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Note

Linear degeneracy of the first-order generalized-harmonic Einstein system

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Abstract
The purpose of this note is to clarify the conditions under which the first-order generalize-harmonic representation of the vacuum Einstein evolution system is linearly degenerate.

Keywords: numerical relativity, Einstein's equation, shock formation

The formation of coordinate shocks is one of the important problems that must be overcome by any representation of Einstein's equation that is to be used successfully in numerical relativity. Poor dynamical gauge conditions can and will lead to the formation of shocks (and consequently coordinate singularities) from the evolution of smooth initial data [1]. Linear degeneracy is a mathematical condition that prevents the formation of shocks in a large class of hyperbolic evolution systems [2–4]. The purpose of this note is to clarify the conditions under which the first-order generalized-harmonic representation of the vacuum Einstein system [5] is linearly degenerate. The original paper on this system claimed, without presenting a proof, that the system was linearly degenerate if a certain constant satisfied the condition $\gamma_1 = -1$ [5]. Here we demonstrate that this claim is correct. While the proof is fairly straightforward, some readers of the original paper have questioned whether that condition is correct [6]. Consequently it seems appropriate to provide a more complete description of the derivation that demonstrates this fact.

The first-order generalized-harmonic representation of Einstein's vacuum equation [5] can be written abstractly as a quasi-linear system,

$$\partial_t u^\alpha + A^{k\alpha}{}_\beta \partial_k u^\beta = F^\alpha, \tag{1}$$

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where $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{iab}\}$ is the collection of dynamical fields: ψ_{ab} the spacetime metric, and its time and space derivatives Π_{ab} and Φ_{iab}. The quantities $A^{k\alpha}{}_\beta$ and F^α depend on u^α but not its derivatives. We use the notation $s^\alpha\,\partial_{u^\alpha}$ for vectors, and $t_\alpha\,\mathrm{d}u^\alpha$ for co-vectors on the space of dynamical fields. The principal parts of the first-order generalized-harmonic vacuum Einstein system, $\partial_t u^\alpha + A^{k\alpha}{}_\beta \partial_k u^\beta \simeq 0$, are given explicitly by

$$\partial_t \psi_{ab} - (1+\gamma_1) N^k \partial_k \psi_{ab} \simeq 0, \tag{2}$$

$$\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + N g^{ki} \partial_k \Phi_{iab} - \gamma_1 \gamma_2 N^k \partial_k \psi_{ab} \simeq 0, \tag{3}$$

$$\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} - \gamma_2 N \partial_i \psi_{ab} \simeq 0, \tag{4}$$

where N and N^k are the lapse and shift associated with the standard $3+1$ representation of the metric ψ_{ab}, and where γ_1 and γ_2 are constants[5]. The characteristic matrix $n_k A^{k\alpha}{}_\beta$ for this system can be written as

$$\begin{aligned} n_k A^{k\alpha}{}_\beta\,\partial_{u^\alpha} \otimes \mathrm{d}u^\beta &= -(1+\gamma_1) n_k N^k\,\partial_{\psi_{ab}} \otimes \mathrm{d}\psi_{ab} \\ &- n_k N^k\,\partial_{\Pi_{ab}} \otimes \mathrm{d}\Pi_{ab} + N n^k\,\partial_{\Pi_{ab}} \otimes \mathrm{d}\Phi_{kab} - \gamma_1\gamma_2 n_k N^k\,\partial_{\Pi_{ab}} \otimes \mathrm{d}\psi_{ab} \\ &- n_k N^k\,\partial_{\Psi_{iab}} \otimes \mathrm{d}\Psi_{iab} + N n_k\,\partial_{\Psi_{kab}} \otimes \mathrm{d}\Pi_{ab} - \gamma_2 N n_k\,\partial_{\Psi_{kab}} \otimes \mathrm{d}\psi_{ab}, \end{aligned} \tag{5}$$

for waves propagating through a surface (chosen arbitrarily) with spacelike unit normal co-vector n_k. Summation over repeated indices, e.g. k, a, and b, is implied. Linear degeneracy is a condition on the eigenvalues and eigenvectors of this characteristic matrix.

The left and right eigenvectors, $\ell^{\hat\alpha}$ and $r_{\hat\alpha}$ respectively, of the characteristic matrix $n_k A^{k\alpha}{}_\beta$ are defined by:

$$\ell^{\hat\alpha}{}_\alpha n_k A^{k\alpha}{}_\beta = v^{(\hat\alpha)} \ell^{\hat\alpha}{}_\beta, \tag{6}$$

$$n_k A^{k\alpha}{}_\beta r_{\hat\alpha}{}^\beta = v_{(\hat\alpha)} r_{\hat\alpha}{}^\alpha. \tag{7}$$

The left eigenvectors $\ell^{\hat\alpha}{}_\beta \mathrm{d}u^\beta$ of the first-order generalized-harmonic vacuum Einstein system are given by

$$\ell^{\hat 0}_{ab\,\beta}\,\mathrm{d}u^\beta = \mathrm{d}\psi_{ab}, \tag{8}$$

$$\ell^{\hat 1\pm}_{ab}{}_\beta\,\mathrm{d}u^\beta = \mathrm{d}\Pi_{ab} \pm n^i \mathrm{d}\Phi_{iab} - \gamma_2\,\mathrm{d}\psi_{ab}, \tag{9}$$

$$\ell^{\hat 2}_{iab\,\beta}\,\mathrm{d}u^\beta = (\delta_i{}^j - n_i n^j)\mathrm{d}\Phi_{jab}, \tag{10}$$

while the right eigenvectors $r_{\hat\alpha}{}^\beta \partial_{u^\beta}$ are given by

$$r^{ab\,\beta}_{\hat 0}\,\partial_{u^\beta} = \partial_{\psi_{ab}} + \gamma_2\,\partial_{\Pi_{ab}}, \tag{11}$$

$$r^{ab\,\beta}_{\hat 1\pm}\,\partial_{u^\beta} = \partial_{\Pi_{ab}} \pm n_i\,\partial_{\Phi_{iab}}, \tag{12}$$

$$r^{iab\,\beta}_{\hat 2}\,\partial_{u^\beta} = (\delta^i{}_j - n^i n_j)\,\partial_{\Phi_{jab}}. \tag{13}$$

[5] The constants γ_1 and γ_2 multiplied by certain constraints of the vacuum Einstein system were added to the equations in [5]. The resulting system is symmetric hyperbolic for any values of these constants. As shown here, the constant γ_1 effects the linear degeneracy of the system. The constant γ_2 effects the growth of small constraint violations, and must be positive, $\gamma_2 > 0$, for numerical stability.

The first-order vacuum Einstein system is symmetric hyperbolic, since there exists a symmetric positive definite tensor $S_{\alpha\beta}$ on the space of fields that satisfies the condition $S_{\alpha\mu}A^{k\mu}{}_{\beta} \equiv A^{k}{}_{\alpha\beta} = A^{k}{}_{\beta\alpha}$ [5]. This implies that the left and right eigenvectors are related (up to normalizations) by $\ell^{\hat\alpha}{}_{\alpha} = S_{\alpha\beta} r_{\hat\alpha}{}^{\beta}$, and the associated eigenvalues must be equal $v_{(\hat\alpha)} = v^{(\hat\alpha)}$. These eigenvalues for the vacuum Einstein system are given by

$$v^{(\hat 0)} = v_{(\hat 0)} = -(1+\gamma_1) n_k N^k, \tag{14}$$

$$v^{(\hat 1\pm)} = v_{(\hat 1\pm)} = \pm N - n_k N^k, \tag{15}$$

$$v^{(\hat 2)} = v_{(\hat 2)} = -n_k N^k. \tag{16}$$

The quasi-linear hyperbolic evolution system, equation (1), is said to be linearly degenerate if all the eigenvalues of the characteristic matrix are constant along the corresponding right eigenvectors of that system, so that

$$r_{\hat\alpha}{}^{\alpha} \frac{\partial v_{(\hat\alpha)}}{\partial u^{\alpha}} = 0, \tag{17}$$

for each $\hat\alpha$ [2]. The eigenvalues of the Einstein system, equations (14)–(16), depend only on the lapse, N, the shift, N^k, and the unit normal vector n^k. These eigenvalues therefore depend only on the metric, ψ_{ab}, and not on its derivatives, Π_{ab} or Φ_{iab}. Thus the derivatives of the eigenvalues in the direction of the right eigenvectors are given by,

$$r_{\hat 0}^{ab\ \alpha} \frac{\partial v_{(\hat 0)}}{\partial u^{\alpha}} = \frac{\partial v_{(\hat 0)}}{\partial \psi_{ab}} = -(1+\gamma_1) \frac{\partial (n_k N^k)}{\partial \psi_{ab}}, \tag{18}$$

$$r_{\hat 1\pm}^{ab\ \alpha} \frac{\partial v_{(\hat 1\pm)}}{\partial u^{\alpha}} = 0, \tag{19}$$

$$r_{\hat 2}^{iab\ \alpha} \frac{\partial v_{(\hat 2)}}{\partial u^{\alpha}} = 0. \tag{20}$$

These derivatives vanish, and consequently the system is linearly degenerate, if and only if $\gamma_1 = -1$.

The original paper on the first-order generalized-harmonic vacuum Einstein system did not explicitly give expressions for either the left or the right eigenvectors [5]. The characteristic fields, $\hat u^{\hat\alpha} \equiv \ell^{\hat\alpha}{}_{\beta} u^{\beta}$, of this system were given, however, and from those the left eigenvectors could easily be inferred. The reported confusion about the correct conditions for linear degeneracy for this system may have arisen from an examination of the quantities $\ell^{\hat\alpha}{}_{\alpha} \partial v^{(\hat\alpha)}/\partial u^{\alpha}$, which do not vanish unless $\gamma_1 = -1$ and $\gamma_2 = 0$ [6]. These quantities involving the left eigenvectors (which are co-vectors, not true vectors) are non-covariant and are therefore meaningless from a fundamental mathematical viewpoint. In any case they are irrelevant because the formal definition of linear degeneracy given by Lax in [2] specifies that the right eigenvectors are to be used in equation (17), and this equation is covariant.

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