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ON THE EVOLUTION OF THE HOMOGENEOUS ELLIPSOIDAL FIGURES

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ABSTRACT

We investigate the combined effects of viscosity and gravitational radiation on the evolution of the homogeneous Newtonian ellipsoidal figures. A quasi-equilibrium approximation scheme is developed which describes the secular evolution of the ellipsoidal figures in the limit of weak viscosity and gravitational radiation. These dissipative effects determine a sequence of Riemann S type ellipsoids along which the evolution slowly occurs. We numerically integrate the equations of motion and show that the evolution of these ellipsoidal figures is qualitatively different from the case where only one or the other of the dissipative effects is present. The results presented here do agree, however, with our previous analyses of the secular instabilities of the Maclaurin spheroids.

Subject heading: rotation

I. INTRODUCTION

In this paper we examine some of the effects of viscosity and of gravitational radiation on the homogeneous ellipsoidal figures. When an astrophysical system undergoes gravitational collapse (e.g., to form a white dwarf or a neutron star), the resulting compact object may be rapidly rotating so that the secular instabilities caused by the dissipative forces could cause evolution away from an axisymmetric state. The purpose of this paper is to examine how that evolution actually occurs, within the approximation of the homogeneous ellipsoidal figures.

The analysis of the secular instabilities of the Maclaurin spheroids illustrates how important the combined effects of viscosity and gravitational radiation can be on the homogeneous ellipsoidal figures. Chandrasekhar (1969 [hereafter referred to as E.F.E.], 1970*a*) demonstrates that the presence of either viscosity or gravitational radiation reaction induces a secular instability in the Maclaurin spheroids beyond the point of bifurcation of the Jacobi and Dedekind sequences. More recently, Lindblom and Detweiler (1977 [Paper I]) show that the presence of both viscosity and gravitational radiation reaction moves the point of the onset of secular instability beyond the point of bifurcation to a point determined by the ratio of the strengths of the dissipative forces. And, for a spheroid of given mass and density, one specific value of the viscosity of the fluid will cause the Maclaurin sequence to be stable all the way to the point of the onset of dynamical instability. Thus, the presence of both gravitational radiation reaction and viscosity drastically changes the discussion of the stability of the Maclaurin spheroids from the case where only one or the other of the dissipative forces is acting.

In the present work we find similar qualitative changes in the evolution of the slowly varying ellipsoids when both dissipative effects are included. In § II we review briefly the general equations of motion which govern the evolution of the ellipsoidal figures, including the effects of viscosity and gravitational radiation reaction.

Miller (1974) and, and Press and Teukolsky (1973) have studied numerically the evolution of the ellipsoidal figures including the effects of gravitational radiation and viscosity, respectively. Their analyses show that a perturbed Maclaurin spheroid, which lies in the region of secular instability, will evolve through a sequence of ellipsoids which lie near the Riemann S family. We use this qualitative feature of their results to derive an approximation scheme which allows the economical large-scale integration of the equations of motion. When the effects of viscosity and gravitational radiation are weak, the general motion of an ellipsoid will consist of (i) a large-scale motion from one place to another along the Riemann S surface, and (ii) small-scale hydrodynamical oscillations about the quasi-equilibrium Riemann S configurations. The second of these effects, although of secondary interest, causes the numerical integration of the equations of motion to be very inefficient. The time step size must be kept small with respect to the oscillation period in order to maintain numerical accuracy. We develop in § III an approximation scheme which suppresses the oscillations, and therefore allows for a quick and efficient integration of the large-scale effects of dissipation on the evolution.

In § IV the results of the numerical integration of the equations of motion for the slowly varying ellipsoidal figures are presented. We illustrate the qualitative features of the evolution (over most of the Riemann S surface) for ellipsoids having varying amounts of viscosity and gravitational radiation reaction. In particular, we illustrate the evolution in the limiting cases of purely viscous or purely radiative evolution. We also examine the critical case where the Maclaurin sequence is stabilized all the way to the point of the onset of a dynamical instability. Several intermediate cases are also examined.

II. THE EQUATIONS OF MOTION

The uniform density ellipsoidal figures are described by the 10 time-dependent parameters a_i, Ω_i, Λ_i ($i = 1, 2, 3$), and p_c (see E.F.E.). The functions a_i are the lengths of the principal axes of the ellipsoid, Ω_i represents the angular velocity of the principal axes with respect to a nonrotating inertial reference frame, Λ_i measures the internal motion of the fluid as seen by an observer in the principal axis frame, p_c represents the central pressure of the fluid, and ρ is the uniform and time-independent mass density. The Riemann-Lebovitz equations describe the hydrodynamical evolution of these functions. They are conveniently written as

$$H_{ij} = 0\,. \tag{1}$$

The elements H_{ij} are defined by cyclically permuting the subscripts 1, 2, 3 in equations (2a, b, c):

$$H_{11} = \frac{d^2a_1}{dt^2} - a_1({\Omega_2}^2 + {\Omega_3}^2 + {\Lambda_2}^2 + {\Lambda_3}^2) + 2(a_2\Lambda_3\Omega_3 + a_3\Lambda_2\Omega_2) + 2\pi G\rho A_1 a_1 - \frac{2p_c}{\rho a_1}\,, \tag{2a}$$

$$H_{12} = a_1\frac{d\Lambda_3}{dt} - a_2\frac{d\Omega_3}{dt} + 2\left(\Lambda_3\frac{da_1}{dt} - \Omega_3\frac{da_2}{dt}\right) + a_1\Lambda_1\Lambda_2 + a_2\Omega_1\Omega_2 - 2a_3\Lambda_1\Omega_2\,, \tag{2b}$$

$$H_{13} = a_3\frac{d\Omega_2}{dt} - a_1\frac{d\Lambda_2}{dt} + 2\left(\Omega_2\frac{da_3}{dt} - \Lambda_2\frac{da_1}{dt}\right) + a_3\Omega_3\Omega_1 + a_1\Lambda_3\Lambda_1 - 2a_2\Omega_3\Lambda_1\,. \tag{2c}$$

In equations (2) the Newtonian gravitational potentials A_i are given by the integral expression,

$$A_i = \int_0^\infty \frac{a_1a_2a_3\,du}{({a_i}^2 + u)[({a_1}^2 + u)({a_2}^2 + u)({a_3}^2 + u)]^{1/2}} \tag{3}$$

(note that $A_1 + A_2 + A_3 = 2$). To the nine equations of motion represented by equation (1) an additional constraint must be added, corresponding to the conservation of mass. Since the mass density is assumed to be a constant, this additional constraint reduces to

$$a_1a_2a_3 = \bar{a}^3 = \text{constant}\,. \tag{4}$$

The unit of length for this paper will be scaled so that $\bar{a} = 1$.

Equation (1) describes the hydrodynamical and the Newtonian gravitational effects on the evolution of the ellipsoid. In this discussion, the effects of viscosity and of gravitational radiation reaction will be of interest; therefore terms which describe those effects must be added to equation (1). The terms which describe the viscous interaction were first written for these ellipsoidal objects by Rosenkilde (1967) and later in a notation more closely related to the one used here by Press and Teukolsky (1973). We describe the effects of viscosity with the average viscosity of the ellipsoid ν and the matrix V_{ij}:

$$V_{11} = \frac{10}{{a_1}^2}\frac{da_1}{dt}\,, \tag{5a}$$

$$V_{12} = \frac{5}{a_2}\left(\frac{a_1}{a_2} - \frac{a_2}{a_1}\right)\Lambda_3\,, \tag{5b}$$

$$V_{13} = \frac{5}{a_3}\left(\frac{a_3}{a_1} - \frac{a_1}{a_3}\right)\Lambda_2\,. \tag{5c}$$

The other elements of V_{ij} can be obtained from equations (5) by cyclically permuting the subscripts 1, 2, 3.

The terms describing gravitational radiation reaction for the ellipsoidal figures were derived by Chandrasekhar (1970*a*, *b*) and modified to the form employed here by Miller (1974). This interaction employs the coupling constant $\mathcal{G} = 2GM/25c^5$ and the matrix G_{ij} defined by

$$G_{ij} = \sum_{\alpha=0}^{5}\sum_{\beta=0}^{\alpha} C^5{}_\alpha C^\alpha{}_\beta R^\beta{}_{in}\frac{d^{5-\alpha}Q_{nk}}{dt^{5-\alpha}}R^{\alpha-\beta}{}_{lk}A_{lj}\,, \tag{6}$$

where

$$C^\alpha{}_\beta = \alpha!/[(\alpha-\beta)!\beta!] \quad \text{and} \quad A_{ij} = \text{diag}\,(a_1, a_2, a_3)\,.$$

The quantity Q_{ij} is defined by

$$Q_{ij} = A_{ik}A_{kj} - \tfrac{1}{3}\delta_{ij}A_{kl}A_{kl}\,,$$

and is proportional to the quadrupole tensor for the ellipsoids. Lowercase Latin indices take the values 1, 2, 3, and summation is implied for repeated indices. The rotation matrices are defined by

$$R^0{}_{ij} = \delta_{ij} \quad \text{and} \quad R^{\alpha+1}{}_{ij} = \left(\delta_{il}\frac{d}{dt} - \epsilon_{ilm}\Omega_m\right)R^\alpha{}_{lj}\,.$$

In terms of the quantities defined above, the complete description of the evolution of the ellipsoidal figures is given by

$$H_{ij} + \nu V_{ij} + \mathscr{G} G_{ij} = 0\,. \tag{7}$$

III. AN APPROXIMATION TO THE EQUATIONS OF MOTION

Equation (7) forms a set of ordinary differential equations which, in principle, may be integrated in a straightforward manner. However, in practice there is a major drawback: these equations govern not only the long-time-scale evolution of an ellipsoid due to the dissipative forces, but also the short-time-scale hydrodynamical oscillations of an ellipsoid which is not in perfect equilibrium. Thus, while we are primarily interested in the long-time-scale evolution near the stationary Riemann S ellipsoids, a numerical integration of equation (7) necessitates an extremely short time step size to refrain from losing information about the hydrodynamical oscillations.

In the derivation of equation (7) it is necessarily assumed that the effects of radiation reaction and of viscosity are small, but cumulative. We are led, therefore, to seek solutions to the equations of motion which are slowly evolving from one quasi-equilibrium configuration to another. Thus, in the equations of motion we assume that the velocities $d(a_i, \Omega_i, \Lambda_i)/dt$ and the dissipative coefficients ν and $\mathscr{G}$ are small, and subsequently drop terms which are higher than first order in these small quantities. These assumptions are equivalent to assuming as an initial configuration an ellipsoid which lies very near the Riemann S surface, and whose subsequent evolution consists of a slow cumulative motion in addition to small amplitude oscillations.

We now expand the terms in equation (7) to first order in the small quantities: ν, $\mathscr{G}$, and $d(a_i, \Omega_i, \Lambda_i)/dt$. The initial configuration is taken to be near the Riemann S surface; in particular we take $\Lambda_1 = \Lambda_2 = \Omega_1 = \Omega_2 = 0$ initially (we set $\Omega_3 = \Omega$, $\Lambda_3 = \Lambda$). The nonzero components of the viscous and radiative matrices to this order are given by

$$V_{12} = a_1 V_{21}/a_2 = 5(a_1{}^2 - a_2{}^2)\Lambda/a_1 a_2{}^2\,, \tag{8}$$

and

$$G_{12} = a_2 G_{21}/a_1 = 16\Omega^5 a_2(a_1{}^2 - a_2{}^2)\,. \tag{9}$$

The resulting equations of motion are the following:

$$H_{11} = -a_1(\Lambda^2 + \Omega^2) + 2a_2\Lambda\Omega + 2\pi G\rho a_1 A_1 - 2p_c/\rho a_1 = 0\,, \tag{10a}$$

$$H_{22} = -a_2(\Lambda^2 + \Omega^2) + 2a_1\Lambda\Omega + 2\pi G\rho a_2 A_2 - 2p_c/\rho a_2 = 0\,, \tag{10b}$$

$$H_{33} = 2\pi G\rho a_3 A_3 - 2p_c/\rho a_3 = 0\,, \tag{10c}$$

$$H_{12} + \nu V_{12} + \mathscr{G} G_{12} = a_1\frac{d\Lambda}{dt} - a_2\frac{d\Omega}{dt} + 2\Lambda\frac{da_1}{dt} - 2\Omega\frac{da_2}{dt} + 5\nu\Lambda\frac{a_1{}^2 - a_2{}^2}{a_1 a_2{}^2} + 16\mathscr{G}\Omega^5 a_2(a_1{}^2 - a_2{}^2) = 0\,, \tag{11a}$$

and

$$H_{21} + \nu V_{21} + \mathscr{G} G_{21} = a_1\frac{d\Omega}{dt} - a_2\frac{d\Lambda}{dt} + 2\Omega\frac{da_1}{dt} - 2\Lambda\frac{da_2}{dt} + 5\nu\Lambda\frac{a_1{}^2 - a_2{}^2}{a_1{}^2 a_2} + 16\mathscr{G}\Omega^5 a_1(a_1{}^2 - a_2{}^2) = 0\,. \tag{11b}$$

The off diagonal equations,

$$H_{13} = H_{31} = H_{23} = H_{32} = 0\,, \tag{12}$$

guarantee that the vorticity and the rotation axes maintain their orientation ($\Lambda_1 = \Lambda_2 = \Omega_1 = \Omega_2 = 0$) as the ellipsoid evolves.

Equations (10) may be solved for Λ, Ω, and p_c in terms of the values of a_i:

$$(\Lambda - \Omega)^2 = 2\pi G\rho\left[\frac{a_1 A_1 + a_2 A_2}{a_1 + a_2} - \frac{a_3{}^2 A_3}{a_1 a_2}\right], \tag{13a}$$

$$(\Lambda + \Omega)^2 = 2\pi G\rho\left[\frac{a_1 A_1 - a_2 A_2}{a_1 - a_2} + \frac{a_3{}^2 A_3}{a_1 a_2}\right], \tag{13b}$$

and

$$2p_c/\rho = 2\pi G\rho a_3{}^2 A_3\,. \tag{13c}$$

The relationships illustrated in equations (13) are identical to those of a Riemann S-type ellipsoid. Thus to the zeroth order in the small quantities the ellipsoid is instantaneously of Riemann S type; and to the first order in the small quantities the evolution of the ellipsoid is governed by equations (11).

The equations of evolution may be cast in a more useful form. To accomplish this, the equations relating Λ and Ω to a_1 and a_2 (eqs. [13a, b]) may be differentiated with respect to time to obtain expressions for $d\Lambda/dt$ and $d\Omega/dt$ as linear combinations of da_1/dt and da_2/dt. These expressions can then be used to eliminate $d\Lambda/dt$ and $d\Omega/dt$ from the equations of motion (11) and the resulting relationships solved to obtain equations for da_1/dt and da_2/dt. The final form of these equations are given by

$$\frac{da_i}{dt} = \nu b_i + \mathscr{G} c_i\,, \quad i = 1, 2, 3\,. \tag{14}$$

The quantities b_i and c_i are functions of a_1, a_2, a_3, Λ, and Ω; thus, these equations can be easily integrated numerically to explore the orbits of the slowly varying ellipsoidal figures. The precise expressions for the coefficients b_i and c_i are given by

$$b_1 = -5\Lambda \frac{a_1^2 - a_2^2}{a_1^2 a_2^2} \frac{Q(2, 1)(a_1 + a_2) + Q(2, -1)(a_1 - a_2)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{15a}$$

$$b_2 = -5\Lambda \frac{a_1^2 - a_2^2}{a_1^2 a_2^2} \frac{Q(1, -1)(a_1 - a_2) - Q(1, 1)(a_1 + a_2)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{15b}$$

$$b_3 = -a_3\left(\frac{b_1}{a_1} + \frac{b_2}{a_2}\right)\,, \tag{15c}$$

$$c_1 = -16\Omega^5(a_1^2 - a_2^2) \frac{Q(2, 1)(a_1 + a_2) + Q(2, -1)(a_2 - a_1)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{16a}$$

$$c_2 = -16\Omega^5(a_1^2 - a_2^2) \frac{Q(1, -1)(a_2 - a_1) - Q(1, 1)(a_1 + a_2)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{16b}$$

$$c_3 = -a_3\left(\frac{c_1}{a_1} + \frac{c_2}{a_2}\right)\,. \tag{16c}$$

In equations (15) and (16) we have made use of the symbols $Q(\alpha, \epsilon)$ which are defined by

$$Q(\alpha, \epsilon) = \frac{\pi G\rho}{\Lambda - \epsilon\Omega}\left[A_\alpha + a_\alpha A_{\alpha,\alpha} + \epsilon a_\beta A_{\beta,\alpha} - a_3 A_{\alpha,3} - \frac{a_3 a_\beta}{a_\alpha} A_{\beta,3} - \frac{a_1 A_1 + \epsilon a_2 A_2}{a_1 + \epsilon a_2} + (\epsilon)^\alpha \frac{a_3^2}{a_\alpha^2 a_\beta}(a_1 + \epsilon a_2)(3A_3 + a_3 A_{3,3} - a_\alpha A_{3,\alpha}) + \frac{2}{\pi G\rho}(\Lambda - \epsilon\Omega)^2\right]\,, \tag{17}$$

with $\alpha, \beta = 1, 2$; $\alpha \neq \beta$; $\epsilon = \pm 1$; and $A_{i,j} \equiv \partial A_i/\partial a_j$.

The Riemann S algebraic constraints, equations (13), along with the evolution equations (14), form the complete description of the slowly varying ellipsoidal figures. The evolution equations are rather complicated; however, the form in which they are presented allows them to be integrated numerically in a straightforward manner. We have performed these integrations and discuss the results in § IV.

IV. NUMERICAL RESULTS

The evolution of the slowly varying ellipsoidal figures is described by equations (14). From these equations it is clear that the evolutionary trajectory of a given initial ellipsoid is determined solely by the ratio of the viscous time scale to the radiation-reaction time scale. (The rate at which the ellipsoid evolves along the trajectory is proportional to the magnitudes of both dissipative time scales, however.) Thus, as in Paper I, it is convenient to introduce a dimensionless constant

$$X = \frac{125(1 - e_0^2)^{2/3}}{2\Omega_0^4} \frac{\nu}{GM\bar{a}^4/c^5}\,, \tag{18}$$

where Ω_0 and e_0 are the angular velocity and the eccentricity of the Maclaurin spheroid at the point of the onset of the dynamical instability; so that

$$\Omega_0^2/\pi G\rho = 0.44022 \quad \text{and} \quad e_0 = 0.95289\,.$$

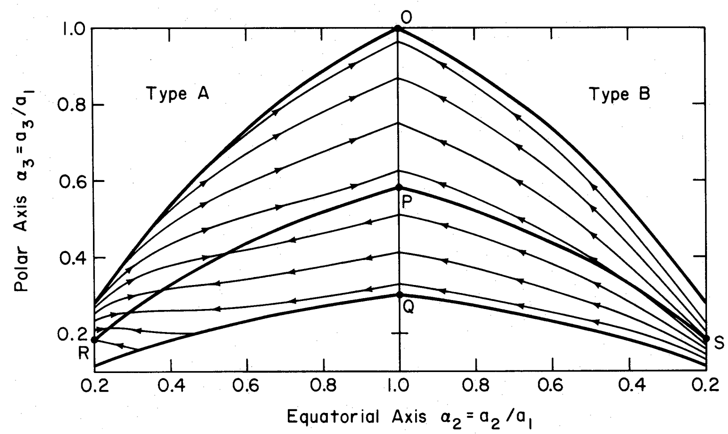

FIG. 1.—The evolutionary paths of the slowly varying ellipsoids for $X = 0$ (only radiation reaction) or for $X = \infty$ (only viscosity). The Jacobi and Dedekind sequences are RP and PS; the stable Maclaurin sequence is OP, and the secularly unstable Maclaurin sequence is PQ.

The constants appearing in equation (18) are chosen such that if $X = 1$, then (as discussed in Paper I) the entire Maclaurin sequence is stable up to the point of the onset of a dynamical instability, point Q in Figure 1. X is a certain ratio of the viscous to the gravitational radiation time scale; thus, the trajectories of the ellipsoids are determined simply by specifying X. The evolution of the ellipsoid will tend to be viscosity dominated if $X > 1$ and radiation reaction dominated if $X < 1$.

We have examined the evolution of the slowly varying ellipsoidal figures by numerically integrating equations (14). As a test of the approximation discussed in § III we compared some of our trajectories with those tabulated by Miller (1974) and found, as expected, that her ellipsoids performed small oscillations about a sequence of Riemann S-type ellipsoids which satisfied equations (14). As a second check we evolved ellipsoids with either the viscosity or the radiation reaction forces absent, and found that either the circulation or the angular momentum, respectively, were conserved as required for these interactions (see E.F.E. and Miller 1974).

The results of our numerical analysis are presented in Figures 1–5. For different choices of the parameter X the evolutionary tracks effectively cover the Riemann S surface. Figure 1 serves the dual roles of illustrating the evolution of an ellipsoid with X either zero or infinite. For $X = 0$ the evolution is caused solely by the radiation reaction; in this case Type A corresponds to the Dedekind-like ($|\Omega| < |\Lambda|$) ellipsoids, Type B to the Jacobi-like ($|\Omega| > |\Lambda|$); the trajectories are contours of constant circulation; all of the evolution is directed either toward the stable portion of the Maclaurin sequence, OP, or toward the Dedekind sequence, RP. Similarly for $X = \infty$, the evolution is caused solely by the viscosity; Type A corresponds to the Jacobi-like ellipsoids and Type B to the Dedekind-like; the trajectories are contours of constant angular momentum; all of the evolution is directed either toward OP or toward the Jacobi sequence, RP.

Figures 2–4 illustrate the evolutionary tracks for values of X ranging from 10^{-4} to 50. For $X = 50$ (Fig. 5) the typical evolution may proceed as follows: starting below the Dedekind sequence on the left half of the diagram, the

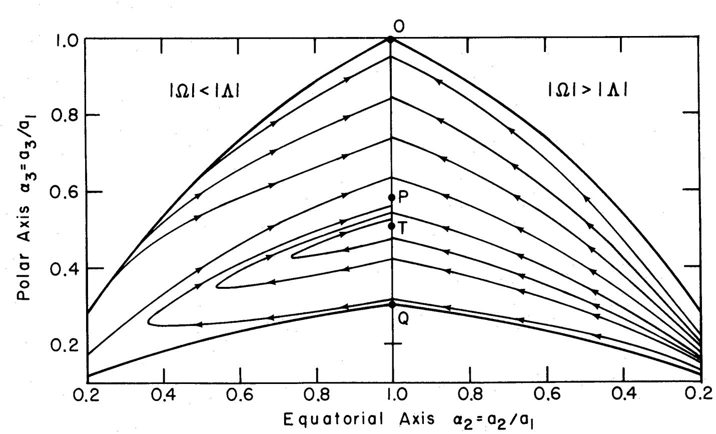

FIG. 2.—The evolutionary paths for $X = 10^{-4}$. The point P is the point of bifurcation of the Jacobi and Dedekind sequences. The stable Maclaurin spheroids are OT; the secularly unstable Maclaurin spheroids are TQ.

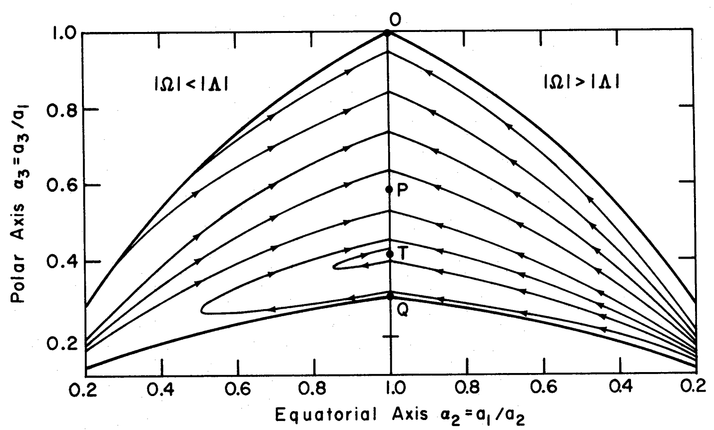

FIG. 3.—The evolutionary paths for $X = 10^{-2}$

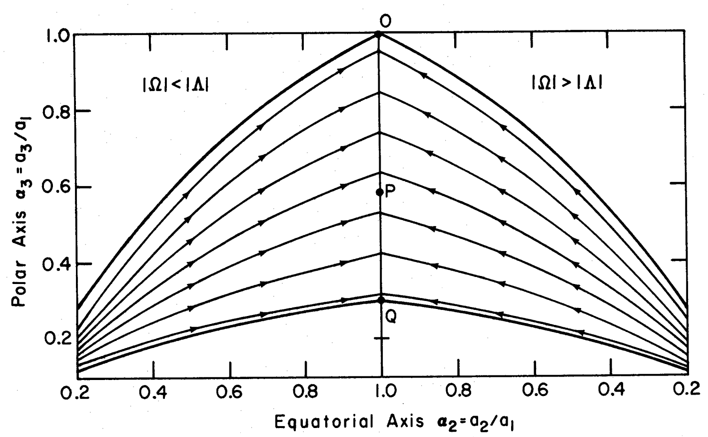

FIG. 4.—The evolutionary paths for $X = 1$. The point Q is the point of the onset of dynamical instability. All of the Maclaurin spheroids along OQ are stable.

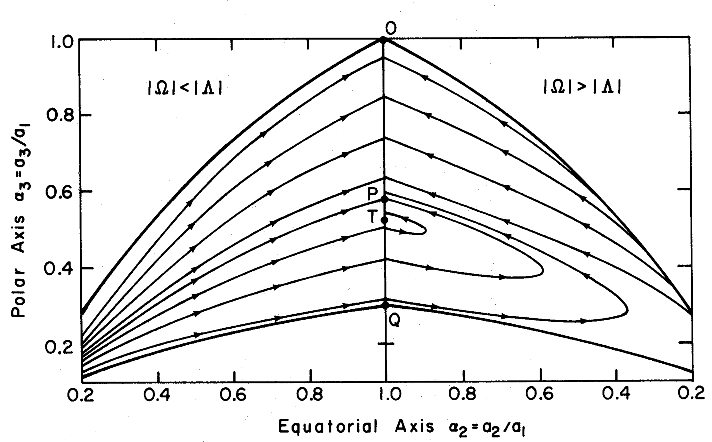

FIG. 5.—The evolutionary paths for $X = 50$

ellipsoid is driven up and toward the Maclaurin sequence nearly along a contour of constant angular momentum. If it approaches the Maclaurin sequence below the point T, the point of the onset of secular instability, it will be evolving toward an unstable Maclaurin spheroid. In a physically realistic situation a small perturbation is now needed to move the ellipsoid across the Maclaurin sequence into the Jacobi-like region. The evolution continues then toward the Jacobi sequence (PS in Fig. 1); the circulation decreases because of viscous dissipation, and the quadrupole moment increases as the ellipsoid tends toward a "rotating cigar" configuration. Eventually viscosity ceases to play the dominant role in the evolution—not because of the lack of viscosity but rather because the ellipsoid is nearly rigidly rotating. With a large rotating quadrupole moment, the ellipsoid now loses angular momentum in the form of gravitational radiation and the evolution proceeds back toward a stable member of the Maclaurin sequence. Analogous evolutionary scenarios describe each of the other figures.

For a choice of viscosity such that $X = 1$ (Fig. 4) all of the evolutionary tracks lead directly toward the Maclaurin sequence implying that the entire line OQ is in stable equilibrium. If a Maclaurin spheroid anywhere along the line OQ is displaced slightly, it simply evolves back toward the Maclaurin sequence.

Figures 1–5 clearly illustrate the results of Paper I. A Maclaurin spheroid which lies on the sequence OT is stable; if perturbed slightly, it simply evolves back toward the Maclaurin sequence. On the other hand, a spheroid on the sequence TQ is unstable; if it is perturbed, it evolves away from the Maclaurin sequence. For the limiting cases of purely viscous or purely radiative dissipation, Figure 1 shows that the point T, the onset of secular instability, coincides with the bifurcation point P. Figure 3 illustrates the case $X = 1$, showing that the point T coincides with Q, the point of dynamical instability.

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