On the analyticity of certain stationary nonvacuum Einstein space–times*

Lee Lindblom

Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
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It is shown that certain nonvacuum solutions of Einstein's general relativistic field equations are analytic space–times, i.e., an analytic atlas exists with respect to which the components of the metric tensor, and all material fields are analytic functions. The two specific cases discussed here are interesting from an astrophysical point of view. The first is the class of space–times containing a source free electromagnetic field: the exterior of a charged black hole, for example. The second is the class of space–times filled with a rigidly moving perfect fluid, often used to describe the interior of a rotating star.

1. INTRODUCTION

Müller zum Hagen[1,2] has shown that every stationary vacuum solution of the Einstein field equations is analytic, i.e., there exists an analytic atlas with respect to which the components of the metric tensor field are analytic functions. His result has proven to be a useful tool for the study of space–times of astrophysical interest, e.g., the exteriors of rotating stars or black holes (see Refs. 3 and 4). The purpose of this paper is to extend those results to certain nonvacuum space–times which have possible astrophysical interpretations. The case of a space–time which contains a source-free electromagnetic field includes the charged black hole solutions. Hawking,[4] in his proof that stationary black holes are axisymmetric, uses the analyticity of the metric tensor. The result presented here, therefore, makes his argument rigorous for the case where electromagnetic fields are present. The other case presented here, space–times containing a rigidly moving barytropic fluid, is often used to model the interiors of rotating stars.

2. BACKGROUND

Analyticity of these space–times is demonstrated by showing that the functions which describe the geometry and the configuration of the matter satisfy systems of elliptic partial differential equations. A theorem of Morrey[5] is then recalled which guarantees the analyticity of such functions. Several definitions and results implied from previous work will be required to effect these proofs; they are simply listed in this section. Discussion and proofs of these points may be found in the references.[1,2,5]

Definition: A coordinate chart in a stationary space–time M is said to be stationary and harmonic if (a) the components of the timelike Killing vector are given by $\eta^\alpha = \delta_0^\alpha$; and if (b) the Christoffel connection satisfies $\Gamma^\alpha_{\mu\nu} g^{\mu\nu} = 0$, $(\alpha, \beta, \cdots = 0, 1, 2, 3)$.

Definition: A function $f(x)$ is said to be Hölder continuous of order $0 < \mu < 1$ (C^μ), on some domain D, if $\exists$ a constant K, such that $\forall\ x, y \in D$, $|f(x) - f(y)| < K|x-y|^\mu$.

Lemma 1 (*Müller zum Hagen*): Assumptions: A space–time M is C^{n+2} for integer $n \geq 2$. It contains a globally timelike vector field, η^α, which is C^{n+1}. The metric tensor is C^n. Assertions: In a neighborhood of each point $x \in M$ there exists a stationary harmonic coordinate chart which is $C^{n+\mu}$, $0 < \mu < 1$, related to the C^{n+2} charts on M.

Lemma 2 (*Müller zum Hagen*): Consider a stationary space–time M in which the components of the metric tensor are analytic functions of the stationary harmonic coordinate systems at each point. The the stationary harmonic coordinate charts form a basis for an analytic atlas on M.

Definition: A system of second order partial differential equations, $\Phi^A(x^\alpha, f^B, \partial_\alpha \partial_\beta f^B) = 0$ $(A, B = 1, 2 \ldots, N)$ is said to be elliptic in some domain D if $\forall\, x^\alpha \in D$ and $\forall$ vectors $\lambda^\alpha \neq 0$,

$$0 \neq \det \left\{ \sum_{\alpha,\beta} \lambda^\alpha \lambda^\beta \left[\frac{\partial}{\partial y^B_{\alpha\beta}} \Phi^A(x^\gamma, y^C, y^C_\gamma, y^C_{\gamma\epsilon}) \right] \right\} ,$$

evaluated at $y^B = f^B$, $y^B_\alpha = \partial_\alpha f^B$, and $y^B_{\alpha\beta} = \partial_\alpha \partial_\beta f^B$.

Theorem (*Morrey*): Assumptions: f^B is a function which is the solution of the system of elliptic differential equations, $\Phi^A(x^\alpha, f^B, \partial_\alpha f^B, \partial_\alpha \partial_\beta f^B) = 0$, $(A, B = 1, 2 \ldots, N)$ on some domain D. f^B is of class $C^{2+\mu}$, $0 < \mu < 1$. The functions $\Phi^A(x^\alpha, y^B, y^B_\alpha, y^B_{\alpha\beta})$ are analytic in the variables $(x^\alpha, y^B, y^B_\alpha, y^B_{\alpha\beta})$. Assertion: The functions f^B are analytic on the domain D.

3. ELECTROMAGNETISM

The electromagnetic field is described by the vector potential A^α. In a source-free space–time, the field equations which govern A^α are

$$\nabla_\alpha \nabla^\alpha A^\beta + R^\beta_\alpha A^\alpha = 0. \tag{1}$$

(The Lorentz gauge condition has been adopted, $\nabla_\alpha A^\alpha = 0$.) These fields are themselves sources for the gravitational field, via the Einstein equations

$$R_{\alpha\beta} = (2g^{\nu\xi}\delta^\mu_\alpha \delta^\sigma_\beta - \tfrac{1}{2} g_{\alpha\beta} g^{\mu\sigma} g^{\nu\xi})(\nabla_\mu A_\nu - \nabla_\nu A_\mu)(\nabla_\sigma A_\xi - \nabla_\xi A_\sigma). \tag{2}$$

These space–times are called stationary if there exists a globally timelike vector field η^α which satisfies

$$L_\eta A^\alpha = \eta^\mu \nabla_\mu A^\alpha - A^\mu \nabla_\mu \eta^\alpha = 0, \tag{3}$$

$$L_\eta g_{\alpha\beta} = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha = 0. \tag{4}$$

For space–times described by Eqs. (1)–(3) we will derive the following:

Proposition 1: A stationary space—time M is assumed to contain a source-free electromagnetic field, described by Eqs. (1)—(3). If M is C^6, the components of the Killing vector field η^α are C^5, the metric tensor $g_{\alpha\beta}$ is C^4, and the vector potential A^α is C^3, then M admits an analytic atlas with respect to which $g_{\alpha\beta}$ and A^α are analytic functions.

Proof: In a harmonic coordinate system the components of the Ricci tensor, and the D'Alembertian of a vector field may be expressed as,

$$R_{\alpha\beta}=\tfrac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta}+B_{\alpha\beta}(g,\partial g), \tag{5}$$

$$\nabla^\mu\nabla_\mu A^\alpha=g^{\mu\nu}\partial_\mu\partial_\nu A^\alpha+R^\alpha_\mu A^\mu+C^\alpha(A,\partial A,g,\partial g). \tag{6}$$

The functions $B_{\alpha\beta}$ and C^α are functions only of the metric $g_{\alpha\beta}$, the vector field A^α, and their first derivatives. Equations (1) and (2) may be rewritten in harmonic coordinates [using Eqs. (5) and (6)] to obtain,

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta}=B'_{\alpha\beta}(g,\partial g,A,\partial A), \tag{7}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu A^\alpha=C'^\alpha(g,\partial g,A,\partial A). \tag{8}$$

When a stationary and harmonic coordinate system (existence is guaranteed by Lemma 1) is assumed, Eqs. (3) and (4) may be rewritten as

$$\partial_0 A^\alpha=0,\quad \partial_0 g_{\alpha\beta}=0.$$

In these coordinates, the operator $g^{\mu\nu}\partial_\mu\partial_\nu$ may be replaced by $g^{ij}\partial_i\partial_j$, with $i,j=1,2,3$, in Eqs. (7) and (8). Since g^{ij} is a positive definite matrix, Eqs. (7) and (8) are elliptic systems of differential equations for A^α and $g_{\alpha\beta}$. Morrey's theorem guarantees the analyticity of A^α and $g_{\alpha\beta}$ with respect to the stationary harmonic coordinate charts. Lemma 2 guarantees the existence of an analytic atlas for M. ∎

4. FLUIDS

Perfect fluids are described via the Einstein equations,

$$R_{\alpha\beta}=8\pi[(\rho+p)u_\alpha u_\beta+\tfrac{1}{2}(\rho-p)g_{\alpha\beta}]. \tag{9}$$

The energy density of the fluid is ρ, the pressure is p, and u^α ($u^\alpha u_\alpha=-1$) is the four-velocity, tangent to the world lines of the fluid. The fluid under consideration here is assumed to have an analytic barytropic equation of state, i.e. $\rho(p)$ is an analytic function of the pressure. Also, the fluid is assumed to be moving rigidly; this condition is given by,

$$(\delta^\mu_\alpha+u_\alpha u^\mu)(\delta^\nu_\beta+u_\beta u^\nu)(\nabla_\mu u_\nu+\nabla_\nu u_\mu)=0. \tag{10}$$

The stationarity of these space—times is expressed by the existence of a timelike vector field η^α, satisfying Eq. (4) and,

$$L_\eta u^\alpha=\eta^\mu\nabla_\mu u^\alpha-u^\mu\nabla_\mu\eta^\alpha=0, \tag{11}$$

$$L_\eta p=\eta^\mu\nabla_\mu p=0. \tag{12}$$

For these space—times, the following proposition holds:

Proposition 2: A stationary space—time M is assumed to contain a rigidly moving barytropic fluid (with analytic equation of state).[6] If M is C^7, the components of the Killing vector field η^α are C^6, the metric $g_{\alpha\beta}$ is C^5 and, the pressure p and the four-velocity u^α are C^3, then M admits an analytic atlas with respect to which $g_{\alpha\beta}$, p, and u^α are analytic functions.

Proof: Equations (9) and (10) and the fact that the fluid is barytropic imply that the following relationships are satisfied:

$$\nabla^\alpha\nabla_\alpha p=-\nabla^\alpha(\rho+p)u^\beta\nabla_\beta u_\alpha+(\rho+p)(u^\alpha u^\beta R_{\alpha\beta}-\nabla_\alpha u_\beta\nabla^\beta u^\alpha), \tag{13}$$

$$\nabla^\alpha\nabla_\alpha u^\beta=(u^\mu\nabla_\mu u_\alpha)(\nabla^\beta u^\alpha-\nabla^\alpha u^\beta)-u^\beta(\nabla_\mu u_\nu)(\nabla^\nu u^\mu). \tag{14}$$

Equations (9), (13), and (14) form a system of second order differential equations for the functions $g_{\alpha\beta}$, u^α, and p. These equations may be written in a stationary harmonic coordinate system, in analogy with Eqs. (7) and (8),

$$g^{ij}\partial_i\partial_j p=A(p,g,\partial g,u,\partial u),$$

$$g^{ij}\partial_i\partial_j u^\alpha=B^\alpha(p,g,\partial g,u,\partial u),$$

$$g^{ij}\partial_i\partial_j g_{\alpha\beta}=C_{\alpha\beta}(p,g,\partial g,u).$$

These equations form an elliptic system, thus the theorem of Morrey and Müller zum Hagen's lemma can be applied to complete the proof. ∎

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[4]S. W. Hawking, Commun. Math. Phys. 25, 152 (1972).
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[6]It has been shown elsewhere (Ref. 3) that a stationary, viscous, heat conducting fluid is necessarily rigidly moving and barytropic. Thus any real, truly stationary, fluid would be expected to satisfy these criteria with the possible exception of the analyticity of the equation of state.