THE ASTROPHYSICAL JOURNAL, 208:873–880, 1976 September 15

STATIONARY STARS ARE AXISYMMETRIC

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Received 1976 January 12

ABSTRACT

The final state of thermonuclear evolution of a star may consist of either a black hole or an "ordinary" star such as a white dwarf or neutron star. For the case of black holes, a great deal is known about the structure of this final stationary state; in particular, Hawking has shown that stationary black holes are axisymmetric. An extension of this result is presented which includes the "ordinary" final state of stars. It is shown that a stationary star, consisting of a viscous heat-conducting general-relativistic fluid, must be axisymmetric. For this proof, the star is assumed to be embedded in an asymptotically Minkowskian spacetime manifold. The functions which describe the geometry and the fluid of the star are assumed to satisfy certain smoothness conditions.

Subject headings: hydrodynamics — relativity — rotation — stars: collapsed — stars: interiors

I. INTRODUCTION

What are the properties of the final state of thermonuclear evolution of a star? According to our present understanding, a star may approach, as its final equilibrium configuration, either a black hole or an "ordinary" star containing degenerate matter—i.e., a white dwarf or neutron star. At this time, a great deal is known about the properties of the final states which may be described as stationary black holes. Work has recently been concluded which shows that the Kerr family of black-hole solutions comprises a complete description of this possible endpoint of stellar evolution (see Carter 1972; Robinson 1975). In comparison, much less is known about the possible properties of the "ordinary" final state of evolution. These "ordinary" stationary stars present a far more complex theoretical problem than do the black holes. As the theorem of Carter and Robinson has shown, a complete description of a black hole is given simply by specifying its mass and angular momentum. In the case of a star, however, to these parameters must be added the chemical equation of state. The equation of state depends on the detailed microphysics of the system. Ordinary stars therefore present complexities not present in the black-hole problem. Furthermore, finding the geometrical structure of a star, once its mass, angular momentum, and equation of state have been specified, is a far more complicated problem than for the case of black holes.

This paper represents a step in gaining an understanding of the possible geometrical configurations allowed for the final stationary configuration of stars. We show that stationary stars, like stationary black holes, must be axisymmetric. This additional symmetry greatly reduces the complexity of the problem of determining the structure of the star together with its gravitational field (for reviews, see Thorne 1969 and Bardeen 1972).

We will consider a star composed of a viscous, heat-conducting general-relativistic fluid. The assumption of stationarity for such a system implies that any dissipation due to viscosity, heat conduction, or gravitational radiation has already occurred. The physical content of the result which is presented here—"stationary stars are axisymmetric"—is that after the dissipative processes have acted, such a star must be axisymmetric in its final equilibrium state. This result has been anticipated by analogous results for similar physical situations. The effects of viscosity and of gravitational radiation reaction have been included recently in the study of uniform-density Newtonian ellipsoids (see Chandrasekhar 1969; Press and Teukolsky 1973; Miller 1974). The result of these analyses indicate that when the dissipative effects are taken into account, any nonaxisymmetric ellipsoid (e.g., the Jacobi and Dedekind families) will evolve toward a member of the axisymmetric Maclaurin family of ellipsoids.

In the ultrarelativistic limit, another analogous result has been obtained by Hawking (1972). He has shown that stationary black holes are axisymmetric. Since the result which is presented here uses the full formalism of the general theory of relativity, the method of proof is far more analogous to Hawking's black-hole results than to the Newtonian ellipsoid results described above. Finally, one other analogous result should be mentioned. Pachner and Miketinac (1972) have examined the general-relativistic equations of perfect-fluid hydrodynamics. They conclude that "stationarity implies axisymmetry." Their definition of stationarity is based on the constancy of certain scalars along the flow lines of the fluid. It bears no simple relationship with the definition which is generally used—the existence of a timelike isometry (see Carter 1972). For this reason, the result which is presented here differs from that of Pachner and

Miketinac; the more standard definition of stationarity is used in the present discussion.

The precise mathematical statement of the result which is derived here is given by the

PROPOSITION ("Stationary Stars are Axisymmetric"):

ASSUMPTIONS: *The spacetime* $\mathscr{M}$ *is stationary (nonstatic), nonsingular, and admits a Cauchy surface.* $\mathscr{M}$ *consists of three regions* $\mathscr{M}_1$, $\mathscr{M}_2$, *and* Σ*: the exterior, interior, and surface of the star, respectively.* $\mathscr{M}_1$ *is empty and asymptotically Minkowskian.* $\mathscr{M}_2$ *is filled with a viscous heat-conducting fluid. The coefficients of viscosity* ζ, η, *and the heat conduction coefficient* κ *are positive. The surface* Σ, *which separates* $\mathscr{M}_1$ *and* $\mathscr{M}_2$, *is defined as the surface on which the pressure vanishes. It is assumed to be "compatible" with the exterior geometry. The manifold* $\mathscr{M}$ *is* C^5, *the components of the timelike Killing vector field* η^α *are* C^4, *and the components of the metric tensor are* C^3 *(except on* Σ*).*

ASSERTION: $\mathscr{M}$ *is axisymmetric.*

The technical details of the mathematical proof will be given in the later portions of the paper. At this time, however, it is appropriate to outline briefly the method of proof.

The proof of the main proposition breaks up roughly into two separate and distinct arguments. The first, which is contained in § II, deals with the symmetries of the interior region of the star. The laws of general-relativistic viscous hydrodynamics are shown to imply that, whenever the fluid is stationary, there must be an additional symmetry which is tangent to the flow lines of the fluid. This result comes about in the following way: The stationarity assumption is shown to imply that the fluid must be in a state of thermodynamic equilibrium. Thermodynamic equilibrium is then shown to imply (*a*) the absence of heat flow, and (*b*) the absence of expansion and shear in the fluid flow. If the fluid motion contained shear or expansion, the viscosity of the fluid would tend to damp out this motion; thus, the only equilibrium state must leave the fluid moving rigidly. A result of Pirani and Williams (1962) is then used to show that the rigid motion of the fluid which is derived here implies the existence of a second symmetry of the spacetime (which is tangent to the flow lines of the fluid). Furthermore, it is shown that the second symmetry is linearly independent of the globally timelike symmetry which defines the stationarity of the spacetime.

The second major point of argument is found in § III. It focuses on the problem of showing that the symmetry found to exist inside the star must also exist in the exterior. Symmetries in general relativity theory are equivalent to the existence of Killing vector fields. The problem of showing that the exterior of the star shares the symmetry of the interior, reduces then to the problem of extending a Killing vector field, ξ^α, across the boundary of the star and into the exterior. This extension is performed by applying several theorems from the literature of partial differential equations to the Cauchy problem for the differential equation $\nabla^\alpha\nabla_\alpha\xi^\beta = 0$, on the initial surface Σ. This equation is necessarily satisfied by any Killing vector field in the exterior region of the star; thus it is a natural one to use for the extension of ξ^α. The surface of the star, Σ, is used as the initial surface, on which the Cauchy data (consisting of the values of the field ξ^α and their first derivatives) are defined by continuity from the interior of the star. The existence of this extension is guaranteed by the Cauchy-Kowalewsky theorem. It is shown that an extension obtained in this way is a Killing vector field which commutes with the globally timelike Killing vector. Once extended a short way past the boundary of the star, the field ξ^α can be analytically continued to cover the remainder of the exterior. In this way, the additional symmetry found in the interior of the star is extended to include the entire spacetime.

The remaining problem considered in § III is to show that the additional symmetry corresponds to a rotation, using an argument similar to Hawking's (1972). The spacetime near spacelike infinity behaves asymptotically as flat Minkowski spacetime, whose symmetries are elements of the Poincaré group. A star is not invariant under spacelike translations or velocity boosts. Thus, asymptotically the star admits only time translations and space rotations. The additional symmetry, being linearaly independent of the time translation symmetry (defined by the stationarity of the space), must be some linear combination of a rotation and a time translation. Therefore the star is rotationally or axially symmetric.

This concludes the outline of the proof which is given explicitly below. Before proceeding to the proof below, we note that most of the assumptions used here simply state the currently accepted theoretical descriptions of stars within the framework of general relativity theory. Einstein's theory of general relativity is used throughout this discussion. However, it is likely that analogous results could be obtained using other relativistic theories of gravity, e. g., Brans-Dicke theory. Certain continuity assumptions are required of the various functions in the problem. These are the standard minimal ones used to ensure that all appropriate differential equations are well defined. In addition, one assumption concerning smoothness of the surface of the star Σ is made without physical motivation. This is the "compatibility" of the surface with the exterior geometry. A precise understanding of this assumption can best be had only in the context of the actual proof of the theorem; a detailed discussion of it is therefore left until then. This requirement is a technical point which is required to guarantee an extension of the Killing vector beyond the boundary of the star. It is shown that the condition is necessary, in that if the star is axisymmetric, then the condition is necessarily satisfied.

There is one assumption which is physically rather strong. This assumption is the requirement of stationarity; it is undoubtedly the most severe physical constraint which is used in this discussion. No one expects any physical system to be exactly stationary; one can only hope that the particular system of interest approaches stationary equilibrium sufficiently rapidly that, after a reasonable length of time, the nonstationary features may be ignored. For the case of solar mass sized black holes this assumption is quite reasonable: the time scales for approaching stationarity are of the order of milliseconds (Carter 1972). For the case of stars, on the other hand, the stationarity assumption is probably not as realistic. Even after a star has evolved to its final state as a white dwarf or neutron star, the rate at which the star approaches complete chemical and thermal equilibrium (or even the rate at which differential rotation is damped out) is very slow. Kippenhahn and Möllenhoff (1974) have shown, for example, that in rapidly rotating white dwarf stars, the differential rotation is damped out with a time scale of about 10^5 years, which they show is shorter than the time scale of about 10^6 years over which the star is expected to cool toward thermal equilibrium. Even though no real star can be expected to behave in a stationary, equilibrium way, it is reasonable to expect that a large class of real stars can be described adequately as perturbations of equilibrium models. In general relativity theory very little is really known about the theory of equilibrium rotating stellar models, and it is the purpose of this paper to provide a small contribution to that understanding.

II. STATIONARITY AND THERMODYNAMIC EQUILIBRIUM

This section will derive two results that make explicit the properties of a viscous heat conducting fluid which is assumed to be stationary. The first result, ***A***, shows that stationarity implies thermodynamic equilibrium. The precise statement of this result is given by the following:

A. *A relativistic fluid moving in a nonsingular spacetime which admits a Cauchy surface, and which is stationary, and asymptotically Minkowskian, must be in a state of thermodynamic equilibrium.*

A stationary spacetime admits a timelike vector field, η^α, along which all physical fields are Lie-transported. In particular the metric tensor, $g_{\alpha\beta}$, and the entropy current vector, s^α, have zero Lie derivatives along $\eta^\alpha(\mathscr{L}_\eta g_{\alpha\beta} = 0, \mathscr{L}_\eta s^\alpha = 0)$. Since the spacetime admits a Cauchy surface, there exists a surface τ_0 which is globally spacelike and which intersects every integral curve of the vector field η^α exactly once. A family of surfaces $\tau(t)$ can be defined by letting $\tau(0) = \tau_0$ and then assigning to the surface $\tau(t)$ those points which have an affine parameter t from τ_0 along the integral curves of η^α.

The total entropy of the fluid may be defined via an integral over one of these surfaces:

$$S(t) = \int_{\tau(t)} \sqrt{-g}\, s^\alpha d^3x_\alpha\,. \tag{1}$$

Since the entropy current is Lie-transported along η^α, the total entropy as defined in equation (1) is independent of which surface $\tau(t)$ the integral is performed over. Now consider a region of spacetime Ω whose boundary consists of the two surfaces $\tau(t_1)$, $\tau(t_2)$ plus a piece at spacelike infinity. There is no matter at infinity, so the entropy current vanishes there. Then the integral of the divergence of the entropy current over Ω is zero:

$$\int_\Omega \sqrt{-g}\, \nabla^\alpha s_\alpha d^4x = \int_{\tau(t_1)} \sqrt{-g}\, s^\alpha d^3x_\alpha - \int_{\tau(t_2)} \sqrt{-g}\, s^\alpha d^3x_\alpha = 0\,. \tag{2}$$

The second law of thermodynamics for relativistic fluids,

$$\nabla^\alpha s_\alpha \geq 0\,, \tag{3}$$

implies that the integrand on the left is positive. Therefore, by equation (2) $\nabla^\alpha s_\alpha = 0$; this is the condition for thermodynamic equilibrium. This concludes the proof of result ***A***.

The second result, to be discussed in this section, derives the implications of thermodynamic equilibrium on a viscous heat conducting fluid. The precise statement of this result is:

B. *If a general-relativistic fluid, having nonzero coefficients of viscosity and heat conduction, is in a state of thermodynamic equilibrium, then the equation of state of the fluid is barotropic and the ratio of the four-velocity of the fluid to the temperature is a Killing vector field.*

We begin by reviewing the standard theory of general-relativistic hydrodynamics. The stress energy tensor for a fluid which satisfies the equations of viscous relativistic hydrodynamics (corresponding to the Navier-Stokes equation) is given by

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + (p - \zeta\theta)p^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} + q^\alpha u^\beta + q^\beta u^\alpha \tag{4}$$

(see, for example, Misner, Thorne, and Wheeler 1973 or Weinberg 1972). In this expression ρ is the energy density of the fluid, p is the pressure, and ζ and η are the coefficients of viscosity, which are in general nonnegative. The fluid moves along the integral curves of the fluid four-velocity, u^α. Heat flow is described by the vector field q^α. The rate of shear, expansion, and the projection operator for the vector field u^α are defined by

$$\sigma^{\alpha\beta} = \tfrac{1}{2}p^{\alpha\mu}\nabla_\mu u^\beta + \tfrac{1}{2}p^{\beta\mu}\nabla_\mu u^\alpha - \tfrac{1}{3}p^{\alpha\beta}\theta\,, \tag{5a}$$

$$\theta = \nabla_\mu u^\mu\,, \tag{5b}$$

$$p^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta\,. \tag{5c}$$

The equations of motion for the fluid are obtained by using the equations of conservation of stress energy,

$$\nabla_\alpha T^{\alpha\beta} = 0\,. \tag{6}$$

These equations must be supplemented by the first law of thermodynamics:

$$\rho + p = Ts + \mu n , \tag{7a}$$

$$\nabla_\alpha \rho = T\nabla_\alpha s + \mu \nabla_\alpha n . \tag{7b}$$

The temperature T, entropy density s, chemical potential μ, and the particle number density n have been introduced in these equations. The particle number density satisfies the additional conservation law:

$$\nabla_\alpha (nu^\alpha) = 0 . \tag{8}$$

One more equation, the heat flow equation, is needed to complete the system of equations governing the motions of the fluid. For the purposes of this discussion, the general-relativistic version of the Fourier law of heat conduction, first proposed by Eckart (1940), will be used,

$$q^\alpha = -\kappa p^{\alpha\beta}(\nabla_\beta T + Tu^\mu \nabla_\mu u_\beta) , \tag{9}$$

where κ is the coefficient of thermal conductivity which is in general nonnegative. The divergence of the entropy current,

$$s^\alpha \equiv su^\alpha + q^\alpha/T , \tag{10}$$

can be computed using equations (6)–(9):

$$T\nabla_\alpha s^\alpha = \zeta\theta^2 + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \kappa q^\alpha q_\alpha/T . \tag{11}$$

The vector q^α is purely spacelike, and $\sigma^{\alpha\beta}$ has nonzero components only orthogonal to u^α. Since ζ, η, and κ are positive definite, the right-hand side of equation (11) is the sum of nonnegative definite terms. Thermal equilibrium ($\nabla_\alpha s^\alpha = 0$) therefore implies that

$$\sigma^{\alpha\beta} = 0 , \qquad q^\alpha = 0 , \qquad \text{and} \qquad \theta = 0 . \tag{12}$$

When the rate of shear and expansion of a congruence of curves vanishes, then the curves are said to represent rigid motion (see Pirani and Williams 1962). Thus, we see that in thermal equilibrium the fluid moves rigidly.

The condition that the heat flow q^α vanish is equivalent to

$$u^\mu \nabla_\mu u^\alpha = -p^{\alpha\mu}\nabla_\mu (\log T) . \tag{13}$$

The expansion of the fluid vanishes so that the conservation of particle number (eq. [8]) and the conservation of energy derived from equations (6) and (12) imply that n and ρ are constants along the integral curves of u^α. Together with the first law of thermodynamics (eq. [7]) this implies that all of the thermodynamic scalars—in particular the temperature—are constant along u^α: $u^\alpha \nabla_\alpha T = 0$. This means that equation (13) becomes

$$u^\mu \nabla_\mu u^\alpha = -\nabla^\alpha (\log T) . \tag{14}$$

This nearly completes the proof of the result. Pirani and Williams (1962) have shown that a shear-free, expansion-free vector field, whose acceleration is the gradient of a scalar, is proportional to a Killing vector field. The proportionality constant is in this case the temperature,

$$u^\alpha = T\xi^\alpha \qquad \text{with } \mathscr{L}_\xi g_{\alpha\beta} = 0 . \tag{15}$$

Euler's equation for a fluid satisfying equation (12) is needed to show that in equilibrium the fluid has a barotropic equation of state. This can be derived from equation (6), if it is recalled that for an expansion-free fluid, $u^\alpha \nabla_\alpha p = 0$. The resulting form of Euler's equation is given by

$$(\rho + p)u^\mu \nabla_\mu u^\alpha = -\nabla^\alpha p . \tag{16}$$

The fact that the fluid is barotropic ($\nabla_\alpha p \nabla_\beta \rho - \nabla_\beta p \nabla_\alpha \rho = 0$) follows directly from equations (14) and (16). This concludes the proof of result ***B***.

Before proceeding further, it is appropriate to mention the relationship between the way in which the thermodynamics is handled in the preceding results and the other approaches which have been taken in the literature. Result ***B*** implies that thermodynamic equilibrium of a viscous fluid exists only if the fluid velocity is proportional to a Killing vector field, and the temperature (and chemical potential) is proportional to the "redshift factor" (see Thorne 1969 for definition). These implications of equilibrium are identical with those derived by Katz and Manor (1975). Their analysis assumes stationarity and axisymmetry from the start; however, it assumes the fluid to be a nonviscous perfect fluid. They define equilibrium via a global variational principle, demanding the total entropy of the star (as in eq. [1]) to be extremized. Another analogous result is obtained by Stewart (1971). He considers collision-dominated equilibrium in a gas which satisfies the relativistic Boltzmann equation. He shows that in equilibrium, the four-velocity of the gas is equal to the temperature multiplied by a Killing vector field. Therefore, starting from rather different mathematical assumptions, we see that the symmetry condition implied by equilibrium is the same as the one found here.

III. STATIONARITY IMPLIES AXISYMMETRY

a) *Preliminaries*

The precise statement of the proposition to be demonstrated in this section can be found in § I. The Killing vector field ξ^α derived in § II represents a symmetry of the spacetime within the star. The discussion of this section concentrates on showing that ξ^α must also exist in the exterior of the star. This work will proceed in four steps. Subsection (*a*) shows that under the assumptions of the proposition, ξ^α is linearly independent of the globally timelike Killing vector field η^α. The junction conditions for the gravitational field at the surface of the star are also reviewed here. Subsection (*b*) proves the existence of an extension of ξ^α into the exterior of the star. Subsection (*c*) shows that the extension obtained in (*b*) is in fact a Killing vector field which commutes with the globally timelike Killing vector field. Subsection (*d*) completes the proof by showing that the symmetry represented by ξ^α is a rotational symmetry of the star.

From the results of § II, it is known that $\mathscr{M}_2$ admits a Killing vector field ξ^α which is proportional to the four-velocity of the fluid. A theorem due to Lichnerowicz (see Carter 1972, p. 151) guarantees that ξ^α will be linearly independent of η^α whenever the timelike Killing vector field η^α is nonstatic. Therefore, the interior of a stationary (nonstatic) star is invariant under *two* isometries.

Before proceeding with the extension of ξ^α, the junction conditions which match the exterior gravitational field to the interior field at the surface of the star Σ must be discussed. Σ is the surface on which the pressure of the fluid vanishes. It is not customary to demand that the equation of state be one for which the density necessarily also vanishes on this surface; therefore, there may be a jump discontinuity in the stress energy tensor, and in the Ricci tensor at this surface. The junction conditions of Synge (see Synge 1966, p. 39) are the appropriate ones to describe the resulting discontinuities in the metric tensor in this situation. In a neighborhood of some point $r \in \Sigma$, an adapted coordinate system is constructed: i.e., a system in which Σ is a level surface of one of the coordinate functions, say x^1. Synge has shown that the discontinuities in the stress energy tensor come from the second derivatives of the metric along x^1; thus $\partial_1\partial_1 g_{\alpha\beta}$ may be discontinuous. In the adapted coordinates discussed above, however, $g_{\alpha\beta}$, $\partial_\alpha g_{\beta\gamma}$, and all second derivatives except $\partial_1\partial_1 g_{\alpha\beta}$ will be continuous. The metric tensor is therefore C^1 in any open set which contains part of the boundary of the star, and C^3 elsewhere.

b) *Extending the Killing Vector Field*

To propagate ξ^α off the surface Σ, the differential equation

$$\nabla_\alpha \nabla^\alpha \xi^\beta = 0 \tag{17}$$

will be used. This equation is chosen to define the extension of ξ^α since it is satisfied by any Killing vector field in $\mathscr{M}_1$. The initial values of the field, ξ^α and $\partial_\alpha \xi^\beta$, will be specified on Σ by taking the limits of the corresponding quantities from the star's interior $\mathscr{M}_2$. These initial values plus the differential equation (17) form a Cauchy initial value problem for ξ^α on the surface Σ. The mathematical tool which is used to show the existence of this extension of ξ^α is the Cauchy-Kowalewsky theorem (see Courant and Hilbert 1962, p. 39). This theorem guarantees the existence of a solution of the Cauchy problem in a small neighborhood of the initial surface if (*a*) the differential equation depends analytically on the unknown functions, their derivatives, and on the coordinates; and if (*b*) the Cauchy data are analytic functions of the coordinates on the initial surface. A theorem of Müller zum Hagen (1970*b*) proves that the components of the metric tensor, $g_{\alpha\beta}$, are analytic functions in the region $\mathscr{M}_1$; therefore, condition (*a*) of the Cauchy-Kowalewsky theorem is satisfied by the equation (17). The next task is to show that the condition (*b*), the analyticity of ξ^α and $\partial_\alpha \xi^\beta$, is also satisfied on the initial surface Σ. To accomplish this, certain constraints must first be placed on the surface Σ.

The surface Σ forms the boundary between the external, analytic vacuum spacetime $\mathscr{M}_1$ and the interior of the star $\mathscr{M}_2$. Therefore, in general, the exterior geometry may be analytically continued past the surface of the star Σ, to form some spacetime $\mathscr{M}_1'$ which is everywhere vacuum.[1] Whenever this analytic continuation is possible, the surface of the star can be viewed as a hypersurface embedded in the manifold $\mathscr{M}_1'$. It will be useful to view the surface Σ in this way. Furthermore, it is natural to suppose that the surface of the star shares the smoothness of the geometry in which it is embedded. Hence, the surface Σ is assumed to be "compatible" with the exterior geometry of the star. This condition is defined precisely as follows:

DEFINITION: The surface of the star, Σ, will be called "compatible" with the external geometry if:

1) there is some open subset $U \subset \Sigma$ of the surface of the star across which the external geometry can be analytically continued;
2) the surface U in the extended vacuum spacetime $\mathscr{M}_1'$ is the level surface of an analytic function: $f(r) = 0, df(r) \neq 0, r \in U$.

Briefly, this condition requires that some (possibly small) portion of the surface of the star be the level surface of an analytic function. This makes the surface "compatible" with the analyticity of the external geometry of the star. The condition (2) may at first seem to be a rather strong one. However, it is easy to see that it is in fact a necessary condition; this necessity is demonstrated in Appendix A. Furthermore, it is not stronger than the smoothness conditions which are generally adopted to describe physical situations. In order for a surface to violate the compatibility condition, it must fail to be analytic everywhere. Such a situation is rather pathological, and probably not physical.

It will now be shown that the components of the Killing vector ξ^α and its first derivatives must be analytic functions on the surface Σ. Begin by noting that the vector field ξ^α within the star is a Killing vector not only of the four-geometry of $\mathscr{M}_2$, but also of the three-geometry intrinsic to each surface of constant pressure. To see this, let n^α represent the unit normal to the surfaces of constant pressure. (Note that $\mathscr{L}_\xi n^\alpha = 0$.) The metric tensor intrinsic to these surfaces is given by $\gamma_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta$. Its Lie derivative along ξ^α vanishes: $\mathscr{L}_\xi \gamma_{\alpha\beta} = 0$. Equivalently, ξ^α satisfies

[1] An example of this extendability is given by static spherical stars. The interior regions of these stars satisfy the Oppenheimer-Volkoff equations; the exterior geometry is simply a piece of the vacuum Schwarzschild geometry. For such stars, the external region $\mathscr{M}_1$ can always be continued past the surface Σ to form $\mathscr{M}_1'$, the complete Schwarzschild solution.

Killing's equation within the surface: $D_i\xi_j + D_j\xi_i = 0$, $i, j = 0, 2, 3$. D_i represents the covariant derivative related to the intrinsic geometry. Furthermore, since ξ_i is a Killing vector field, it must satisfy

$$D_iD^i\xi^j = {}^3R^j{}_i\xi^i\,. \tag{18}$$

Equation (18) must hold on each surface of constant pressure within the star. In particular, then, it must hold on the surface of the star Σ.

It has been assumed that the surface of the star, Σ, is "compatible" with the exterior geometry. This requires that an analytic function f exists, one of whose level surfaces is the surface Σ. Müller zum Hagen (1970*b*) has shown that the metric tensor is analytic in suitable coordinates on the manifold $\mathscr{M}_1'$. Since the function f is assumed to be analytic, it may be used to replace one of the analytic coordinates constructed by Müller zum Hagen. The components of the metric in the resulting adapted analytic coordinate system will be analytic functions. Furthermore, the intrinsic metrics, which describe the geometry of the surfaces of constant f, will have analytic components. In particular, the intrinsic geometry of the surface Σ must be analytic. Therefore, on Σ, equation (18) is an analytic equation for ξ_i. Using an argument presented in Appendix B, it follows that solutions of equation (18) must be analytic functions, since equation (18) forms an elliptic system of analytic partial differential equations. Thus the functions ξ^α must be analytic functions on the surface of the star Σ.

All that remains to establish condition (*b*) of the Cauchy-Kowalewsky theorem is to show that the first derivatives $\partial_\alpha\xi^\beta$ are also analytic functions on Σ. Let n^α be the components of the unit normal vector to Σ, and e^α be the components of an arbitrary analytic vector field which is tangent to Σ. Since ξ^α are analytic functions, it follows that $e^\alpha\partial_\alpha\xi^\beta$ will also be analytic. To learn about the derivatives of ξ^α in the direction normal to the surface, the four-dimensional Killing's equation is used:

$$0 = \xi^\mu\partial_\mu g_{\alpha\beta} + g_{\alpha\mu}\partial_\beta\xi^\mu + g_{\beta\mu}\partial_\alpha\xi^\mu\,.$$

The inner products of this equation with the vectors n^α and e^α give expressions for the normal derivatives:

$$n_\alpha n^\beta\partial_\beta\xi^\alpha = -\tfrac{1}{2}n^\alpha n^\beta\xi^\mu\partial_\mu g_{\alpha\beta}\,, \tag{19a}$$

$$e_\alpha n^\beta\partial_\beta\xi^\alpha = -n^\alpha e^\beta\xi^\mu\partial_\mu g_{\alpha\beta} - n_\alpha e^\beta\partial_\beta\xi^\alpha\,. \tag{19b}$$

The left-hand side of equation (19) gives all possible components of the normal derivatives of ξ^α. The right-hand side is composed entirely of functions which are known to be analytic. Thus, we conclude that the Cauchy data ξ^α, $\partial_\alpha\xi^\beta$ are analytic functions on the initial surface Σ. The Cauchy-Kowalewsky theorem therefore guarantees the existence of a solution of equation (17) with the initial data given above. The vector field ξ^α is thereby extended at least a small distance into the exterior of the star, $\mathscr{M}_1$.

c) *Properties of the Extension*

The vector field ξ^α has been extended a small way into $\mathscr{M}_1$ in the previous section. It is now shown that this extension is a Killing vector field which commutes with the globally timelike Killing vector field η^α. The following identity is satisfied by any vector field in a vacuum spacetime:

$$\nabla_\alpha\nabla^\alpha(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) - 2(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)R^\alpha{}_\mu{}^\beta{}_\nu = \nabla_\mu\nabla_\alpha\nabla^\alpha\xi_\nu + \nabla_\nu\nabla_\alpha\nabla^\alpha\xi_\mu\,. \tag{20}$$

When ξ^α is extended using equation (17), the right-hand side of equation (20) vanishes. The left-hand side then becomes an equation for $t_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha$, the Killing tester (which vanishes if and only if ξ^α is a Killing vector):

$$\nabla_\alpha\nabla^\alpha t_{\mu\nu} - 2R_\mu{}^\alpha{}_\nu{}^\beta t_{\alpha\beta} = 0\,. \tag{21}$$

To prove that ξ^α is a Killing vector field, it must be shown that the only solution to equation (21) which is consistent with the boundary conditions is $t_{\alpha\beta} = 0$. The boundary conditions, on Σ must therefore be examined so that the values of $t_{\alpha\beta}$ and $\partial_\alpha t_{\beta\gamma}$ may be evaluated on Σ.

It will now be shown that the tensor $t_{\alpha\beta}$, and its first derivatives, $\partial_\alpha t_{\beta\gamma}$, vanish on the surface Σ. This fact follows from the continuity required by the junction conditions at Σ. The tensor $t_{\alpha\beta}$ is a function of the vector ξ^α, the metric $g_{\alpha\beta}$, and their first derivatives:

$$t_{\alpha\beta} = \xi^\mu\partial_\mu g_{\alpha\beta} + g_{\alpha\mu}\partial_\beta\xi^\mu + g_{\beta\mu}\partial_\alpha\xi^\mu\,.$$

The metric and its first derivatives are required to be continuous by Synge's junction conditions. The components of the vector field ξ^α and its first derivatives $\partial_\alpha\xi^\beta$ were required to be continuous in the extension of ξ^α. Therefore, $t_{\alpha\beta}$ must be continuous across the surface Σ. Since it vanishes in $\mathscr{M}_2$, it must vanish on Σ. The derivatives $\partial_\alpha t_{\beta\gamma}$ must also be continuous; this is not as easily seen. These derivatives depend on the metric, the vector field, plus their first and second derivatives:

$$\partial_\alpha t_{\beta\gamma} = \partial_\alpha(\xi^\mu\partial_\mu g_{\beta\gamma} + g_{\beta\mu}\partial_\gamma\xi^\mu + g_{\mu\gamma}\partial_\beta\xi^\mu)\,. \tag{22}$$

The only term in equation (22) involving second derivatives of the metric is $\xi^\mu\partial_\alpha\partial_\mu g_{\beta\gamma}$. (The vector field ξ^α is tangent to the surface Σ. This follows from the fact that ξ^α is a Killing vector field within $\mathscr{M}_2$, which implies that the gradient of the pressure is orthogonal to ξ^α: $\xi^\mu\nabla_\mu p = 0$; therefore, ξ^α must be tangent to Σ.) The Synge junction conditions require that only the second derivatives of the form $n^\alpha n^\beta\partial_\alpha\partial_\beta g_{\mu\nu}$ have discontinuities. None of these terms are present in equation (22) since ξ^μ is tangent to Σ.

It must also be shown that the second derivatives of the form $\partial_\alpha\partial_\beta\xi^\gamma$ are continuous. The second derivatives of the form $e^\mu\partial_\mu\partial_\alpha\xi^\beta$ will be continuous whenever e^μ is tangent to the surface Σ. Only the second derivatives of the form $n^\alpha n^\beta\partial_\alpha\partial_\beta\xi^\mu$ need to be considered. These derivatives are determined not by the junction conditions, but by the differential equations governing ξ^α. In $\mathscr{M}_1$, ξ^α satisfies equation (17) by construction; in $\mathscr{M}_2$ (since ξ^α is a Killing vector field) it must satisfy

$$\nabla_\alpha \nabla^\alpha \xi^\beta = R^\beta{}_\alpha \xi^\alpha . \quad (23)$$

Since the region $\mathscr{M}_1$ is vacuum, equation (23) is satisfied by ξ^α everywhere. The left-hand side of equation (23) can be expanded out in terms of coordinates to obtain

$$\nabla_\alpha \nabla^\alpha \xi^\beta = g^{\mu\nu}\partial_\mu \partial_\nu \xi^\beta + R^\beta{}_\mu \xi^\mu + \xi^\mu \partial_\mu (g^{\alpha\nu}\Gamma^\beta{}_{\alpha\nu}) + B^\beta(\xi, \partial\xi, g, \partial g) . \quad (24)$$

The term B^β is a function of only the quantities ξ^α, $g_{\alpha\beta}$, and their first derivatives; therefore, it is continuous across Σ. Using equations (23) and (24) together, the second derivatives of ξ^α may be expressed as

$$g^{\mu\nu}\partial_\mu \partial_\nu \xi^\beta = -\xi^\mu \partial_\mu (g^{\alpha\nu}\Gamma^\beta{}_{\alpha\nu}) - B^\beta(\xi, \partial\xi, g, \partial g) . \quad (25)$$

The right-hand side of equation (25) is continuous across Σ: The first term contains only second derivatives of the metric of the form $\xi^\mu \partial_\mu \partial_\alpha g_{\beta\gamma}$, which have been shown to be continuous. The second term B^β is continuous by construction. Thus, $g^{\mu\nu}\partial_\alpha \partial_\nu \xi^\beta$ are continuous functions at Σ. This implies that all of the second derivatives $\partial_\alpha \partial_\beta \xi^\gamma$ must be continuous. This completes the argument showing that the functions $t_{\alpha\beta}$ and $\partial_\alpha t_{\beta\gamma}$ are continuous across Σ; therefore, since each vanishes in $\mathscr{M}_2$, they must vanish on Σ.

The functions $t_{\alpha\beta}$ and $\partial_\alpha t_{\beta\gamma}$ form Cauchy data for the linear differential equation (21). In stationary vacuum spacetimes such as $\mathscr{M}_1$, the components of the metric tensor are analytic functions when expressed in suitable coordinates. Therefore, equation (21) forms a linear system of partial differential equations with analytic coefficients, for the quantities $t_{\alpha\beta}$. A theorem due to Holmgrem (see Courant and Hilbert 1962, p. 237) guarantees the uniqueness of the solutions of this Cauchy problem. Since $t_{\alpha\beta} = 0$ is a solution, it must be the unique solution. Thus, we finally conclude that when ξ^α is extended according to equation (17), the extension must be a Killing vector field.

It is also useful to show that the extension of ξ^α via equation (17) commutes with the timelike Killing vector n^α. Let us define $l^\alpha = \mathscr{L}_\eta \xi^\alpha$, the commutator of ξ^α and η^α. Since η^α is a Killing vector field, it is straightforward to show that

$$\nabla_\alpha \nabla^\alpha l^\beta = \mathscr{L}_\eta(\nabla_\alpha \nabla^\alpha \xi^\beta) . \quad (26)$$

The right-hand side vanishes whenever equation (17) is satisfied. It is therefore possible to use equation (26) and the initial values of l^α and its derivatives as a Cauchy problem on Σ. The initial-value data l^α and $\partial_\alpha l^\beta$ are functions of ξ^α, η^α, and their first and second derivatives. It has already been shown that ξ^α and its first two derivatives are continuous at Σ. The vector field η^α is assumed to be C^4. Thus both l^α and $\partial_\alpha l^\beta$ must be continuous functions at Σ. Each vanishes within the star; therefore each must vanish on Σ. As before, the theorem of Holmgren guarantees that $l^\alpha = 0$ is the unique solution of this problem. Thus, the extension of ξ^α commutes with η^α.

d) Axisymmetry

The remainder of the proof of this proposition—"stationary stars are axisymmetric"—follows exactly the final steps of Hawking's (1972) theorem—"stationary black holes are axisymmetric." It is appropriate briefly to summarize those steps here. The Killing vector ξ^α has been shown to exist inside the star, and in at least a small open neighborhood in the exterior. Since the exterior geometry of the star is analytic, the components ξ^α must also be analytic functions (see Appendix B). These can be extended to cover the entire exterior spacetime by analytic continuation.

The integral curves of the now globally defined Killing vector field ξ^α are the orbits of an isometry of the spacetime $\mathscr{M}$. What is this symmetry? Near spacelike infinity, the spacetime $\mathscr{M}$ is assumed to behave asymptotically like Minkowski spacetime. Minkowski spacetime is isometric under transformations of the Poincaré group. Therefore, the symmetry represented by the Killing vector ξ^α must behave near infinity as a Poincaré transformation. The spacetime containing the star is clearly not invariant under spacelike translations, or velocity boosts. The only remaining symmetries are time translations or rotations. The globally timelike Killing vector field η^α represents the time translations. Therefore the vector field ξ^α must represent some linear combination of a rotation and time translation. The spacetime $\mathscr{M}$ is therefore invariant under a rotation; it is axisymmetric.

I would like to express my appreciation to the people who helped and encouraged me during the course of this work, especially: D. Brill, S. Detweiler, J. Isenberg, B. Mashhoon, C. Misner, J. Nester, and P. Yasskin. I would also like to thank the Max-Plank-Institut für Physik und Astrophysik, München, for the hospitality extended to me while I completed part of this work. The financial support of the NSF under grant GP-25548 is also gratefully acknowledged.

APPENDIX A

In this Appendix it is shown that the surface of a rigidly rotating axisymmetric star must be the level surface of an analytic function. If a Killing vector ξ^α exists in the region $\mathscr{M}_1$, then the argument presented in Appendix B shows that the components ξ^α will be analytic functions. This fact, together with Müller zum Hagen's (1970*b*) result, shows that the function $h^2 = -g_{\alpha\beta}\xi^\alpha\xi^\beta$ is analytic in $\mathscr{M}_1$. This Killing vector is proportional to the fluid velocity within the star, $\xi^\alpha = -hu^\alpha$. This fact, and Euler's equation (16), imply that

$$\nabla_\alpha(\log h) = -(\nabla_\alpha p)/(\rho + p) .$$

Therefore, the surfaces of constant h correspond to the surfaces of constant pressure. This means that the surface Σ of the star is the level surface of the analytic function h in the region $\mathscr{M}_1$. If $dh = 0$ on the surface, then $dp = 0$ also. This would correspond to the Poincaré limit on the rotation of the star, and any such star would be unstable to mass shedding. Thus, we have shown that the "compatibility" condition is a necessary one for a rotating axisymmetric star.

APPENDIX B

The purpose of this Appendix is to show that the solutions of the equation,

$$\nabla_\alpha \nabla^\alpha \xi^\beta = R_\alpha{}^\beta \xi^\alpha , \tag{26}$$

must be analytic functions if (*a*) they are at least of class C^3, (*b*) they commute with the globally timelike Killing vector field η^α, $\mathscr{L}_\eta \xi^\alpha = 0$, and (*c*) the metric tensor is analytic. Müller zum Hagen (1970*a*, *b*) has shown that in a neighborhood of every point in a stationary spacetime, coordinates may be chosen which are both stationary and harmonic. In stationary coordinates the components of the timelike Killing vector are $\eta^\alpha = \delta_0{}^\alpha$. In harmonic coordinates the Christoffel connection must satisfy $g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} = 0$. In such a coordinate system equation (26) may be expanded as

$$g^{\mu\nu}\partial_\mu\partial_\nu\xi^\alpha + B^\alpha(\xi, \partial\xi, g, \partial g) = 0 . \tag{27}$$

The function B^α depends algebraically on its arguments, the vector field ξ^α, the metric $g_{\alpha\beta}$, and their first derivatives. The metric tensor is assumed to be analytic; thus, this equation forms an analytic system of differential equations for ξ^α. The timelike Killing vector commutes with ξ^α, so that $\partial_0\xi^\alpha = 0$. In this case the equation (27) may be rewritten as

$$g^{ij}\partial_i\partial_j\xi^\alpha + B^\alpha(\xi, \partial\xi, g, \partial g) = 0 \quad (i, j = 1, 2, 3) . \tag{28}$$

The matrix g^{ij} is positive definite. Therefore, equation (28) forms an elliptic system of differential equations.

It can now be demonstrated that ξ^α are analytic functions by recalling a theorem of Morrey (see Morrey 1958, or Müller zum Hagen 1970*a*). The specific case of that theorem which is applicable here states:

MORREY'S THEOREM: Consider an elliptic system of second-order partial differential equations:

$$\Phi^A(x^\alpha, f^B, \partial_\alpha f^B, \partial_\alpha\partial_\beta f^B) = 0 \quad (A, B = 1, \ldots, N) . \tag{29}$$

If f^B is a function of class C^3 which is a solution of equation (29) in some domain D, and if the function $\Phi^A(x^\alpha, y^B, y_\alpha{}^B, y_{\alpha\beta}{}^B)$ is analytic in the variables $(x^\alpha, y^B, y_\alpha{}^B, y_{\alpha\beta}{}^B)$, then the function f^B is analytic in the domain D.

It has been shown that the equation (28) forms an elliptic system of analytic partial differential equations. The vector field ξ^α has been assumed to be of at least class C^3. Therefore by Morrey's theorem, the components ξ^α are analytic functions. The argument in this Appendix assumes that the geometry is four-dimensional; however, essentially the same argument applies to the three-dimensional equation (18).

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