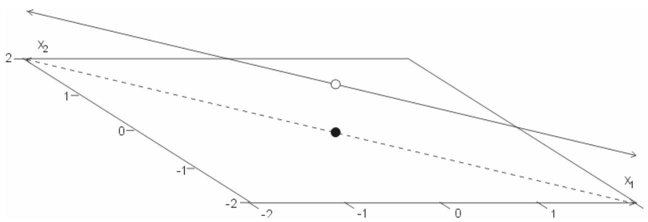
Examples of different types of differentiability.

(iv) Frechet Differentiable $\Rightarrow$ (iii) Gateaux Differentiable $\Rightarrow$ (ii) All Directional Derivatives Exist $\Rightarrow$ (i) Partial Derivatives Exist
However, the converses of the above three implications are not true. Below are counterexamples to disprove all three.

Example 1. To disprove (i) $\Rightarrow$ (ii). Consider $f_2 = \left\{ \begin{matrix} \frac{1}{2} & \text{for } x_1 + x_2 = 0 \text{ and } (x_1, x_2) \neq 0 \\ 0 & \text{for } x_1 + x_2 \neq 0 \text{ or } (x_1, x_2) = 0 \end{matrix} \right\}$.

$D_1 f_2 = D_2 f_2 = 0$. Therefore we have all the partial derivatives at $(0,0)$.

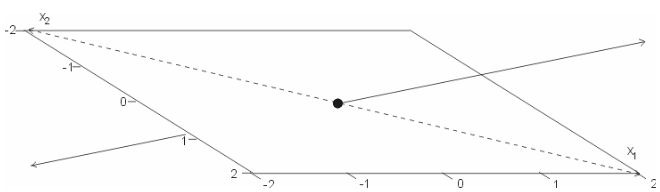
However, we don't have all the directional derivatives at $(0,0)$. $D_u f_2(0,0)$ doesn't exist for $u = \frac{1}{\sqrt{2}}[1 \quad -1]^T \in \mathbb{R}^2$



Example 2. To disprove (ii) $\Rightarrow$ (iii). Consider $f_3 = \left\{ \begin{matrix} \frac{1}{2} x_1 & x_1 - x_2 = 0 \\ 0 & x_1 - x_2 \neq 0 \end{matrix} \right\}$.

$D_u f_3 = 0$ for $u \neq v = \frac{1}{\sqrt{2}}[1 \quad -1]^T \in \mathbb{R}^2$ and $D_v f_3 = \frac{1}{\sqrt{2}}$. Therefore we have all the directional derivatives.

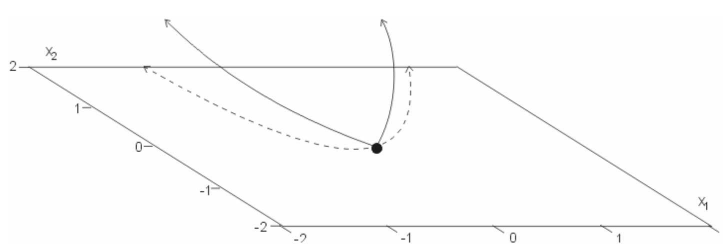
However the Gateaux derivative does not exist at $(0,0)$ since there exists no $g \in L(\mathbb{R}^2, \mathbb{R})$ such that $g(w) = D_w f_3(0,0) \quad \forall w \in \mathbb{R}^2$



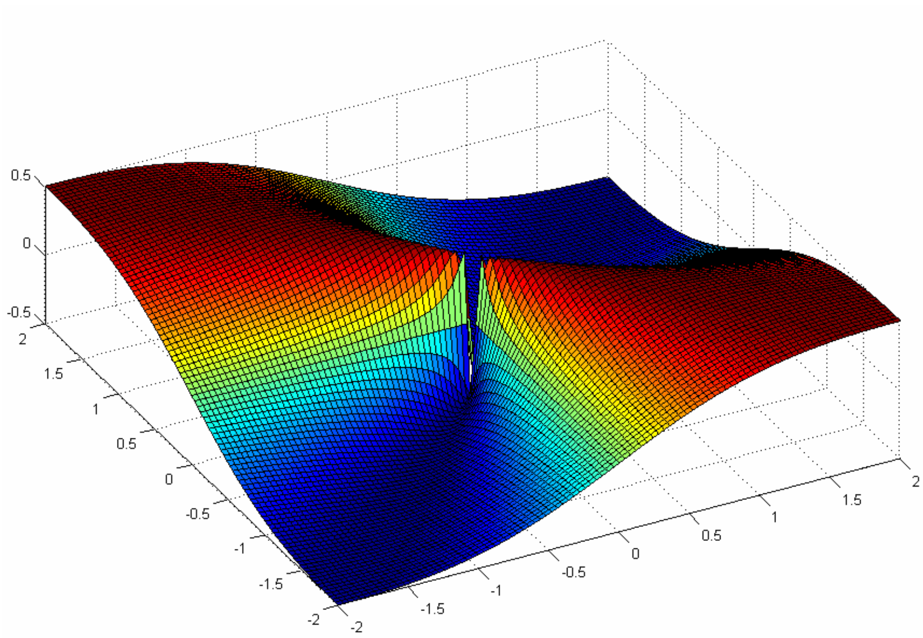
Example 3. To disprove $(iii) \Rightarrow (iv)$. Consider $f_4 = \left\{ \begin{matrix} \frac{1}{2}|x_1| & \text{for } x_2 - x_1^2 = 0 \\ 0 & \text{for } x_2 - x_1^2 \neq 0 \end{matrix} \right\}$.

$D_u f_4 = 0 \ \forall w \in \mathbb{R}^2$. Therefore the Gateaux derivative exists, $f_4'(0,0) = 0 \in L(\mathbb{R}^2, \mathbb{R})$.

However the Frechet derivative does not exist at $(0,0)$ since $\lim_{|h| \to 0} \frac{|f((0,0)+h) - f(0,0)|}{|h|} = \frac{1}{2} \neq 0$ for $h(t) = (t, t^2)$ with $t \to 0+$

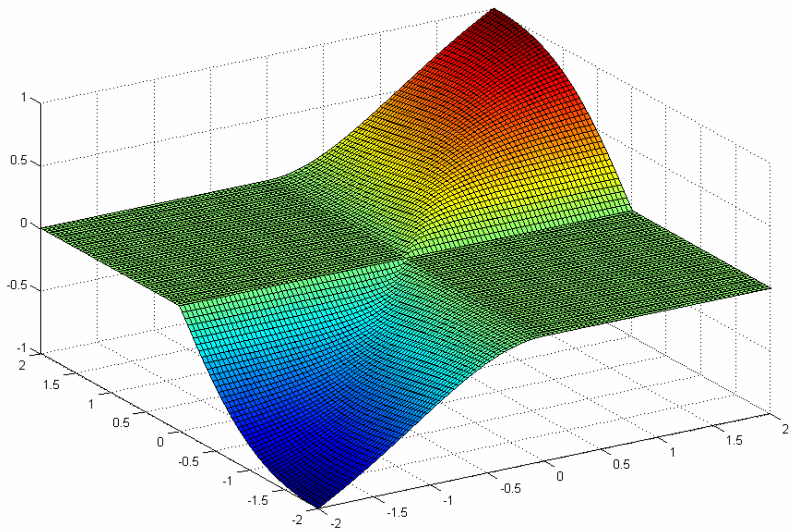


On the next page, you will find functions described that extend continuity to these three counterexamples.



$$f(x)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

At the origin, this function has all of its partial derivatives but no other directional derivatives for the same reasons as example 1.



$$g(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2} & xy>0\\ 0 & xy\leq 0\end{cases}$$

At the origin, this function has all of its directional derivatives but is not Gateaux Differentiable for reasons similar to example 2.

$$h(x,y)=\begin{cases}\dfrac{\left|x^3\right|y}{x^4+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

At the origin, this function is Gateaux Differentiable and continuous but is not Frechet Differentiable for the same reasons as example 3.