PHYSICAL REVIEW D VOLUME 10, NUMBER 10 15 NOVEMBER 1974

Inertial effects in the gravitational collapse of a rotating shell*

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(Received 8 July 1974)

Inertial dragging effects of slowly rotating masses in asymptotically flat spaces are well known for the case of a stationary distribution of matter. In the present work we investigate a more general example in which angular acceleration of the matter is present. We solve the Einstein field equation for the case of the free-fall collapse of a rotating dust shell. The solution is exact, through terms of first order in ω (the angular velocity of the shell), for shells of arbitrary rest mass and radial velocity. The inertial properties of the flat interior region of this solution are discussed. Among the problems elucidated by this solution is the question of whether the inertial effects are instantaneous or retarded as viewed from infinity.

I. INTRODUCTION

The analysis of inertial-frame behavior near rotating bodies has been motivated by two closely related considerations. One is the desire to understand in a qualitative and physical way the effects of rotation in general relativity. The second is that such configurations appear well suited to test to what extent the elusive Mach's principle is contained in general relativity.

In the past, discussions of rotating configurations of mass m and angular momentum J have confined their attention to (a) "slow rotation," $J/m^2 \ll 1$ (Refs. 1–3), and (b) stationary or adiabatically collapsing configurations.[4] In the present paper we retain restriction (a) but allow rapid collapse. We consider a particular idealized case of collapsing, slowly rotating matter, which contains some features of the more complicated problem of the gravitational field of a rotating collapsing star. The relevance of this example to Mach's principle is less apparent and deserves some explanation.

Although today there are still different and incompatible views of Mach's principle,[5] there is considerable agreement on what constitutes Machian effects. Such effects are usually found in their purest form in a region of flat space, where gravitational waves are guaranteed to be absent. We therefore consider the flat region inside a slowly rotating, collapsing shell of matter. In this region there is no question of what we mean by inertial frames; they are the Lorentz frames of special relativity.

Mach proposed that the relative rotation of such inertial frames should be measured with respect to directions defined by the light received from the "fixed stars," i.e., objects at rest in inertial frames of some asymptotic regions. In the present discussion we assume that the asymptotic region is also flat; more realistically, one could in principle replace it by a Friedmann-like cosmology joined smoothly onto the external field of the shell at some finite radius.[6] One would expect then, on the basis of Mach's ideas, that the inertial frames within the shell would be affected both by the rotation of the nearby shell and by the distant matter of the universe.

The discussions in the literature have shown that according to general relativity the inertial frames inside the shell are affected by the motion of the shell, and in fact rotate with respect to the inertial frames at infinity. Some communication with infinity, typically via light signals from the fixed stars, is necessary to define the rotation of the inertial frames within the shell.[3] The Machian rotation of local inertial frames is therefore a highly nonlocal effect. It makes sense, therefore, to ask whether the inertial-frame rotation is related to matter in the usual retarded "causal" way,[7,8] or in the "instantaneous" fashion[9,10] suggested by the spacelike formulation used in some versions of Mach's principle.

The stationary or quasistationary examples discussed in the literature cannot, however, easily distinguish the causality character of different formulations of Mach's principle. Therefore, we consider here a case where the frame dragging changes quickly with time: the free-fall expansion or collapse of a slowly rotating spherical shell of matter. We solve Einstein's equations, to first order in the angular velocity of the shell, for the corresponding metric, assuming asymptotic flatness and no incoming waves. We can then determine which view of inertia gives the simplest account of the inertial dragging exhibited by this solution in the interior of the shell.

II. SPACE-TIME INSIDE AND OUTSIDE THE SHELL

The slowly rotating, nearly spherical, collapsing shells to be considered here are first-order perturbations of nonrotating, spherical, collapsing shells. The unperturbed geometry can therefore be taken to be spherically symmetric, and by Birkhoff's theorem it must be the Schwarzschild geometry in the vacuum exterior region. Regularity at the origin requires flat space-time in the vacuum interior of the shell.

The perturbations, due to the shell's rotation, of the stress-energy and metric tensors can be expanded in tensor spherical harmonics. We use here the stress-energy tensor of a shell of dust $[T^{\mu\nu}=\rho(r,t)u^\mu u^\nu]$. When this stress-energy tensor is expanded to include only the first-order terms in the angular velocity of the shell, ω, we find that this perturbation contains only $l=1$, $m=0$ magnetic parity components (for a discussion of spherical tensor harmonics see Zerilli[11]). Since the perturbed Einstein equations decouple various l, m and parity modes, one obtains for this problem only $l=1$, $m=0$ magnetic parity contributions to the perturbed metric in the absence of external gravitational radiation from infinity. Zerilli[11] has shown that all perturbations of order $l=0$ or 1 are stationary. (However, to describe the stationary interior region we shall choose a nonstationary rotating coordinate system as described below.) Thus, in the entire vacuum region the geometry has the familiar[2] form appropriate to stationary slowly rotating configurations,

$$\begin{aligned} ds^2 &= {}^4g_{\mu\nu}dx^\mu dx^\nu \\ &= -V^2dt^2 \\ &\quad +\psi^4[dr^2+r^2d\theta^2+r^2\sin^2\theta(d\phi-\Omega dt)^2]. \end{aligned} \tag{1}$$

Here the functions V and ψ contain no perturbation terms, because the perturbed metric can be cast in a gauge in which ${}^4g_{\phi t}$ contains the only nonzero perturbation to this order.[11] Thus V and ψ are the Schwarzschild metric functions. For Ω we choose the solution of the first-order perturbation equations which is regular at the origin inside the shell, and the solution which falls to zero at infinity outside the shell. Thus we have (with m = total shell mass, J = total shell angular momentum[4])

inside the shell:

$$\begin{aligned} &\psi=\psi_-=\text{constant}, \\ &V=V_-(t), \\ &\Omega=\Omega_-(t); \end{aligned} \tag{2a}$$

outside the shell:

$$\begin{aligned} &\psi=\psi_+=1+m/2r, \\ &V=V_+=(2r-m)/(2r+m), \\ &\Omega=\Omega_+=2J/r^3\psi^6. \end{aligned} \tag{2b}$$

The arbitrary functions $V_-(t)$ and $\Omega_-(t)$ appearing in the inside solution allow description of this flat space in terms of arbitrarily rotating coordinates. To fix these coordinates, and hence these functions, we demand that the t, θ, and ϕ coordinates be continuous across the shell. Thus Ω_- measures the rotation of the interior inertial frame as seen from infinity.

III. MATCHING CONDITIONS ACROSS THE SHELL

Our aim is to determine the interior geometry, i.e., $V_-(t)$, $\Omega_-(t)$, in terms of the behavior of the shell. Let the shell's angular velocity (as measured from infinity) be $\omega(t)$, its density be $\sigma(t)$, and its radius be $R(t)$. These functions can be determined by matching the inside and outside solution across the shell, if the initial values (in the outside coordinates) $R(t_0)=R_0$, $(dR/dt)(t_0)=0$, $\sigma(t_0)=\sigma_0$, and $\omega(t_0)=\omega_0$ are given. In addition we assume $\psi_+(R_0)=\psi_-(R_0)$, i.e., that the r coordinate be continuous at $t=t_0$. (Note that in general the r coordinate will not be continuous.) The matching conditions are most easily expressed in the formalism developed by Israel[12]: Let g_{ij} and K_{ij} be the intrinsic and extrinsic geometry, respectively, of the timelike hypersurface Σ which represents the shell [$r=R(t)$ in "outside" coordinates], and let S_{ij} be the surface energy tensor of Israel[12] defined on Σ. For the dust shells considered here, $S^{ij}=\sigma u^i u^j$, where u^i are the components of the four-velocity of the dust projected into the coordinate system of Σ, $u^i=(-g_{tt})^{1/2}(1,0,\omega)$. Then the matching conditions and the coordinate continuity conditions of Sec. II demand that the intrinsic geometry be continuous, and that the discontinuity in the extrinsic curvature be determined by

$$K^+_{ij}-K^-_{ij}=8\pi(S_{ij}-\tfrac{1}{2}g_{ij}S). \tag{3}$$

The unperturbed case of a nonrotating collapsing shell has been treated via this formalism by Israel.[12] In the slowly rotating case we therefore expect to rederive his results, expressed in our coordinates, from the diagonal components of the equations; new results will come from the off-diagonal components. Because of the spherical symmetry, the diagonal components reduce to two intrinsic equations, and one of the extrinsic equations can be replaced by the conservation law within the surface

$$S^{ij}{}_{;j}=0. \tag{4}$$

The equations then become

$$\begin{aligned} &g^+_{tt}=g^-_{tt}, \\ &g^+_{\theta\theta}=g^-_{\theta\theta}, \\ &K^+_{\theta\theta}-K^-_{\theta\theta}=4\pi\sigma R^2(1+m/2R)^4. \end{aligned} \tag{5}$$

Detailed expressions for the quantities considered here in terms of the metric functions of Eq. (1) are found in the Appendix, as well as a few words about the notation used here. To write their solutions we define

$$\psi(r) \equiv 1 + m/2r\,,\qquad V(r) \equiv (2r - m)/(2r + m)\,, \tag{6}$$

and find

$$\psi_- = \psi(R_0)\,,$$
$$V_- = V^2(R)\left(1 + \frac{m}{2R}\,\frac{\psi^2(R_0)}{\psi^2(R)}\right)\left(1 - \frac{m}{2R}\,\frac{\psi^2(R_0)}{\psi^2(R)}\right)^{-1},$$
$$\sigma = \frac{m}{4\pi}\,\psi(R_0) R^{-2}\psi^{-4}(R)\,. \tag{7}$$

The radial equation of motion of the shell is also derived from these equations and is given by

$$\dot{R}^2 = \frac{V^2(R)}{\psi^4(R)}\left(1 - \frac{m^2}{4R^2}\,\frac{\psi^2(R_0)}{\psi^2(R)}\right)\left(1 - \frac{\psi^2(R_0)}{\psi^2(R)}\right) \times \left(1 - \frac{m}{2R}\,\frac{\psi^2(R_0)}{\psi^2(R)}\right)^{-2}. \tag{8}$$

This solution agrees with that of Israel,[12] transformed to our isotropic coordinates.

The essentially new information in the rotating case comes from the t, ϕ components of the matching conditions:

$$g^+_{t\phi} = g^-_{t\phi}\,,\qquad K^+_{t\phi} - K^-_{t\phi} = 4\pi\sigma_0(\Omega - 2\omega)\sin^2\theta\,. \tag{9}$$

These equations determine (a) the value of the constant J (total angular momentum) in terms of the initial values,

$$J = \tfrac{1}{2}\omega_0[R_0\psi^2(R_0)]^3\left[1 + \frac{3(2R_0 - m)}{8m}\right]^{-1}, \tag{10}$$

(b) the time dependence of $\omega(t)$ necessary to keep J constant,

$$\omega(t) = \frac{2J}{[R\psi^2(R)]^3} \times \left\{1 + \frac{3RV^2(R)\psi^2(R)}{4m}\left[1 - \frac{m}{2R}\,\frac{\psi^2(R_0)}{\psi^2(R)}\right]^{-1}\right\}, \tag{11}$$

and (c) the "induced rotation" of the interior inertial frames,

$$\Omega_-(t) = 2J[R\psi^2(R)]^{-3}\,. \tag{12}$$

Equations (10)–(12) represent the exact solution, for the case of a shell, of the slow-rotation dynamic problem which was considered in the limit of adiabatically slow collapse by Cohen.[4]

IV. DISCUSSION

This solution allows us to trace the history of a slowly rotating collapsing dust shell down to the vicinity of the horizon, where strong gravitational fields are present. Since we have neglected terms of order Ω^2 in comparison with all diagonal metric components, our solution ceases to be valid when $V^2 \sim m^2\Omega^2$, i.e., near the "ergosphere." The new features of this solution, the effects of collapse on rotation, are expressed in terms of the quantities $\omega(t)$ and $\Omega(r, t)$ from Eqs. (11), (12), and (2b). These quantities can be related to physical measurements as follows: We place an isotropic light source in the center of the collapsing rotating sphere. Its emitted light has zero conserved angular momentum. We use this light to project a marker on the sphere (such as a small hole) to infinity. If the sphere were not collapsing, an observer at infinity could measure the rotation rate $\omega(t)$ by simply counting flashes coming from the shell, much as we measure pulsar rotation rates. In the stationary case, we could similarly measure $\Omega_-(t)$, the inertial frame dragging inside the shell: We place a searchlight at rest in the inertial frame at the center of the shell. We project its light beam in the equatorial plane. If the shell were transparent, the observer at infinity could observe this light beam, and measure $\Omega_-(t)$ by counting flashes.

When the shell is collapsing, in addition to rotating, the situation is not as simple. The coordinate system which we have used here employs a ϕ coordinate which is analogous to the ϕ coordinate used in the Boyer-Lindquist form of the Kerr metric. It is well known[13] that, in this coordinate system, photons with vanishing angular momentum will loop around the origin many times if they originate near the ergosphere. Nevertheless, even in the collapsing case the Ω_{obs} observed at infinity as described above equals the Ω_- of the solution; the ω_{obs} differs from the coordinate angular velocity ω only by an amount which vanishes when the shell reaches its maximum radius R_0, and when it nears the horizon $\frac{1}{2}m$. To show this, we consider two light geodesics which cross the shell at t and $t+\delta t$, when the marker is at position R, ϕ and $R+\delta R$, $\phi+\delta\phi$, respectively. The difference in angle $\Delta\phi$ (as observed at infinity) between the geodesics consists of (a) the angle $\delta\phi$ through which the shell has rotated, and (b) the additional angle $\delta\phi_A$ through which the second light beam must rotate in traveling from $R+\delta R$ to R. The difference in time of reception (at infinity) consists of (a) the time δt between the geodesics crossing the shell, and (b) the additional time δt_A needed by the second light beam to travel from $R+\delta R$ to R. The observed angular velocity at infinity is therefore given by

$$\omega_{\rm obs} = \frac{\delta\phi + \delta\phi_A}{\delta t + \delta t_A}$$

$$= \left[\omega - \dot{R}\,\frac{\psi^2(R)\Omega_-}{V(R)}\right]\left[1 - \dot{R}\,\frac{\psi^2(R)}{V(R)}\right]^{-1}. \quad (13)$$

The observed dragging of inertial frames $\Omega_{\rm obs}$ can be shown in the same manner to be given by

$$\Omega_{\rm obs} = \Omega_- . \quad (14)$$

From Eq. (13) we see that, after the time t_0 of maximum expansion, $\omega_{\rm obs}(t)$ continually increases, as expected from angular momentum conservation, reaching the finite value $\omega(R = \frac{1}{2}m)$ as the shell approaches the horizon. The inertial-frame rotation $\Omega_{\rm obs}(t)$ follows a similar behavior, staying always less than $\omega_{\rm obs}$ but approaching the latter near the horizon. In particular, $\Omega_{\rm obs}(t)$ is determined by the "instantaneous" radius of the shell $R(t)$. That is, as seen from infinity, the inertial frames within the shell rigidly rotate at the angular velocity $\Omega_{\rm obs}$: There are no retardation effects between the shell and the inertia of a gyroscope at its center. This of course does not contradict any physical causality principle, since Ω_- can be considered to be merely the angular velocity of a coordinate system for the interior flat region. However, it is this coordinate system which is most directly related to effects observable from infinity, as explained above. Thus another view, more closely related to Machian ideas, is equally consistent, in which $\Omega_-(t)$ is observable but highly nonlocal, so that a local causality principle does not apply to it.

The observed relative angular velocity of the interior inertial frame and the matter of the shell $\omega_{\rm obs} - \Omega_{\rm obs}$ approaches zero as the shell crosses the horizon. Nothing unusual, however, is seen by an observer inside, or falling with, the shell. The relative angular velocity as measured by an observer in the shell is given by

$$V_-^{-1}(\Omega_-(R) - \omega(t)) = -\left(\frac{3R}{4m}\right)\psi^2(R)\Omega_-(R) \times\left(1 + \frac{m}{2R}\,\frac{\psi^2(R_0)}{\psi^2(R)}\right). \quad (15)$$

This quantity is finite at the horizon. In a Machian interpretation, the inside observer would have to attribute the inertia seen by him as determined both by the presence of the shell and by the distant matter in the universe (or spacelike infinity in our example here). Again, the measured rotation rates are determined by the instantaneous parameters of the shell. We emphasize again that our results may need modification in the region $V^2 \sim m^2\omega^2$ where our perturbation expansion breaks down.

It is interesting to note that the results for the relative angular velocity $\Omega - \omega$ which we have derived here for the collapsing case differ qualitatively from those derived by Brill and Cohen[2] for the stationary case. In the solution reported here, $\Omega - \omega$ measured by an observer within the shell is shown to approach a finite nonzero value, as the shell approaches the horizon. In the stationary case it was shown that if one considers a sequence of stationary solutions with constant mass and angular momentum, $\Omega - \omega$ vanishes as seen by an observer inside the shell as the radius of the shell is reduced to the "Schwarzschild" radius. This difference was anticipated and discussed by Cohen.[4] The difference lies in the fact that the stresses in the stationary sequence of shells become very large for shells near the horizon. These stresses affect the coupling between ω and Ω and force $\Omega - \omega$ to go to zero in this limit.

These results fit most simply with the "spacelike" formulation of Mach's principle. In this formulation it is, in general, the coordinate functions, N, and N^i that are to be determined in a Machian fashion from the dynamical variables on a spacelike hypersurface. The constraint equations determining N and N^i are purely spacelike equations, and in this sense all Machian effects will be related to the instantaneous values of the dynamical variables.[9]

ACKNOWLEDGMENT

The authors would like to thank Robert M. Wald for several helpful discussions pertaining to this work.

APPENDIX

A word about notation: The 4-space indices μ, ν run over the values t, r, θ, and ϕ. As coordinates in the 3-space Σ we use the coordinates t, θ, and ϕ restricted to Σ; the indices i, j run over these values. Since the 4-vector $\partial/\partial t$ is not tangent to Σ, care must be taken to distinguish components of 4-tensors, like ${}^4g_{tt}$, from components of 3-tensors on Σ, like g^+_{tt}. The only 4-tensor occurring in the text is ${}^4g_{\mu\nu}$; all the rest are 3-tensors. The 4-coordinates t, r, θ, and ϕ are used both for the interior and exterior region, since it is clear from context which coordinate patch is meant. However, the inner and outer coordinates yield two different sets of coordinates when restricted to Σ as described above. We distinguish the corresponding components by superscript $\pm$. For compactness of notation we have left the coordinate r in some 3-

tensor expressions in cases when some r differentiations, denoted by primes, are still to be carried out. After such differentiation, all 3-tensor components are to be evaluated on the surface Σ, $r_+=R(t)$ or $r_-=R\psi^2(R)\psi^{-2}(R_0)$. We use the notation of an overdot in a similar fashion to denote differentiation with respect to the 4-space time coordinate. Detailed expressions for the 3-tensor components used in the text are given below:

$$g^{\pm}_{tt}=[\dot{R}^2\psi^4-V^2]^{\pm},$$
$$g^{\pm}_{\theta\theta}=[R^2\psi^4]^{\pm},$$
$$g^{\pm}_{t\phi}=-[R^2\psi^4\Omega\sin^2\theta]^{\pm},$$
$$K^{\pm}_{\theta\theta}=-\tfrac{1}{2}[V\psi^{-2}(r^2\psi^4)'(V^2-\dot{R}^2\psi^4)^{-1/2}]^{\pm},$$
$$K^{\pm}_{t\phi}=\tfrac{1}{2}[V^{-1}\psi^{-2}(V^2-\dot{R}^2\psi^4)^{-1/2}\times\sin^2\theta(V^2(r^2\psi^4\Omega)'-r^2\psi^8\dot{R}^2\Omega')]^{\pm}.$$

*Work supported in part by the National Science Foundation under Grant No. GP-25548.

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Behavior of spacelike geodesics in the extended Schwarzschild manifold

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(Received 24 July 1974)

We demonstrate that there exist unstable, spacelike, circular orbits in the Schwarzschild field for all radii in the range $0<r<3m$. The conditions under which spacelike trajectories bend toward or away from the source of the field are derived for the entire r-ϕ plane. We show that any nonradial spacelike geodesic with turning point less than $2m$ will appear spacelike over the entire u-v plane. The scattering and capture cross sections for a particle on a spacelike trajectory are evaluated. Finally, we suggest that there are compelling reasons for rejecting the usual assumption of a global past-future relation in the extended Schwarzschild manifold.

I. INTRODUCTION

In a recent paper Raychaudhuri[1] has found that a tachyon moving radially in a Schwarzschild field experiences an "inverse force of repulsion" and has claimed that a tachyon moving radially inwards "turns back after penetrating inside the Schwarzschild singularity." He has also found that circular orbits with $r<3m$ exist for tachyons, in agreement with Hettel and Helliwell,[2] who found that circular tachyon orbits in the Schwarzschild field were unstable and restricted to the range $2m<r<3m$. Moreover, they have found that in the approximation $m/r\ll1$ tachyons are deflected toward the source of the gravitational field.

Here we shall be concerned with clarifying the properties of spacelike geodesics in the extended Schwarzschild manifold. In particular, we find