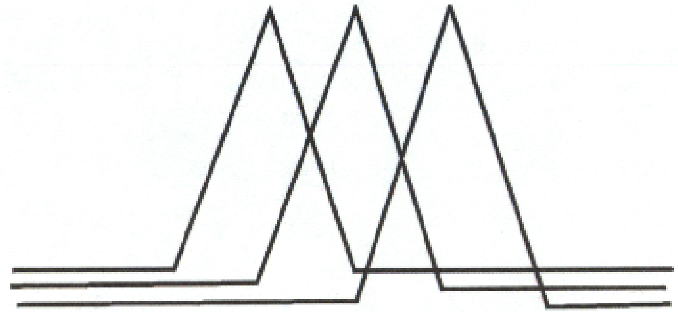
Modifying Finite Elements to use the Computational Efficiency of Diagonally Implicit Runge-Kutta Methods.

Outline

- Finite Elements with Linear Basis
- Finite Elements with Quadratic Basis
- Modifying Finite Elements
- New Method in Action
- Error Results from Simulations
- Hypothesized Error Bound
- Future Work

Finite Elements

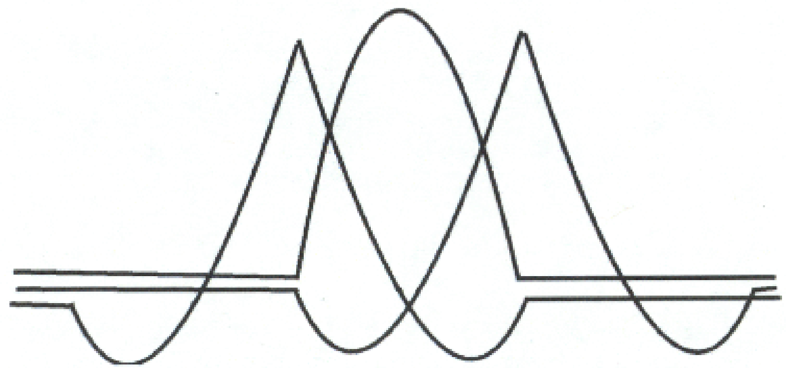
Solve Differential Equation with:



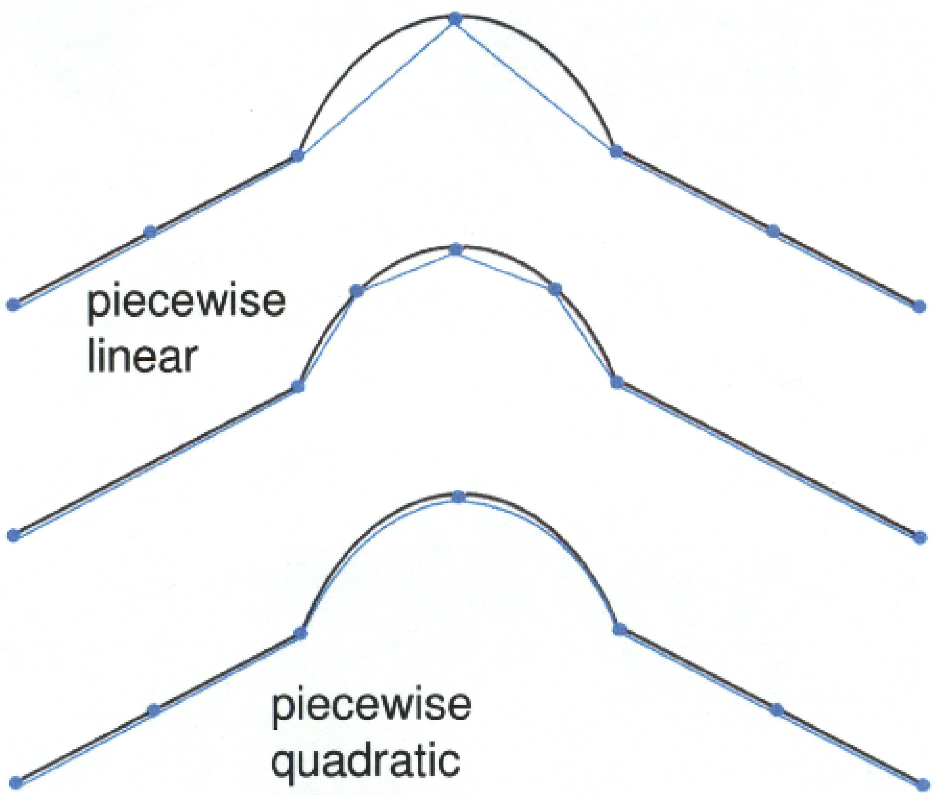
ϕ_i are piecewise polynomial basis functions with overlapping support.

$$u_h = \sum_{i=1}^{N} \alpha_i \phi_i$$

finite approximation to true solution *u*



Basis Functions



$$u_1(x) = \sum_{i=1}^{7} \alpha_i \phi_i$$

$$u_2(x) = \sum_{i=1}^{7} \tilde{\alpha}_i \tilde{\phi}_i$$

$$u_3(x) = \sum_{i=1}^{7} \tilde{\alpha}_i \psi_i$$

Each blue line (solution) is the linear combination of 7 basis functions, but the error is different for each. The true solution is the black line.

FE with Linear Basis Functions

$$u_t = f(t,u), \quad u(0) = 0$$

$$\text{Find } u = \sum_{j=1}^{N} \alpha_j \phi_j, \quad \int u_t \phi = \int f\phi \quad \forall \phi_i$$

Solving with linear basis is simple. Since we know alpha_0, Matrix A is effectively lower triangle and therefore we can solve for one alpha at a time easily.

$$\begin{bmatrix} x_0 & x_1 & 0 & 0 & 0 & 0 \\ x_2 & x_3 & x_4 & 0 & 0 & 0 \\ 0 & x_5 & x_6 & x_7 & 0 & 0 \\ 0 & 0 & x_8 & x_9 & x_{10} & 0 \\ 0 & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 0 & 0 & 0 & x_{14} & x_{15} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}$$

$$A\alpha = B$$

$$x_5\alpha_1 + x_6\alpha_2 + x_7\alpha_3 = B_2 = \int_{t_1}^{t_3} f\phi_2 \approx hf(t_2) \longrightarrow \alpha_3 \approx -\frac{x_5}{x_7}\alpha_1 + \frac{h}{x_7} f(t_2)$$

Quadratic Basis Functions

$$u_t = f(t,u), \quad u(0)=0$$

$$\text{Find } u = \sum_{j=1}^{N} \tilde{\alpha}_j \tilde{\phi}_j, \quad \int u_t \tilde{\phi} = \int f \tilde{\phi} \; \forall \tilde{\phi}_i$$

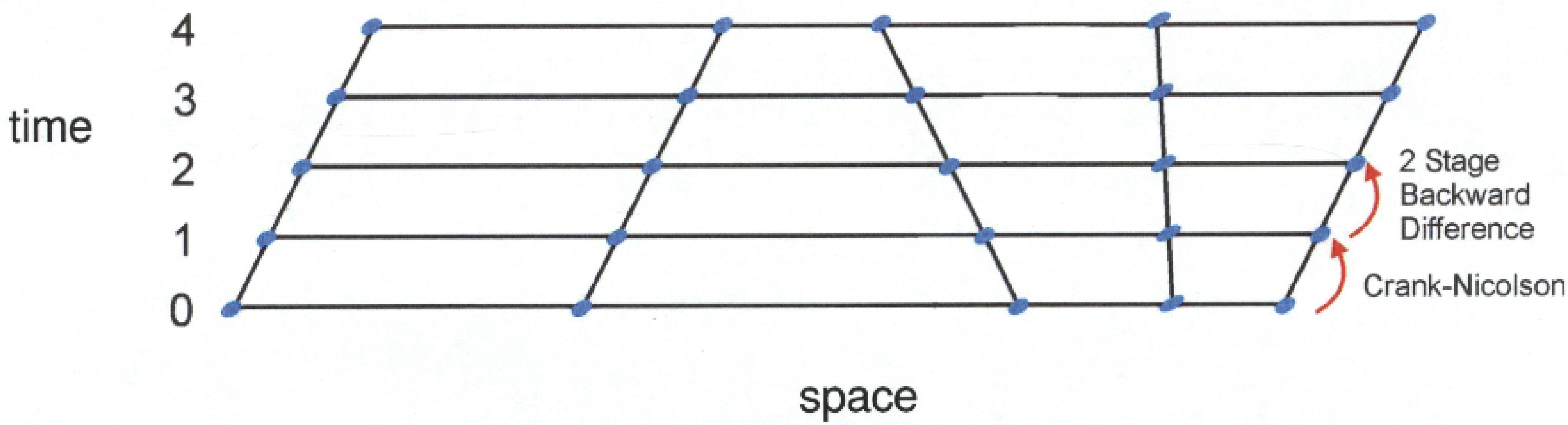
$$\tilde{A}\tilde{\alpha} = \tilde{B}$$

$$\begin{bmatrix} y_0 & y_1 & y_2 & 0 & 0 & 0 \\ y_3 & y_4 & y_5 & 0 & 0 & 0 \\ y_6 & y_7 & y_8 & y_9 & y_{10} & 0 \\ 0 & 0 & y_{11} & y_{12} & y_{13} & 0 \\ 0 & 0 & y_{14} & y_{15} & y_{16} & y_{17} \\ 0 & 0 & 0 & 0 & y_{18} & y_{19} \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \\ \tilde{\alpha}_3 \\ \tilde{\alpha}_4 \\ \tilde{\alpha}_5 \end{bmatrix} = \begin{bmatrix} \tilde{B}_0 \\ \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ \tilde{B}_4 \\ \tilde{B}_5 \end{bmatrix}$$

Solving for alphas with quadratic basis is computationally harder than using linear basis functions. We have effectively lower triangle with one upper diagonal so we must solve for two alphas at a time.

Finding two alphas at a time makes solving FE with quadratic basis functions essentially a 2 Stage Fully Implicit Runge-Kutta Method.

Diagonally Implicit Runge Kutta

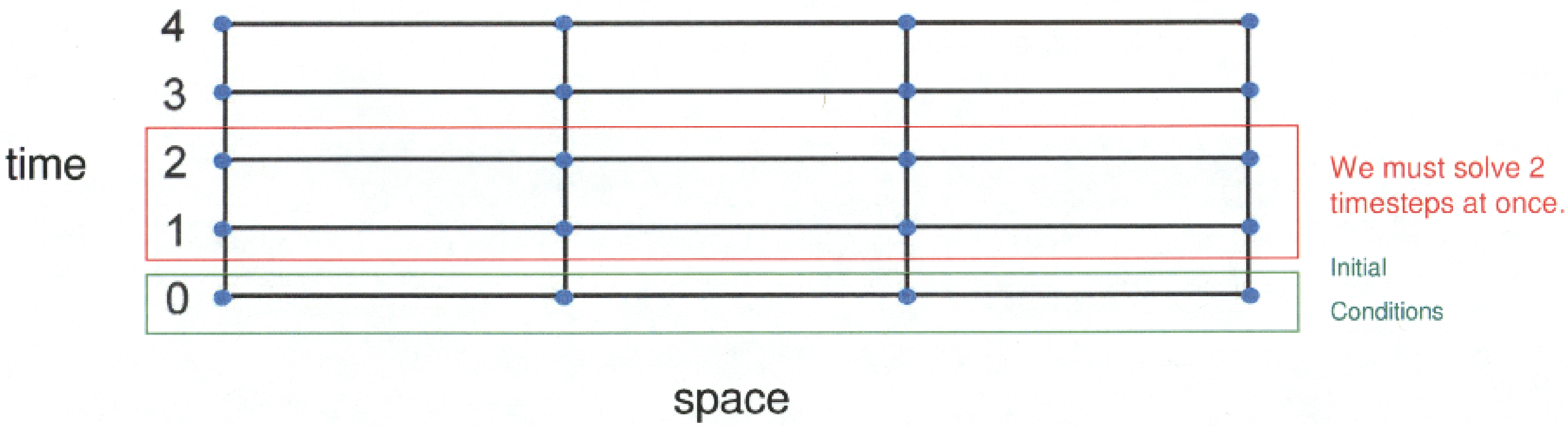


Let's use TR-BDF in time and see what happens.

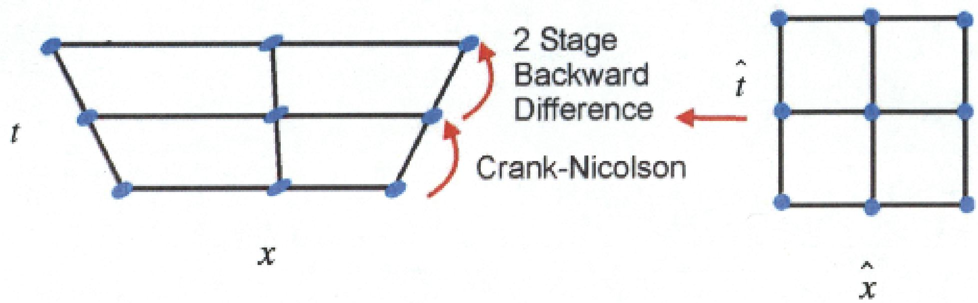
Time and 1-D Space

When using quadratic finite elements in time, we must simultaneously solve for 2 timesteps of unknown space variables at once.

We have seen that finite elements in time is similar to a Fully Implicit Runge-Kutta method. What happens if we use a Diagonally Implicit Runge-Kutta method instead?



Isoparametric Elements



$$u_t = \frac{1}{\Delta t} u_{\hat{t}} - \frac{1}{h(t)} \frac{\partial x}{\partial t} u_{\hat{x}}$$

$$u_t - u_{xx} = f \quad \longrightarrow \quad \int u_t \phi + \int u_x \phi_x = \int f \phi$$

$$\int \tfrac{1}{\Delta t} u_{\hat{t}} \phi = -\int \left(u_x \phi_x - \tfrac{\partial x}{\partial t} u_x \phi \right) + \int f \phi \quad \longrightarrow \quad M\alpha'(t) = -\tilde{A}\alpha(t) + B$$

TR-BDF Runge-Kutta Method

$$u'(t) = F(t,u)$$

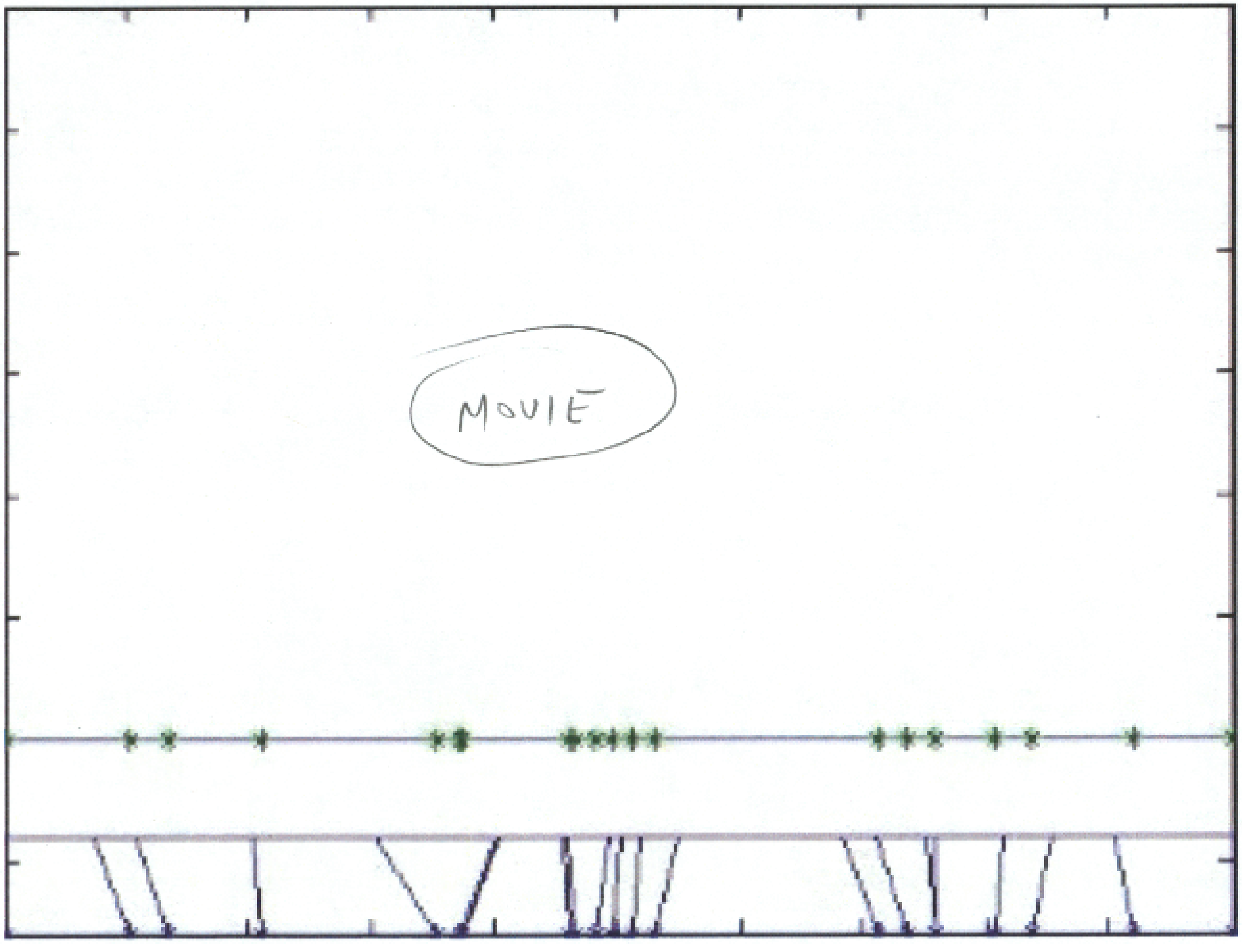
First, Crank-Nicolson

$$u^{k+1} = u^k + \tfrac{1}{2}\Delta t\left(F\left(t^{k+1},u^{k+1}\right) + F\left(t^k,u^k\right)\right)$$

Then 2 Stage Backward Difference

$$u^{k+2} = \tfrac{4}{3}u^{k+1} - \tfrac{1}{3}u^k + \tfrac{2}{3}\Delta t \cdot F\left(t^{k+2},u^{k+2}\right)$$

Simple Problem 1

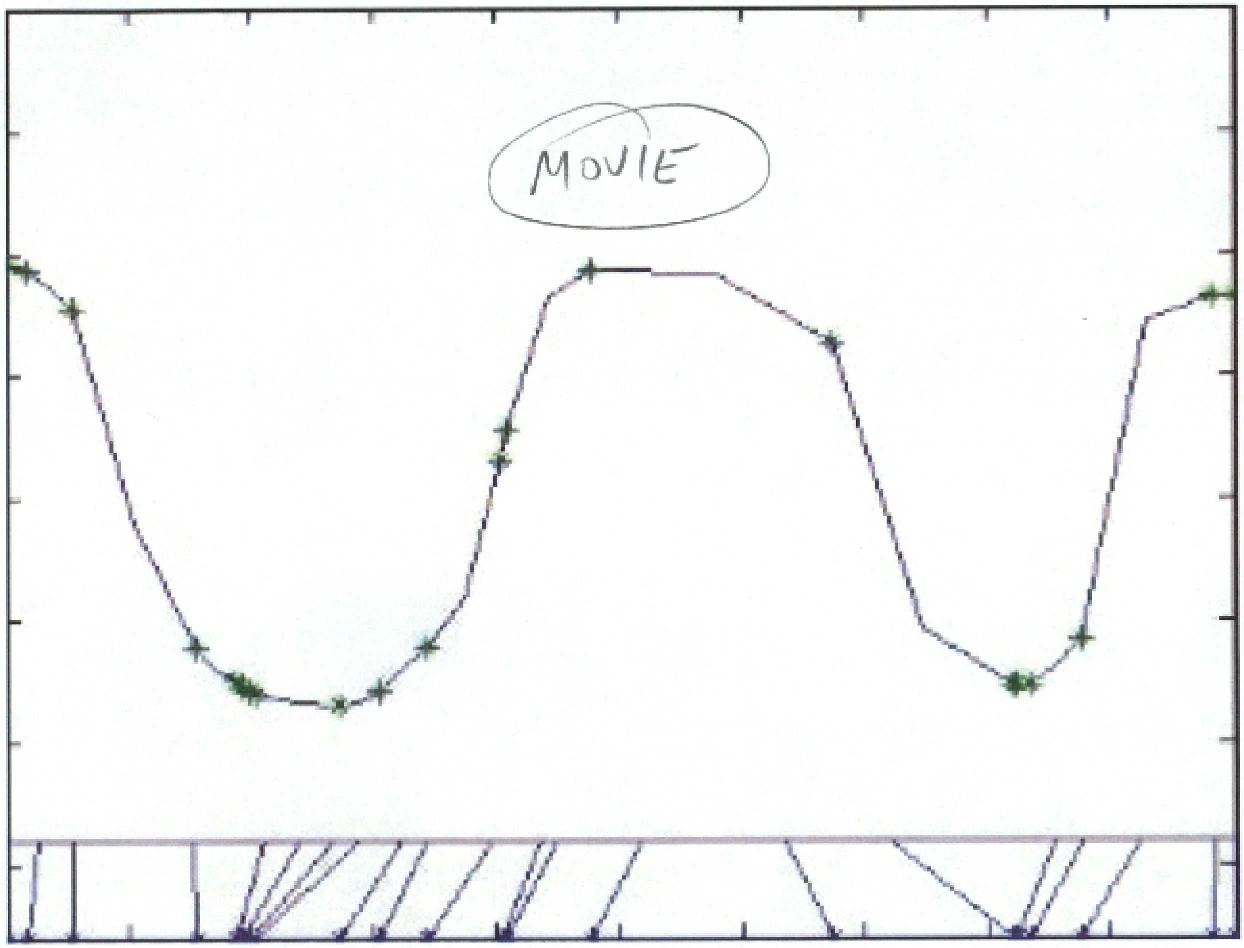


$$u_t - \Delta u = 16$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = 0$$

Simple Problem 2

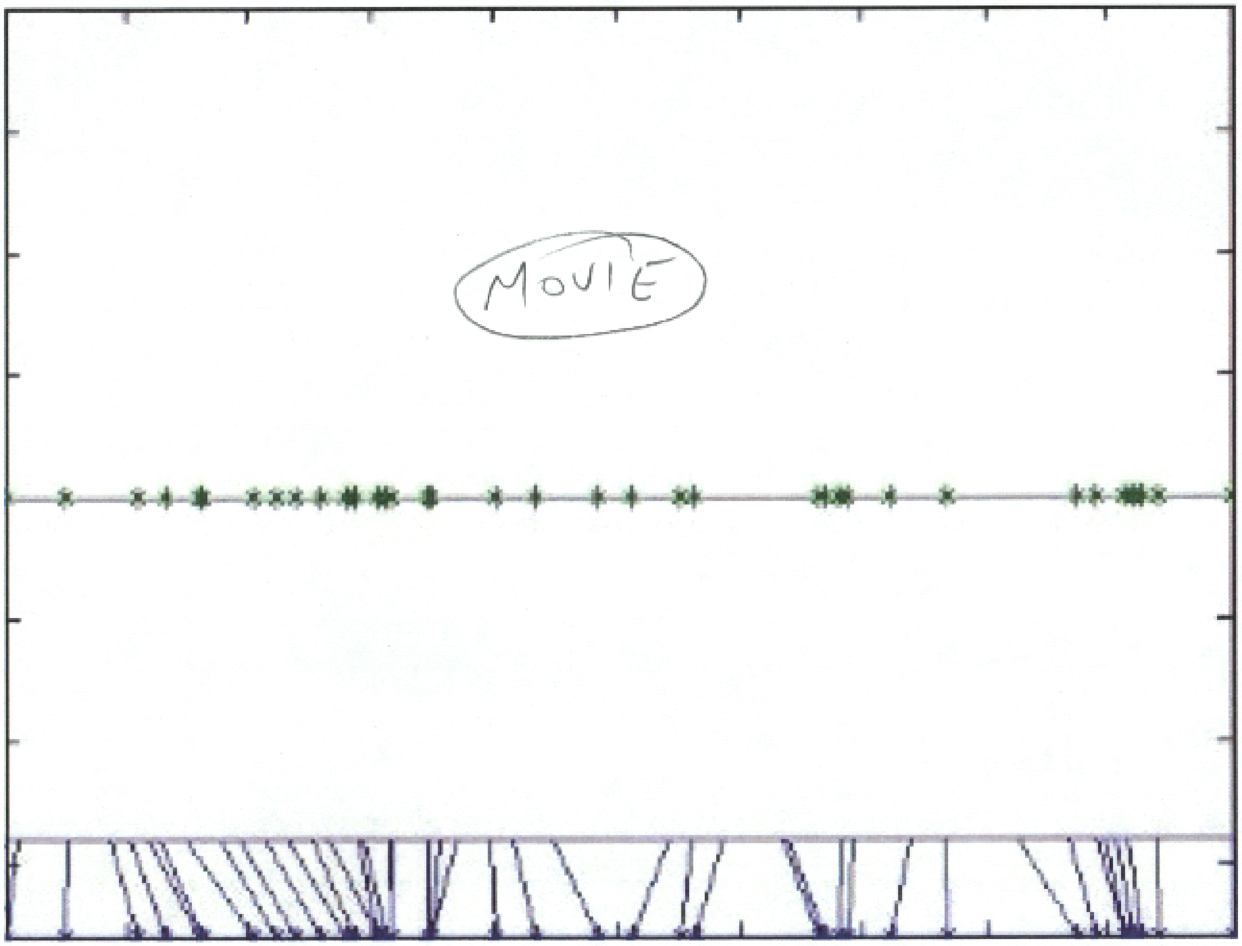


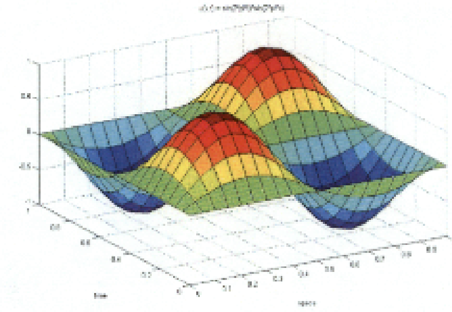
$$u_t - \Delta u = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = v(x)$$

Hard Problem 1



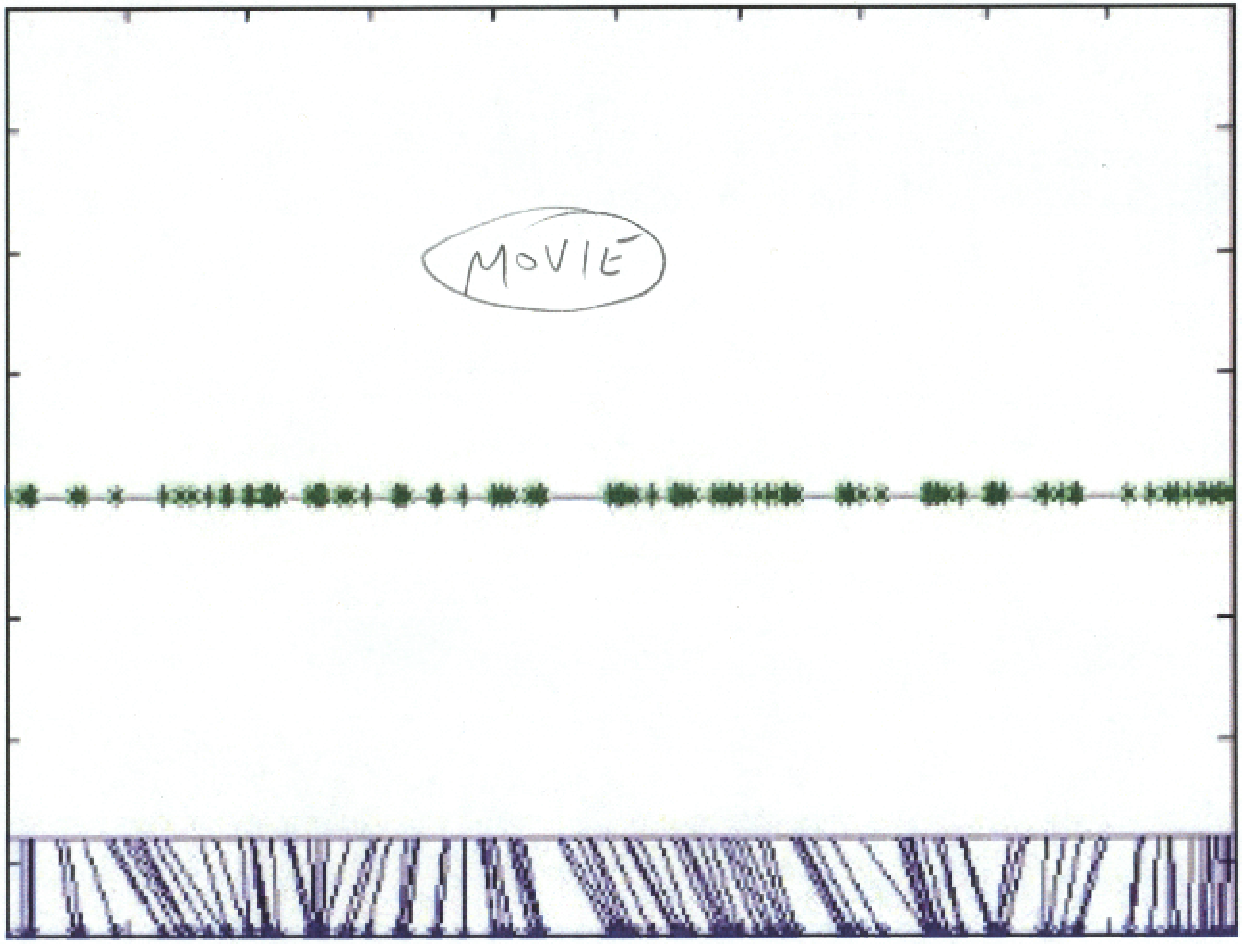


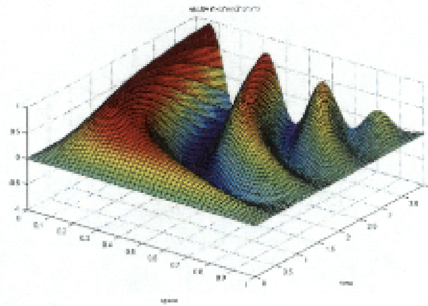
$$u_t - \Delta u =$$
$$2\pi \sin(2\pi x)\cos(2\pi t)$$
$$+4\pi^2 \sin(2\pi x)\sin(2\pi t)$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = 0$$

Exact Solution:

$$u(x,t) = \sin(2\pi x)\sin(2\pi t)$$

Hard Problem 2





$$u_t - \Delta u =$$
$$2\pi x(1-x)\cos(2\pi xt)$$
$$+4\pi t\cos(2\pi xt)$$
$$+(1-x)\left(2\pi t\right)^2\sin(2\pi xt)$$

$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = 0$$

Exact Solution:

$$u\left(x,t\right) = (1-x)\sin(2\pi xt)$$

Error Norms

$$\left\| u(x,t) - u_h(x,t) \right\|_{L_\infty^{\text{time}} L_2^{\text{space}}} = \left\| \sqrt{\int (u - u_h)^2 \, dx} \right\|_{L_\infty^{time}}$$

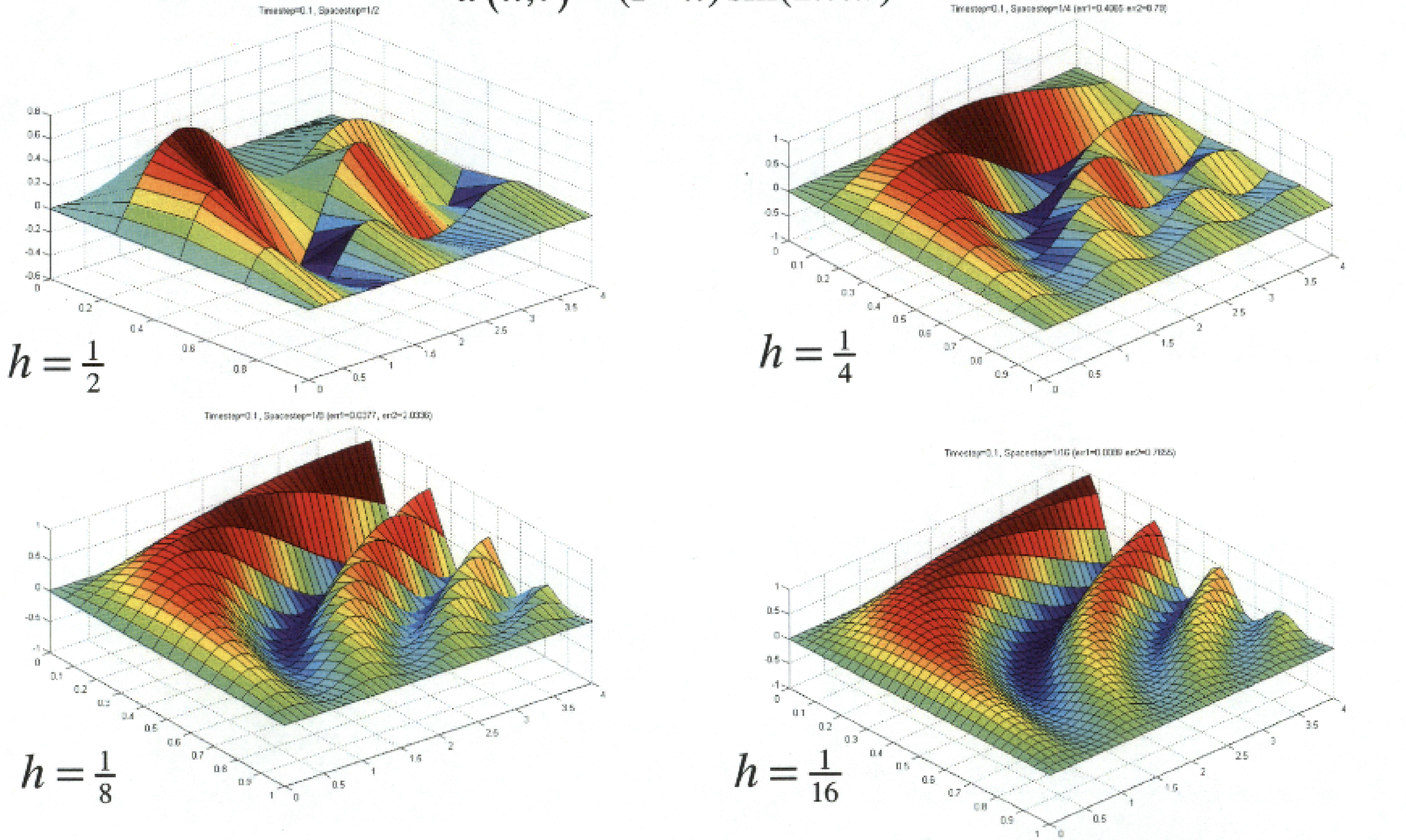
$$\left\| u(x,t) - u_h(x,t) \right\|_{L_2^{\text{time}} H_1^{\text{space}}} = \sqrt{\int \int (\nabla u - \nabla u_h)^2 + (u - u_h)^2 \, dx dt}$$

$$\left\| u(x,t) - u_h(x,t) \right\|_{H_1^{\text{time}} L_2^{\text{space}}} = \sqrt{\int \int \left(u_t - (u_h)_t\right)^2 + (u - u_h)^2 \, dx dt}$$

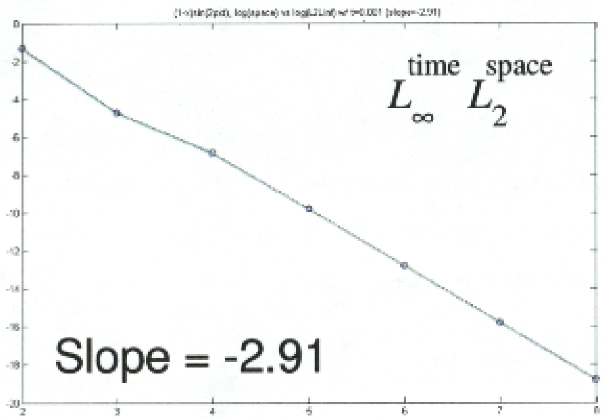
Quadratic interpolation achieves $\left\| (u - u_h)^{(k)} \right\| \le C h^{3-k} \left\| u \right\|_{H_3}$

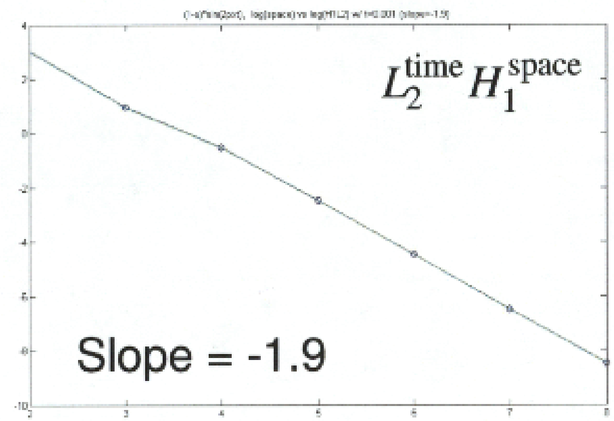
Fix $\Delta t = 0.001$ decrease h

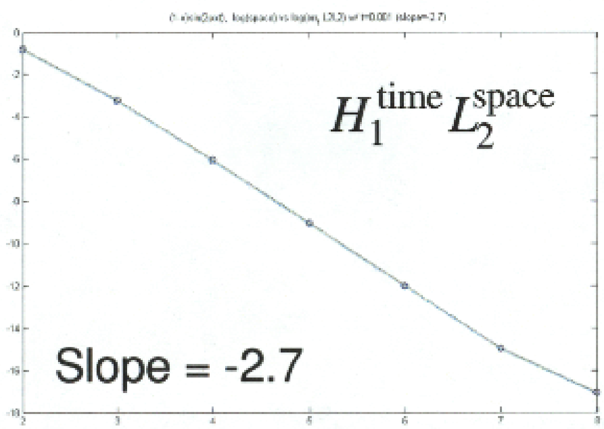
$$u(x,t) = (1-x)\sin(2\pi xt)$$



Log Space Elements vs. Log Err





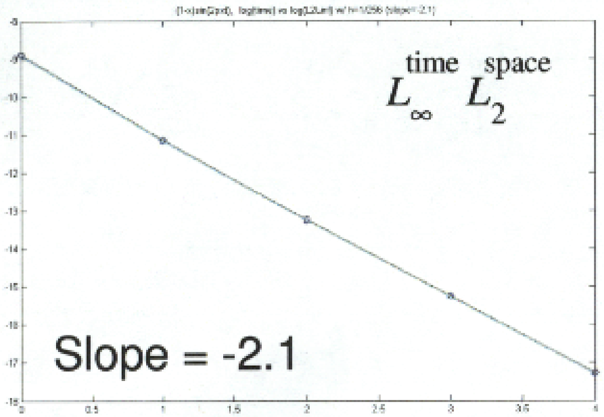


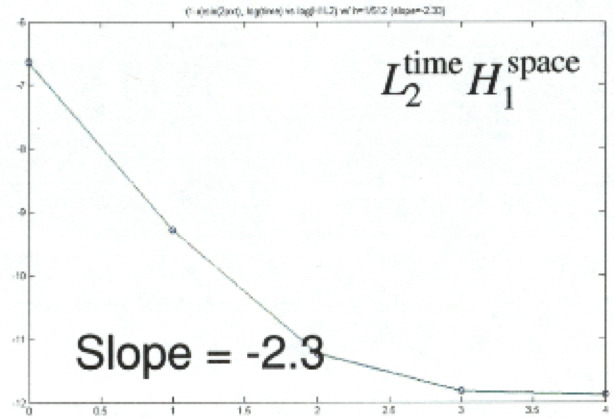
$$u(x,t) = (1-x)\sin(2\pi x t)$$

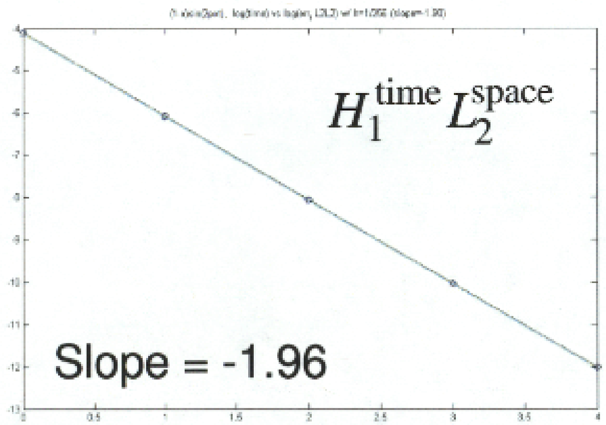
Fix $\Delta t = 0.001$ decrease h

$$h = \left\{\tfrac{1}{4}, \tfrac{1}{8}, \ldots, \tfrac{1}{256}\right\}$$

Log Time Elements vs. Log Err





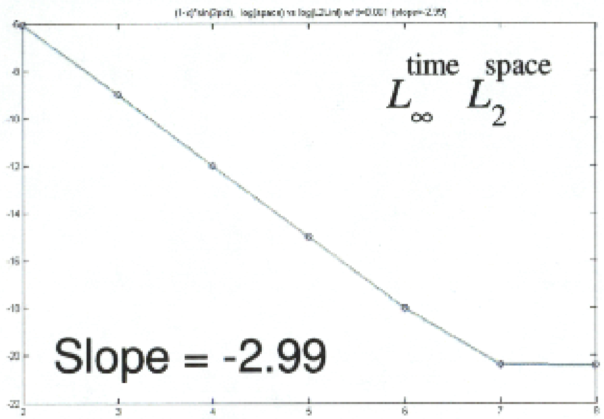


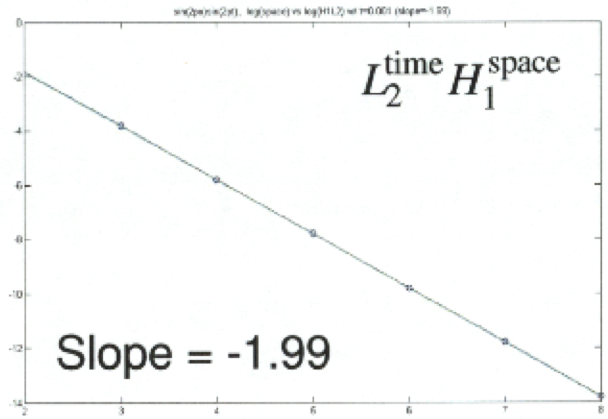
$$u(x,t) = (1-x)\sin(2\pi x t)$$

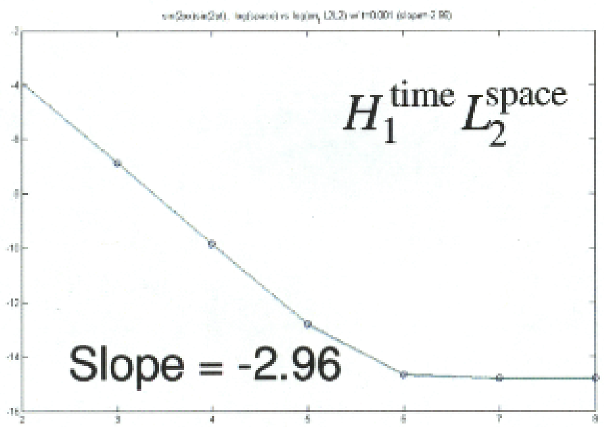
Fix $h = 1/2000$ decrease Δt

$$\Delta t = \left\{0.1, \tfrac{0.1}{2}, \tfrac{0.1}{4}, \tfrac{0.1}{8}, \tfrac{0.1}{16}\right\}$$

Log Space Elements vs. Log Err





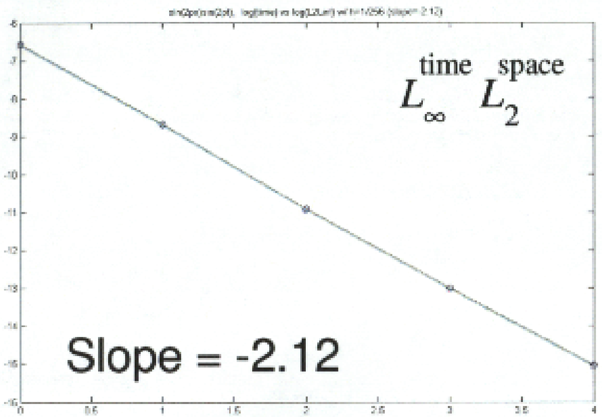


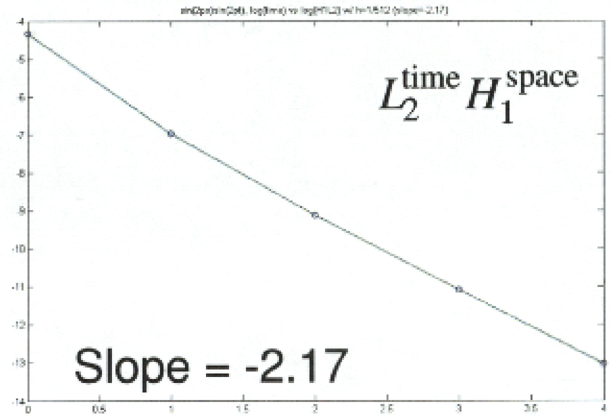
$$u(x,t) = \sin(2\pi x)\sin(2\pi t)$$

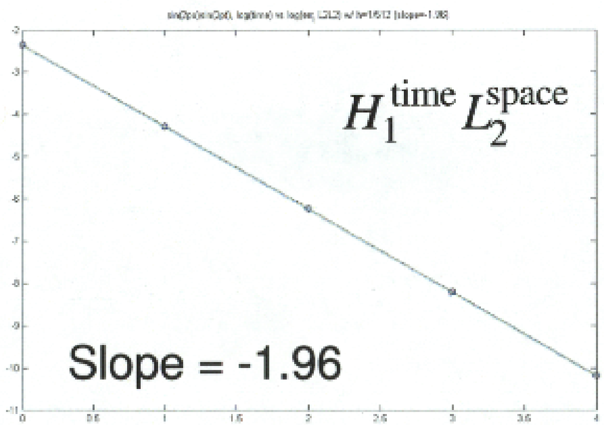
Fix $\Delta t = 0.001$ decrease h

$$h = \left\{\tfrac{1}{4}, \tfrac{1}{8}, \ldots, \tfrac{1}{256}\right\}$$

Log Time Elements vs. Log Err







$$u(x,t) = \sin(2\pi x)\sin(2\pi t)$$

Fix $h = 1/1000$ decrease Δt

$$\Delta t = \left\{0.1, \tfrac{0.1}{2}, \tfrac{0.1}{4}, \tfrac{0.1}{8}, \tfrac{0.1}{16}\right\}$$

Hypothesized Error Bound

$$\left\|(u-u_h)^{(k)}\right\| \le Ch^{3-k}\left\|u\right\|_{H_3}$$

for space derivatives

k=0 and k=1

$$\left\|(u-u_h)^{(k)}\right\| \le Ch^{2}\left\|u\right\|_{H_3}$$

for time derivatives

k=0 and k=1

Future Work

Instead of using second order TR-BDF, use a third order Diagonally Implicit Runge-Kutta Method to see if we can get better accuracy in time.

$$v_1 = f\left(u_k + h\beta_{11}v_1\right)$$
$$v_2 = f\left(u_k + h\beta_{21}v_1 + h\beta_{22}v_2\right)$$
$$u_{k+1} = u_k + h\gamma_1 v_1 + h\gamma_2 v_2$$

$$\gamma_1 = \tfrac{12}{12+\delta^2},\quad \gamma_2 = \tfrac{\delta^2}{12+\delta^2}$$
$$\beta_{11} = \tfrac{1}{2} - \tfrac{\delta}{12},\quad \beta_{21} = \tfrac{2}{\delta},\quad \beta_{22} = \tfrac{1}{2} - \tfrac{1}{\delta}$$

Let $\delta = 3$

$$u_{k+\frac{1}{2}} = u_k + \tfrac{h}{2} f\left(\tfrac{1}{2}u_{k+\frac{1}{2}} + \tfrac{1}{2}u_k\right)$$

$$u_{k+1} = -\tfrac{1}{7}u_k + \tfrac{8}{7}u_{k+\frac{1}{2}} + \tfrac{3h}{7} f\left(-\tfrac{5}{18}u_k + \tfrac{8}{9}u_{k+\frac{1}{2}} + \tfrac{7}{18}u_{k+1}\right)$$