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Scalar, vector and tensor harmonics on the flat compact orientable three-manifolds

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Abstract. Observations suggest that our universe is spatially flat on the largest observable scales. Exactly six different compact orientable three-dimensional manifolds admit flat metrics. These six manifolds are therefore the most natural choices for building cosmological models based on the present observations. This paper briefly describes these six manifolds and the harmonic basis functions previously developed for representing arbitrary scalar fields on them. The principal focus of this paper is the development of new harmonics for representing arbitrary vector and second-rank tensor fields on these manifolds. These new harmonics are designed to be useful tools for analyzing the dynamics of electromagnetic and gravitational fields on these spaces.

Keywords: cosmological simulations, gravity, physics of the early universe

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Contents

1 Introduction

Current observations of the cosmic microwave radiation, together with observations of lower-redshift distance indicators, show that the large-scale average density of the universe differs by no more than 0.4% (the present observational error estimate) from the critical value that implies the geometry of our universe is spatially flat on these largest observable scales [1]. There are eighteen different three-dimensional manifolds that admit flat metrics [2–5], so these are the most natural manifolds on which to construct realistic cosmological models. Ten of these flat manifolds are orientable, while eight are non-orientable. Spacetimes having non-orientable spatial slices are not parallelizable (i.e. they do not admit collections of smooth non-vanishing linearly independent vector fields), and (consequently) such manifolds do not admit spinor structures [6, 7]. Representations of fermions depend on the existence of these spinor structures, so unless (or perhaps until) new theoretical approaches for defining spinors are developed, the non-orientable three-manifolds appear to be less physically relevant than the orientable ones. A purely emperical approach to the analysis of cosmological observations could justify the inclusion of the non-orientable cases anyway. But given the theoretical spinor structure argument for excluding them, we have chosen to limit our analysis here to the orientable cases. Of the ten orientable flat three-manifolds, six are compact while the remaining four are non-compact. Current observations do not allow us to see the entire universe, so we have no way of knowing whether or not our universe is spatially compact. For computational convenience, we choose to limit consideration here to models having compact spatial slices, i.e. models with finite spatial volumes.

Each of the six compact orientable three-dimensional manifolds that admits a flat metric can be obtained as a quotient $\mathbb{E}^3/\Gamma$ of three-dimensional Euclidean space $\mathbb{E}^3$ by an isometry group Γ of symmetries of $\mathbb{E}^3$. The classification of these spaces has long been known [2–5]. We use the notation $E_1, E_2, \ldots, E_6$ to refer to these spaces. The group action that defines each of these spaces can be thought of as a particular representation of $\mathbb{E}^3$ as a periodic lattice of polytopes. The individual flat compact manifolds can be thought of as one of these polytopes with identifications between its faces. Figure 1 of ref. [8] illustrates these polytope-with-identifications representations of these manifolds. The spaces E_1, E_2 and E_3 are based on rectangular lattice representations of $\mathbb{E}^3$. E_1 is the simple three-torus, obtained by identifying opposite faces of a rectangular solid. E_2 and E_3 are obtained by identifying the opposite x and y faces of a rectangular solid, but twisting by π before identifying opposite z faces for the half-turn space E_2, or by $\pi/2$ for the quarter-turn space E_3. E_4 and E_5 are obtained from representations of $\mathbb{E}^3$ as a lattice of hexagonal prisms. These hexagonal prisms have six rectangular faces, and two hexagonal faces. The opposing rectangular faces of these prisms are identified in E_4 and E_5, while the hexagonal faces are twisted by $2\pi/3$ for the third-turn space E_4, or by $\pi/3$ for the sixth-turn space E_5. The Hantzsche-Wendt space, E_6, is based on a representation of $\mathbb{E}^3$ by a lattice of rhombic dodecahedrons. The space E_6 is formed by a particular identification of the rhombic faces of one of these dodecahedrons. Explicit descriptions of the symmetries used to create each of the spaces, $E_1, E_2, \ldots, E_6$, are given in section 2 as part of our discussion of the scalar harmonics on these spaces.

We use the term harmonics in this paper to refer to the eigenfunctions of the covariant Laplace operator. These eigenfunctions form a complete set of smooth functions on these compact orientable manifolds, and can therefore be used as a basis for representing arbitrary square integrable functions on them. These harmonics can be thought of as generalizations of the Fourier basis functions used to construct representations of functions: $f(\mathbf{x}) = \sum_{\mathbf{k}} f_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$. Scalar harmonics have been developed for each of the six flat compact orientable three-dimensional manifolds, and these harmonics have been used to model the temperature variations in the cosmic microwave background radiation that would be observed on these spaces [8]. Scalar harmonics, however, are not adequate to model the full dynamics of the gravitational or the electromagnetic fields. For example, an analysis of the polarization properties of the cosmic microwave background radiation would require representations of the full vector structure of the electromagnetic field. A number of studies have been carried out on the vector and tensor harmonics on the (less physically relevant) manifolds with the topology of the three-sphere, $\mathbb{S}^3$ [9–15]. But little attention has been paid to such matters on the less-familiar three-manifolds that admit flat metrics. There has been some work on constructing second-rank tensor basis functions for E_1 in the context of studying the effects of inhomogeneous initial conditions on inflation [16], or the possibility of gravitational wave turbulence in the early universe [17]. Here we significantly generalize these studies by constructing complete vector harmonic and second-rank tensor harmonic basis functions on all six flat compact orientable three-manifolds. To facilitate the analysis of the dynamics of the gravitational and electromagnetic fields on these manifolds, we have organized these new harmonics into subsets that maximize the number of classes having vanishing divergence and trace. These new harmonics were also constructed to satisfy nice orthonormality conditions that make it easy to represent arbitrary scalar and second-rank tensor fields on these manifolds.

The remainder of this paper is organized as follows. Explicit expressions for the symmetries used to construct each of the flat compact orientable three-manifolds, $E_1, E_2, \ldots, E_6$, are given in section 2, along with explicit expressions for the (suitably re-normalized) scalar harmonics developed in ref. [8]. The analogous vector and anti-symmetric second-rank tensor harmonics are developed in section 3. Two classes of these new vector harmonics are divergence free, so they provide a natural way to represent the transverse parts of dynamical electromagnetic fields. Symmetric second-rank tensor harmonics are developed in section 4. These new tensor harmonics include five classes of trace-free harmonics, two of which are also divergence-free. These new tensor harmonics are well suited therefore for representing the transverse-traceless parts (i.e. the dynamical gravitational wave parts) of the gravitational fields on these flat compact orientable three-manifolds. Section 5 contains a brief summary and discussion of the new results. Several useful technical lemmas needed in the construction of the new vector and tensor harmonics are given in an appendix.

2 Scalar harmonics

The scalar harmonics are defined here to be eigenfunctions of the covariant Laplace operator:

$$\nabla^a\nabla_a Y = -\kappa^2 Y. \tag{2.1}$$

We use the notation $Y[E_j]_{\mathbf{k}}$ to denote the harmonics on the manifold E_j (for $j = 1, 2, \ldots, 6$), where $\mathbf{k} = k_a = (k_1, k_2, k_3)$ are parameters that identify a particular harmonic. On the three-torus, E_1, these harmonics are (up to normalizations) just the Fourier basis functions:

$$Y[E_1]_{\mathbf{k}} = \frac{e^{\,i\,\mathbf{k}\cdot\mathbf{x}}}{\sqrt{L_1 L_2 L_3}}, \tag{2.2}$$

where L_1, L_2, and L_3, are the lengths of the three principal axes of E_1. The parameters $\mathbf{k}$ are chosen to ensure that the harmonics have the appropriate periodicities on $\mathbb{E}^3$ to make them smooth functions on E_1:

$$\mathbf{k} = k_a = 2\pi\left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3}\right). \tag{2.3}$$

This requires the n_a (for $a = 1, 2, 3$) to be integers, $n_a \in \mathbb{Z}$. The eigenvalues for these solutions to eq. (2.1) are given by

$$\kappa^2 = k^a k_a = k_1^2 + k_2^2 + k_3^2. \tag{2.4}$$

The normalization has been chosen in eq. (2.2) to ensure that these harmonics satisfy the orthonormality conditions,

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} Y[E_1]_{\mathbf{k}}\, Y[E_1]^*_{\mathbf{k}'}\, dx\, dy\, dz = \delta_{n_1 n_1{}'}\,\delta_{n_2 n_2{}'}\,\delta_{n_3 n_3{}'}. \tag{2.5}$$

The harmonics for the remaining flat manifolds, $E_2, \ldots, E_6$, were derived in ref. [8] by using the fact that each scalar harmonic $Y_{\mathbf{k}}$ of $\mathbb{E}^3/\Gamma$ lifts to a $\Gamma-$periodic harmonic $Y_{\mathbf{k}}$ of $\mathbb{E}^3$, i.e. to a scalar harmonic of $\mathbb{E}^3$ that is invariant under the action of the isometry group Γ. Finding the scalar harmonics of the flat space $\mathbb{E}^3/\Gamma$ is equivalent, therefore, to finding the Γ-periodic scalar harmonics of $\mathbb{E}^3$. Each element of an isometry group Γ of Euclidean space $\mathbb{E}^3$, can be written as a rotation/reflection $\mathbf{M}$ followed by a translation $\mathbf{T}$: i.e. these isometries map points $\mathbf{x} \in \mathbb{E}^3$ to the points $\mathbf{x}' = \mathbf{M}\cdot\mathbf{x} + \mathbf{T}$, or equivalently in component notation $x'^a = M^a{}_b\, x^b + T^a$. These transformations are isometries so they preserve the forms of the metric $\mathbf{g} = g_{ab} = \mathrm{diag}(1,1,1)$ and the inverse metric $\mathbf{g}^{-1} = g^{ab} = \mathrm{diag}(1,1,1)$. This implies that $g_{ab} = M^c{}_a\, M^d{}_b\, g_{cd}$ and $g^{ab} = M^a{}_c\, M^b{}_d\, g^{cd}$, and consequently that $\kappa^2 = k^a k_a = \mathbf{g}^{-1}(\mathbf{k},\mathbf{k})$ and $\kappa^2 = k_a M^a{}_c\, k_b\, M^b{}_d\, g^{cd} = \mathbf{g}^{-1}(\mathbf{kM},\mathbf{kM})$ for these isometries.

The isometry group Γ of the half-turn space, E_2, consists of pure translations by L_1, L_2 or L_3 in the principal directions, plus a compound rotation-translation defined by

$$\mathbf{M}[E_2] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{T}[E_2] = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}L_3 \end{pmatrix}. \tag{2.6}$$

The scalar harmonics invariant under these transformations are given by [8],

$$Y[E_2]_{\mathbf{k}} = \frac{1}{\sqrt{2}}\left[Y[E_1]_{\mathbf{k}} + (-1)^{n_3} Y[E_1]_{\mathbf{kM}[E_2]}\right]$$
$$\text{for}\quad \left(n_1 \in \mathbb{Z}^+, n_2, n_3 \in \mathbb{Z}\right) \quad \text{or} \quad \left(n_1 = 0, n_2 \in \mathbb{Z}^+, n_3 \in \mathbb{Z}\right), \tag{2.7}$$

$$Y[E_2]_{(0,0,k_3)} = Y[E_1]_{(0,0,k_3)} \quad \text{for} \quad (n_1 = n_2 = 0, n_3 \in 2\mathbb{Z}), \tag{2.8}$$

where the $Y[E_1]_{\mathbf{k}}$ are the basic E_1 harmonics given in eq. (2.2), and $\mathbf{k} = (k_1, k_2, k_3) = 2\pi\left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3}\right)$. These harmonics have the same eigenvalues as the E_1 harmonics, eq. (2.4), and satisfy orthonormality conditions analogous to eq. (2.5).

The isometry group of the quarter-turn space, E_3, consists of pure translations by L_1, L_2 or L_3 in the principal directions, plus a compound rotation-translation defined by

$$\mathbf{M}[E_3] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{T}[E_3] = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4}L_3 \end{pmatrix}. \tag{2.9}$$

These symmetries imply that $L_1 = L_2$ in the E_3 case. The scalar harmonics invariant under these transformations are given by [8],

$$Y[E_3]_{\mathbf{k}} = \frac{1}{2}\left[Y[E_1]_{\mathbf{k}} + i^{\,n_3} Y[E_1]_{\mathbf{k}\mathbf{M}[E_3]} + i^{\,2n_3} Y[E_1]_{\mathbf{k}\mathbf{M}[E_3]^2} + i^{\,3n_3} Y[E_1]_{\mathbf{k}\mathbf{M}[E_3]^3}\right]$$
$$\text{for} \quad \left(n_1 \in \mathbb{Z}^+, n_2 \in \mathbb{Z}^+ \cup \{0\}, n_3 \in \mathbb{Z}\right), \tag{2.10}$$
$$Y[E_3]_{(0,0,k_3)} = Y[E_1]_{(0,0,k_3)} \quad \text{for} \quad (n_1 = n_2 = 0, n_3 \in 4\mathbb{Z}), \tag{2.11}$$

where the $Y[E_1]_{\mathbf{k}}$ are the basic E_1 harmonics given in eq. (2.2), and $\mathbf{k} = (k_1, k_2, k_3) = 2\pi\left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3}\right)$. These harmonics have the same eigenvalues as the E_1 harmonics, eq. (2.4), and satisfy orthonormality conditions analogous to eq. (2.5).

The spaces E_4 and E_5 are constructed from a hexagonal prism lattice representation of $\mathbb{E}^3$ that is generated by the four pure-translation symmetries

$$\mathbf{T}_1 = \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{T}_2 = \begin{pmatrix} -\frac{1}{2}L_1 \\ \frac{\sqrt{3}}{2}L_2 \\ 0 \end{pmatrix}, \qquad \mathbf{T}_3 = \begin{pmatrix} -\frac{1}{2}L_1 \\ -\frac{\sqrt{3}}{2}L_2 \\ 0 \end{pmatrix}, \qquad \mathbf{T}_4 = \begin{pmatrix} 0 \\ 0 \\ L_3 \end{pmatrix}. \tag{2.12}$$

The isometry group of the third-turn space, E_4, consists of the pure translations given in eq. (2.12) plus a compound rotation-translation defined by

$$\mathbf{M}[E_4] = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{T}[E_4] = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3}L_3 \end{pmatrix}. \tag{2.13}$$

These symmetries imply that $L_1 = L_2$ in the E_4 case. The scalar harmonics invariant under these transformations are given by [8],

$$Y[E_4]_{\mathbf{k}} = \frac{1}{\sqrt{3}}\left[Y[E_1]_{\mathbf{k}} + \omega^{\,2n_3} Y[E_1]_{\mathbf{k}\mathbf{M}[E_4]} + \omega^{\,4n_3} Y[E_1]_{\mathbf{k}\mathbf{M}[E_4]^2}\right]$$
$$\text{for} \quad \left(n_1 \in \mathbb{Z}^+, n_2 \in \mathbb{Z}^+ \cup \{0\}, n_3 \in \mathbb{Z}\right), \tag{2.14}$$
$$Y[E_4]_{(0,0,k_3)} = Y[E_1]_{(0,0,k_3)} \quad \text{for} \quad (n_1 = n_2 = 0, n_3 \in 3\mathbb{Z}), \tag{2.15}$$

where the $Y[E_1]_{\mathbf{k}}$ are the basic E_1 harmonics given in eq. (2.2), and $\omega = e^{\,i\pi/3}$. To preserve the hexagonal translation symmetry in this case we must also take $\mathbf{k} = (k_1, k_2, k_3) = 2\pi\left(-\frac{n_2}{L_1}, \frac{2n_1 - n_2}{\sqrt{3}L_2}, \frac{n_3}{L_3}\right)$. These harmonics have the same eigenvalues as the E_1 harmonics, eq. (2.4), and satisfy orthonormality conditions analogous to eq. (2.5).

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The isometry group of the sixth-turn space, E_5, consists of the pure hexagonal lattice translations given in eq. (2.12), plus a compound rotation-translation defined by

$$\mathbf{M}[E_5] = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{T}[E_5] = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{6}L_3 \end{pmatrix}. \tag{2.16}$$

These symmetries imply that $L_1 = L_2$ in the E_5 case. The scalar harmonics invariant under these transformations are given by [8],

$$\begin{aligned} Y[E_5]_{\mathbf{k}} = \frac{1}{\sqrt{6}}\Big[& Y[E_1]_{\mathbf{k}} + \omega^{n_3} Y[E_1]_{\mathbf{k}\mathbf{M}[E_5]} + \omega^{2n_3} Y[E_1]_{\mathbf{k}\ \mathbf{M}[E_5]^2} + \omega^{3n_3} Y[E_1]_{\mathbf{k}\ \mathbf{M}[E_5]^3} \\ & + \omega^{4n_3} Y[E_1]_{\mathbf{k}\ \mathbf{M}[E_5]^4} + \omega^{5n_3} Y[E_1]_{\mathbf{k}\mathbf{M}[E_5]^5}\Big] \\ & \text{for} \quad \big(n_1 \in \mathbb{Z}^+, n_2 \in \mathbb{Z}^+ \cup \{0\}, n_2 < n_1, n_3 \in \mathbb{Z}\big)\,, \end{aligned} \tag{2.17}$$

$$Y[E_5]_{(0,0,k_3)} = Y[E_1]_{(0,0,k_3)} \quad \text{for} \quad (n_1 = n_2 = 0, n_3 \in 6\mathbb{Z})\,, \tag{2.18}$$

where the $Y[E_1]_{\mathbf{k}}$ are the basic E_1 harmonics given in eq. (2.2), and $\omega = e^{i\pi/3}$. To preserve the hexagonal translation symmetry in this case we must also take $\mathbf{k} = (k_1, k_2, k_3) = 2\pi\left(-\frac{n_2}{L_1}, \frac{2n_1 - n_2}{\sqrt{3}L_2}, \frac{n_3}{L_3}\right)$. These harmonics have the same eigenvalues as the E_1 harmonics, eq. (2.4), and satisfy orthonormality conditions analogous to eq. (2.5).

The isometry group of the Hantzsche-Wendt space, E_6, consists of pure translations by L_1, L_2 or L_3 in the principal directions, plus compound rotation-translations defined by

$$\begin{aligned} \mathbf{M}_1[E_6] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & \mathbf{T}_1[E_6] &= \begin{pmatrix} \frac{1}{2}L_1 \\ \frac{1}{2}L_2 \\ 0 \end{pmatrix}, \\ \mathbf{M}_2[E_6] &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & \mathbf{T}_2[E_6] &= \begin{pmatrix} 0 \\ \frac{1}{2}L_2 \\ \frac{1}{2}L_3 \end{pmatrix}, \\ \mathbf{M}_3[E_6] &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathbf{T}_3[E_6] &= \begin{pmatrix} \frac{1}{2}L_1 \\ 0 \\ \frac{1}{2}L_3 \end{pmatrix}. \end{aligned} \tag{2.19}$$

The scalar harmonics invariant under these transformations are given by [8],

$$\begin{aligned} Y[E_6]_{\mathbf{k}} = \frac{1}{2}\Big[& Y[E_1]_{\mathbf{k}} + (-1)^{n_1 - n_2} Y[E_1]_{\mathbf{k}\mathbf{M}_1[E_6]} \\ & + (-1)^{n_2 - n_3} Y[E_1]_{\mathbf{k}\mathbf{M}_2[E_6]} + (-1)^{n_3 - n_1} Y[E_1]_{\mathbf{k}\mathbf{M}_3[E_6]}\Big] \\ & \text{for} \quad \big(n_1, n_2 \in \mathbb{Z}^+, n_3 \in \mathbb{Z}\big) \quad \text{or} \quad \big(n_1 = 0, n_2, n_3 \in \mathbb{Z}^+\big) \\ & \text{or} \quad \big(n_2 = 0, n_1, n_3 \in \mathbb{Z}^+\big)\,, \end{aligned} \tag{2.20}$$

$$Y[E_6]_{(k_1,0,0)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]_{(k_1,0,0)} + Y[E_1]_{(-k_1,0,0)}\Big] \quad \text{for} \quad \big(n_1 \in 2\mathbb{Z}^+, n_2 = n_3 = 0\big)\,, \tag{2.21}$$

$$Y[E_6]_{(0,k_2,0)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]_{(0,k_2,0)} + Y[E_1]_{(0,-k_2,0)}\Big] \quad \text{for} \quad \big(n_2 \in 2\mathbb{Z}^+, n_1 = n_3 = 0\big)\,, \tag{2.22}$$

$$Y[E_6]_{(0,0,k_3)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]_{(0,0,k_3)} + Y[E_1]_{(0,0,-k_3)}\Big] \quad \text{for} \quad \big(n_3 \in 2\mathbb{Z}^+, n_1 = n_2 = 0\big)\,, \tag{2.23}$$

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where the $Y[E_1]_{\mathbf{k}}$ are the basic E_1 harmonics given in eq. (2.2), and $\mathbf{k} = (k_1, k_2, k_3) = 2\pi\left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3}\right)$. These harmonics have the same eigenvalues as the E_1 harmonics, eq. (2.4), and satisfy orthonormality conditions analogous to eq. (2.5).

3 Vector harmonics

This section constructs vector harmonics on the six flat compact orientable three-manifolds. While these harmonics are not unique and can be chosen in a variety of different ways, our goal here is to choose harmonics having three useful properties: first, the vector harmonics constructed here will be eigenfunctions of the covariant Laplace operator:

$$\nabla^b \nabla_b Y^a = -\kappa^2 Y^a. \tag{3.1}$$

This ensures that these harmonics will form a complete basis for the (square integrable) vector fields on these manifolds. We use the notation $Y[E_j]^a_{(A)\,\mathbf{k}}$ (for $A = 0, 1, 2$) to denote the three linearly independent classes of vector harmonics that satisfy eq. (3.1) on the space E_j (for $j = 1, 2, \ldots, 6$). Second, the vector harmonics constructed here will satisfy nice orthonormality conditions, i.e. $Y[E_j]^a_{(A)\,\mathbf{k}}$ and $Y[E_j]^a_{(A')\,\mathbf{k}'}$ will be orthogonal under the standard L_2 inner product unless $\mathbf{k} = \mathbf{k}'$ and $A = A'$. And third, the vector harmonics constructed here in classes $A = 1$ and $A = 2$ will be divergence free.

It is easy to construct one nice class of vector harmonics on all the flat compact orientable three-manifolds. These harmonics, which we call the class $A = 0$ harmonics, are defined as the gradients of the scalar harmonics on each manifold:

$$Y[E_j]^a_{(0)\,\mathbf{k}} = \kappa^{-1} \nabla^a Y[E_j]_{\mathbf{k}}, \tag{3.2}$$

where $\nabla^a = g^{ab}\nabla_b$ and κ^2 is the corresponding eigenvalue of the covariant scalar Laplace operator, eq. (2.4). Since the scalar harmonics $Y[E_j]_{\mathbf{k}}$ are smooth eigenfunctions of the Laplace operator, their gradients are automatically smooth eigenfunctions having the same eigenvalues on these flat manifolds. The normalization factor in eq. (3.2) is chosen to ensure that these harmonics satisfy the nice orthonormality conditions

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ab} Y[E_j]^a_{(0)\,\mathbf{k}} Y[E_j]^{b*}_{(0)\,\mathbf{k}'}\, dx\, dy\, dz = \delta_{n_1 n_1{}'}\,\delta_{n_2 n_2{}'}\,\delta_{n_3 n_3{}'}\,. \tag{3.3}$$

The divergences of these class $A = 0$ harmonics are given by

$$\nabla_a Y[E_j]^a_{(0)\,\mathbf{k}} = -\kappa Y[E_j]_{\mathbf{k}}. \tag{3.4}$$

The construction of the class $A = 1$ and $A = 2$ vector harmonics is less straightforward. We have developed three different approaches that produce vector harmonics satisfying the three useful properties listed above. The first approach produces the simplest expressions for the vector harmonics, but this approach only works in the space E_1. A somewhat more general approach can be used to derive fairly simple expressions for these vector harmonics in the spaces $E_1, \ldots, E_5$, but not in E_6. We have also developed an even more general approach capable of constructing vector harmonics in all the E_j spaces, but the resulting expressions produced in this way are quite complicated. Here we report the results of this general approach only for the otherwise intractable E_6 case.

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The first approach to constructing the needed vector harmonics is based on the fact that any covariently constant vector field c^a is invariant under the pure translation symmetry group of the three-torus, E_1. Any vector field of the form $c^a Y[E_1]_{\mathbf{k}}$ is an eigenfunction of the covariant Laplace operator, and therefore a candidate vector harmonic. The choices for three linearly independent vectors c^a will define the three classes of vector harmonics for this case. The class $A = 0$ vector harmonics defined in eq. (3.2) for the space E_1 have this form, $Y[E_1]^a_{(0)\,\mathbf{k}} = i\,\kappa^{-1} g^{ab} k_b Y[E_1]_{\mathbf{k}}$ with $c^a = i\,\kappa^{-1} g^{ab} k_b$. All that is needed to complete this simple approach is to choose two additional constant vectors to define the class $A = 1$ and $A = 2$ harmonics. Any unit vectors orthogonal to k_a will do. One choice is $\ell_a = (-k_2 - k_3, k_1 - k_3, k_1 + k_2)$ and $m_a = g_{ab}\epsilon^{bcd} k_c \ell_d$, where ϵ^{abc} is the covariantly constant, $\nabla_a \epsilon^{bcd} = 0$, totally antisymmetric tensor volume element. For convenience, these vectors can be normalized by setting $\hat{k}_a = \kappa^{-1} k_a$, $\hat{\ell}_a = \lambda^{-1}\ell_a$, and $\hat{m}_a = \kappa^{-1}\lambda^{-1} m_a$, where $\lambda^2 = \ell^a \ell_a = (k_2 + k_3)^2 + (k_1 - k_3)^2 + (k_1 + k_2)^2$. Vector harmonics defined in terms of the orthonormal constant vectors $\hat{k}^a = g^{ab}\hat{k}_b$, $\hat{\ell}^a = g^{ab}\hat{\ell}_b$, and $\hat{m}^a = g^{ab}\hat{m}_b$ are given by

$$Y[E_1]^a_{(0)\,\mathbf{k}} = i\,\hat{k}^a Y[E_1]_{\mathbf{k}}, \tag{3.5}$$

$$Y[E_1]^a_{(1)\,\mathbf{k}} = i\,\hat{\ell}^a Y[E_1]_{\mathbf{k}}, \tag{3.6}$$

$$Y[E_1]^a_{(2)\,\mathbf{k}} = i\,\hat{m}^a Y[E_1]_{\mathbf{k}}. \tag{3.7}$$

Special forms for $\hat{k}_a$, $\hat{\ell}_a$, and $\hat{m}_a$ are needed when $\kappa = 0$ or $\lambda = 0$. In the $\kappa = 0$ case we can define $\hat{k}_a = (1, 0, 0)$, $\hat{\ell}_a = (0, 1, 0)$ and $\hat{m}_a = (0, 0, 1)$. In the $\lambda = 0$ but $\kappa \neq 0$ case $\hat{k}_a = \frac{1}{\sqrt{3}}(1, -1, 1)$ so we can simply define $\hat{\ell}_a = \frac{1}{\sqrt{2}}(1, 1, 0)$ and $\hat{m}_a = \frac{1}{\sqrt{6}}(-1, 1, 2)$. All these vector harmonics defined in eqs. (3.5)–(3.7) satisfy each of the useful properties listed above. They are eigenfunctions of the covariant Laplace operator with eigenvalue $-\kappa^2$, and satisfy the following orthonormality conditions,

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ab} Y[E_1]^a_{(A)\,\mathbf{k}} Y[E_1]^{b*}_{(B)\,\mathbf{k}'}\, dx\, dy\, dz = \delta_{AB}\ \delta_{n_1 n_1'}\,\delta_{n_2 n_2'}\,\delta_{n_3 n_3'}\,. \tag{3.8}$$

The divergences of these vector harmonics on the space E_1 satisfy,

$$\nabla_a Y[E_1]^a_{(0)\,\mathbf{k}} = -\kappa Y[E_1]_{\mathbf{k}}, \tag{3.9}$$

$$\nabla_a Y[E_1]^a_{(1)\,\mathbf{k}} = \nabla_a Y[E_1]^a_{(2)\,\mathbf{k}} = 0. \tag{3.10}$$

The divergences of these class $A = 1$ and $A = 2$ vector harmonics vanish whenever $\hat{\ell}_a$ and $\hat{m}_a$ are chosen to be orthogonal to $\hat{k}_a$.

The second approach to constructing vector harmonics uses the fact that the unit vector along the z-axis, $\hat{z}^a$, is the only unit vector field invariant under all the symmetry groups of the manifolds $E_1, \ldots, E_5$. Therefore $\hat{z}^a Y[E_j]_{\mathbf{k}}$ (for $j = 1, \ldots, 5$) is an eigenfunction of the covariant Laplace operator, and can be used in the construction of the vector harmonics on these spaces in much the same way the constant vectors c^a were used in the first approach. We note that $\hat{z}^a \nabla_a Y[E_j]_{\mathbf{k}} = i\,k_3 Y[E_j]_{\mathbf{k}}$ in all of these cases. The vector harmonics for these manifolds can therefore be taken to be

$$Y[E_j]^a_{(0)\,\mathbf{k}} = \frac{1}{\kappa}\,\nabla^a Y[E_j]_{\mathbf{k}}, \tag{3.11}$$

$$Y[E_j]^a_{(1)\,\mathbf{k}} = \frac{1}{\kappa k_3\sqrt{\kappa^2-k_3^2}}\left(\kappa^2\hat{z}^a\hat{z}^b - k_3^2 g^{ab}\right)\nabla_b Y[E_j]_{\mathbf{k}}, \tag{3.12}$$

$$Y[E_j]^a_{(2)\,\mathbf{k}} = \frac{1}{\sqrt{\kappa^2-k_3^2}}\,\epsilon^{abc}\,\hat{z}_b\nabla_c Y[E_j]_{\mathbf{k}}, \tag{3.13}$$

so long as $\kappa^2\neq 0$, $k_3^2\neq 0$, and $\kappa^2\neq k_3^2$. When $\kappa^2=0$, the vector harmonics must be spatially constant vector fields. Three linearly independent spatially constant vector fields exist in the space E_1, so any orthonormal set can be used as $\kappa=0$ harmonics in that space. In the spaces $E_2,\ldots,E_5$ the only spatially constant unit vector field is $\hat{z}^a$, so it becomes the only $\kappa=0$ vector field on those spaces. When $k_3=0$ and $\kappa\neq 0$ the vector harmonics are independent of z, i.e. they depend only on x and y. In this case the class $A=0$ and $A=2$ harmonics are given by eqs. (3.11) and (3.13), but the expression for the $A=1$ harmonic must be replaced by $Y[E_j]^a_{(1)\,\mathbf{k}}=\hat{z}^a Y[E_j]_{\mathbf{k}}$. Finally if $\kappa^2=k_3^2$ and $\kappa\neq 0$ the vector harmonics are independent of x and y, i.e. the harmonics depend only on z. In this case there is only the single class of vector harmonics given by eq. (3.11). It is straightforward to show that the vector harmonics defined in eqs. (3.11)–(3.13) are eigenfunctions of the covariant Laplace operator with eigenvalue $-\kappa^2$, satisfy orthonormality conditions that are the analogs of those given in eq. (3.8), and satisfy divergence identities that are the analogs of those given in eqs. (3.9) and (3.10).

The third (most general) approach to constructing vector harmonics on these compact orientable flat spaces, E_j, is based on the fact that these spaces are quotients, $\mathbb{E}^3/\Gamma$, of Euclidean space $\mathbb{E}^3$ and an isometry group Γ. Therefore the problem of finding vector harmonics on the E_j spaces is equivalent to finding the Γ-invariant vector harmonics on $\mathbb{E}^3$. This can be done using the method developed in ref. [8] to derive expressions for the scalar harmonics. The *Vector Action Lemma* and the *Vector Invariance Lemma* described in the appendix to this paper provide the tools needed to construct linear combinations of the $\mathbb{E}^3$ vector harmonics that are invariant under all the elements of the symmetry groups Γ. The vector harmonics obtained using this method on the spaces $E_1,\ldots,E_5$ are more complicated than those derived using the first two approaches. So here we report only the vector harmonics $Y[E_6]^a_{(A)\mathbf{k}}$ obtained in this way for the otherwise intractable E_6 case:

$$\begin{aligned}Y[E_6]^a_{(A)\,\mathbf{k}} = \frac{1}{2}\Big[&Y[E_1]^a_{(A)\,\mathbf{k}} + (-1)^{n_1-n_2}M_1[E_6]^a{}_bY[E_1]^b_{(A)\,\mathbf{k}\mathbf{M}_1[E_6]}\\ &+(-1)^{n_2-n_3}M_2[E_6]^a{}_bY[E_1]^b_{(A)\,\mathbf{k}\mathbf{M}_2[E_6]}+(-1)^{n_3-n_1}M_3[E_6]^a{}_bY[E_1]^b_{(A)\,\mathbf{k}\mathbf{M}_3[E_6]}\Big],\\ &\text{for}\quad (n_1,n_2\in\mathbb{Z}^+,n_3\in\mathbb{Z})\quad\text{or}\quad (n_1=0,n_2,n_3\in\mathbb{Z}^+)\\ &\text{or}\quad (n_2=0,n_1,n_3\in\mathbb{Z}^+),\end{aligned} \tag{3.14}$$

$$Y[E_6]^a_{(A)\,(k_1,0,0)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]^a_{(A)\,(k_1,0,0)}+Y[E_1]^a_{(A)\,(-k_1,0,0)}\Big],\ \text{for}\ \big(n_1\in 2\mathbb{Z}^+,n_2=n_3=0\big), \tag{3.15}$$

$$Y[E_6]^a_{(A)\,(0,k_2,0)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]^a_{(A)\,(0,k_2,0)}+Y[E_1]^a_{(A)\,(0,-k_2,0)}\Big],\ \text{for}\ \big(n_2\in 2\mathbb{Z}^+,n_1=n_3=0\big), \tag{3.16}$$

$$Y[E_6]^a_{(A)\,(0,0,k_3)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]^a_{(A)\,(0,0,k_3)}+Y[E_1]^a_{(A)\,(0,0,-k_3)}\Big],\ \text{for}\ \big(n_3\in 2\mathbb{Z}^+,n_1=n_2=0\big), \tag{3.17}$$

where $Y[E_1]^a_{(A)\mathbf{k}}$ are the vector harmonics on E_1 described above, and $A=0,1,2$. We note that the expressions in eqs. (3.14)–(3.17) for the $A=0$ case are equivalent to eq. (3.2).

It is straightforward to show that the vector harmonics defined in eqs. (3.14)–(3.17) are eigenfunctions of the covariant Laplace operator with eigenvalue $-\kappa^2$, satisfy orthonormality conditions that are the analogs of those given in eq. (3.8), and satisfy divergence identities that are the analogs of those given in eqs. (3.9) and (3.10). The proofs of these properties use the fact that the matrices $\mathbf{M}$ that define the symmetries of these spaces preserve the inner product of vectors, e.g. for arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}\cdot\mathbf{v} = g(\mathbf{u},\mathbf{v}) = g(\mathbf{M}\cdot\mathbf{u},\mathbf{M}\cdot\mathbf{v}) = (\mathbf{M}\cdot\mathbf{u})\cdot(\mathbf{M}\cdot\mathbf{v})$.

Anti-symmetric tensor fields, $w_{ab} = -w_{ba}$, on orientable three-manifolds are dual to the vector fields v^a. Thus, for every w_{ab} there exists a vector field v^a so that $w_{ab} = \epsilon_{abc}v^c$. Therefore, any anti-symmetric tensor field can be represented as a sum of vector harmonics.

4 Tensor harmonics

This section constructs symmetric second-rank tensor harmonics on the six flat compact orientable three-manifolds. While these harmonics are not unique and can be chosen in a variety of ways, our goal here is to choose harmonics having three useful properties: first, the tensor harmonics constructed here will be eigenfunctions of the covariant Laplace operator:

$$\nabla^c\nabla_c Y^{ab} = -\kappa^2 Y^{ab}. \tag{4.1}$$

This ensures that these harmonics will form a complete basis for the (square integrable) symmetric second-rank tensor fields on these manifolds. We use the notation $Y[E_j]^{ab}_{(A)\,\mathbf{k}}$ (for $A = 0,\ldots,5$) to denote the six linearly independent classes of tensor harmonics that satisfy eq. (4.1) on the space E_j (for $j = 1,\ldots,6$). Second, the tensor harmonics constructed here will satisfy nice orthonormality conditions, i.e. $Y[E_j]^{ab}_{(A)\,\mathbf{k}}$ and $Y[E_j]^{ab}_{(A')\,\mathbf{k}'}$ will be orthogonal under the standard L_2 inner product unless $\mathbf{k} = \mathbf{k}'$ and $A = A'$. And third, the tensor harmonics constructed here in classes $A = 1,\ldots,5$ will be trace free, and those in classes $A = 4$ and $A = 5$ will be divergence free.

It is easy to construct four classes of tensor harmonics that satisfy these properties on all the flat compact orientable three-manifolds. These harmonics, which we call the class $A = 0,\ldots,3$ harmonics, are defined as,

$$Y[E_j]^{ab}_{(0)\,\mathbf{k}} = \frac{1}{\sqrt{3}} g^{ab} Y[E_j]_{\mathbf{k}}, \tag{4.2}$$

$$Y[E_j]^{ab}_{(1)\,\mathbf{k}} = \frac{1}{\sqrt{6}} \left(3\kappa^{-2}\nabla^a\nabla^b Y[E_j]_{\mathbf{k}} + g^{ab}Y[E_j]_{\mathbf{k}}\right), \tag{4.3}$$

$$Y[E_j]^{ab}_{(2)\,\mathbf{k}} = \frac{1}{\kappa\sqrt{2}} \left(\nabla^a Y[E_j]^b_{(1)\,\mathbf{k}} + \nabla^b Y[E_j]^a_{(1)\,\mathbf{k}}\right), \tag{4.4}$$

$$Y[E_j]^{ab}_{(3)\,\mathbf{k}} = \frac{1}{\kappa\sqrt{2}} \left(\nabla^a Y[E_j]^b_{(2)\,\mathbf{k}} + \nabla^b Y[E_j]^a_{(2)\,\mathbf{k}}\right), \tag{4.5}$$

where κ^2 is the corresponding eigenvalue of the covariant scalar Laplace operator, eq. (2.4). Since the scalar harmonics $Y[E_j]_{\mathbf{k}}$ are smooth eigenfunctions of the Laplace operator, their gradients are automatically smooth eigenfunctions having the same eigenvalues on these flat manifolds. The normalization factors in eqs. (4.2)–(4.5) are chosen to ensure that these harmonics satisfy the nice orthonormality conditions

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ac}g_{bd}Y[E_j]^{ab}_{(A)\,\mathbf{k}} Y[E_j]^{cd*}_{(B)\,\mathbf{k}'}\,dx\,dy\,dz = \delta_{AB}\;\delta_{n_1n_1'}\,\delta_{n_2n_2'}\,\delta_{n_3n_3'}\,, \tag{4.6}$$

for $A = 0, \ldots, 3$ and $B = 0, \ldots, 3$. The traces of these tensor harmonics are given by,

$$g_{ab}Y[E_j]^{ab}_{(0)\mathbf{k}} = \sqrt{3}Y[E_j]_{\mathbf{k}}, \tag{4.7}$$

$$g_{ab}Y[E_j]^{ab}_{(A)\mathbf{k}} = 0, \quad \text{for} \quad A = 1, \ldots, 3, \tag{4.8}$$

while the divergences are given by

$$\nabla_a Y[E_j]^{ab}_{(0)\mathbf{k}} = \frac{\kappa}{\sqrt{3}} Y[E_j]^{b}_{(0)\,\mathbf{k}}, \tag{4.9}$$

$$\nabla_a Y[E_j]^{ab}_{(1)\mathbf{k}} = -\frac{2\kappa}{\sqrt{6}} Y[E_j]^{b}_{(0)\,\mathbf{k}}, \tag{4.10}$$

$$\nabla_a Y[E_j]^{ab}_{(2)\,\mathbf{k}} = -\frac{\kappa}{\sqrt{2}} Y[E_j]^{b}_{(1)\,\mathbf{k}}\,, \tag{4.11}$$

$$\nabla_a Y[E_j]^{ab}_{(3)\,\mathbf{k}} = -\frac{\kappa}{\sqrt{2}} Y[E_j]^{b}_{(2)\,\mathbf{k}}\,. \tag{4.12}$$

The construction of the class $A = 4$ and $A = 5$ symmetric second-rank tensor harmonics is less straightforward. Our approach to finding these harmonics is analogous to the methods described in section 3 for deriving the class $A = 1$ and $A = 2$ vector harmonics. We have developed three different approaches that produce tensor harmonics satisfying the three useful properties listed above. The first approach produces the simplest expressions for the tensor harmonics, but this approach only works on the space E_1. A somewhat more general approach can be used to derive fairly simple expressions for these tensor harmonics on the spaces $E_1, \ldots, E_5$, but not on E_6. We have also developed an even more general approach capable of constructing tensor harmonics in all the E_j spaces, but the resulting expressions produced in this way are quite complicated. Here we report the results of this general approach only for the otherwise intractable E_6 case.

The first approach to constructing the needed tensor harmonics is based on the fact that any covariantly constant tensor field c^{ab} is invariant under the pure translation symmetry group of the three-torus, E_1. Therefore any tensor field of the form $c^{ab}Y[E_1]_{\mathbf{k}}$ is an eigenfunction of the covariant Laplace operator, and therefore a candidate tensor harmonic. The choices for six linearly independent symmetric tensors c^{ab} will define the six classes of tensor harmonics for this case. The class $A = 0$ and $A = 1$ tensor harmonics defined in eqs. (4.2) and (4.3) for the space E_1 have this form: $Y[E_1]^{ab}_{(0)(\mathbf{k}} = \frac{1}{\sqrt{3}} g^{ab} Y[E_1]_{\mathbf{k}}$ and $Y[E_1]^{ab}_{(0)(\mathbf{k}} = \frac{1}{\sqrt{6}}\left(g^{ab} - 3\hat{k}^a\hat{k}^b\right)Y[E_1]_{\mathbf{k}}$. All that is needed to complete this simple approach are choices for four additional constant tensors to define the class $A = 2, \ldots, 5$ harmonics. These choices are easy to construct using the set of orthonormal vectors $\hat{k}^a$, $\hat{\ell}^a$, and $\hat{m}^a$ constructed in section 3:

$$Y[E_1]^{ab}_{(0)\,\mathbf{k}} = \frac{1}{\sqrt{3}}\left(\hat{k}^a\hat{k}^b + \hat{\ell}^a\hat{\ell}^b + \hat{m}^a\hat{m}^b\right)Y[E_1]_{\mathbf{k}}, \tag{4.13}$$

$$Y[E_1]^{ab}_{(1)\,\mathbf{k}} = \frac{1}{\sqrt{6}}\left(\hat{\ell}^a\hat{\ell}^b + \hat{m}^a\hat{m}^b - 2\hat{k}^a\hat{k}^b\right)Y[E_1]_{\mathbf{k}}, \tag{4.14}$$

$$Y[E_1]^{ab}_{(2)\,\mathbf{k}} = -\frac{1}{\sqrt{2}}\left(\hat{k}^a\hat{\ell}^b + \hat{k}^b\hat{\ell}^a\right)Y[E_1]_{\mathbf{k}}, \tag{4.15}$$

$$Y[E_1]^{ab}_{(3)\,\mathbf{k}} = -\frac{1}{\sqrt{2}}\left(\hat{k}^a\hat{m}^b + \hat{k}^b\hat{m}^a\right)Y[E_1]_{\mathbf{k}}, \tag{4.16}$$

$$Y[E_1]^{ab}_{(4)\,\mathbf{k}} = \frac{1}{\sqrt{2}}\left(\hat{\ell}^a\hat{\ell}^b - \hat{m}^a\hat{m}^b\right)Y[E_1]_{\mathbf{k}}, \tag{4.17}$$

$$Y[E_1]^{ab}_{(5)\,\mathbf{k}} = \frac{1}{\sqrt{2}}\left(\hat{\ell}^a\hat{m}^b + \hat{\ell}^b\hat{m}^a\right)Y[E_1]_{\mathbf{k}}. \tag{4.18}$$

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The class $A = 0, \ldots, 3$ harmonics given in eqs. (4.13)–(4.16) are equivalent to the expressions given in eqs. (4.2)–(4.5). All these tensor harmonics defined in eqs. (4.13)–(4.18) satisfy each of the useful properties listed above. They are eigenfunctions of the covariant Laplace operator with eigenvalue $-\kappa^2$, and satisfy the following orthonormality conditions,

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ac}g_{bd}Y[E_1]^{ab}_{(A)\,\mathbf{k}}Y[E_1]^{cd*}_{(B)\,\mathbf{k}'}\,dx\,dy\,dz = \delta_{AB}\;\delta_{n_1n_1'}\,\delta_{n_2n_2'}\,\delta_{n_3n_3'}\,. \tag{4.19}$$

The traces of these tensor harmonics on E_1 are given by,

$$g_{ab}Y[E_1]^{ab}_{(0)\,\mathbf{k}} = \sqrt{3}Y[E_1]_{\mathbf{k}}, \tag{4.20}$$

$$g_{ab}Y[E_1]^{ab}_{(A)\,\mathbf{k}} = 0, \quad \text{for} \quad A = 1, \ldots, 5, \tag{4.21}$$

and their divergences are given by,

$$\nabla_a Y[E_1]^{ab}_{(0)\,\mathbf{k}} = \frac{\kappa}{\sqrt{3}}Y[E_1]^{b}_{(0)\,\mathbf{k}}\,, \tag{4.22}$$

$$\nabla_a Y[E_1]^{ab}_{(1)\,\mathbf{k}} = -\frac{2\kappa}{\sqrt{6}}Y[E_1]^{b}_{(0)\,\mathbf{k}}\,, \tag{4.23}$$

$$\nabla_a Y[E_1]^{ab}_{(2)\,\mathbf{k}} = -\frac{\kappa}{\sqrt{2}}Y[E_1]^{b}_{(1)\,\mathbf{k}}\,, \tag{4.24}$$

$$\nabla_a Y[E_1]^{ab}_{(3)\,\mathbf{k}} = -\frac{\kappa}{\sqrt{2}}Y[E_1]^{b}_{(2)\,\mathbf{k}}\,, \tag{4.25}$$

$$\nabla_a Y[E_1]^{ab}_{(4)\,\mathbf{k}} = \nabla_a Y[E_1]^{ab}_{(5)\,\mathbf{k}} = 0. \tag{4.26}$$

The second approach to constructing tensor harmonics uses the fact that the vector $\hat{z}^a$, the metric g^{ab} and the tensor $\hat{z}^a\hat{z}^b$ (where $\hat{z}^a$ is the unit vector along the z-axis) are the only covariantly constant vector and tensor fields invariant under all the symmetry groups of the manifolds $E_1, \ldots, E_5$. Therefore tensors like $g^{ab}Y[E_j]_{\mathbf{k}}$, $\hat{z}^a\hat{z}^bY[E_j]_{\mathbf{k}}$, and $\hat{z}^aY[E_j]^b_{(A)\mathbf{k}} + \hat{z}^bY[E_j]^a_{(A)\mathbf{k}}$ (for $A = 0, 1, 2$ and $j = 1, \ldots, 5$) are eigenfunctions of the covariant Laplace operator that can be used in the construction of the tensor harmonics on these spaces. We note that $\hat{z}^a\nabla_aY[E_j]_{\mathbf{k}} = i\,k_3Y[E_j]_{\mathbf{k}}$ and $\hat{z}^a\nabla_aY[E_j]^b_{(A)\,\mathbf{k}} = i\,k_3Y[E_j]^b_{(A)\,\mathbf{k}}$ in all of these cases. The tensor harmonics for these manifolds can therefore be taken to be

$$Y[E_j]^{ab}_{(0)\,\mathbf{k}} = \frac{1}{\sqrt{3}}g^{ab}Y[E_j]_{\mathbf{k}}, \tag{4.27}$$

$$Y[E_j]^{ab}_{(1)\,\mathbf{k}} = \frac{1}{\kappa^2\sqrt{6}}\left(3\nabla^a\nabla^bY[E_j]_{\mathbf{k}} + g^{ab}\kappa^2Y[E_j]_{\mathbf{k}}\right), \tag{4.28}$$

$$Y[E_j]^{ab}_{(2)\,\mathbf{k}} = \frac{1}{\kappa\sqrt{2}}\left(\nabla^aY[E_j]^b_{(1)\,\mathbf{k}} + \nabla^bY[E_j]^a_{(1)\,\mathbf{k}}\right), \tag{4.29}$$

$$Y[E_j]^{ab}_{(3)\,\mathbf{k}} = \frac{1}{\kappa\sqrt{2}}\left(\nabla^aY[E_j]^b_{(2)\,\mathbf{k}} + \nabla^bY[E_j]^a_{(2)\,\mathbf{k}}\right), \tag{4.30}$$

$$\begin{aligned}Y[E_j]^{ab}_{(4)\,\mathbf{k}} = \frac{1}{2\kappa\,k_3^2\sqrt{2(\kappa^2-k_3^2)}}\Big[&k_3\left(\kappa^2+k_3^2\right)\left(\nabla^aY[E_j]^b_{(1)\,\mathbf{k}} + \nabla^bY[E_j]^a_{(1)\,\mathbf{k}}\right)\\ &+i\kappa^2\left(\kappa^2-3k_3^2\right)\left(\hat{z}^aY[E_j]^b_{(1)\,\mathbf{k}} + \hat{z}^bY[E_j]^a_{(1)\,\mathbf{k}}\right)\\ &+2\kappa\sqrt{\kappa^2-k_3^2}\left(\kappa^2\hat{z}^a\hat{z}^b - k_3^2g^{ab}\right)Y[E_j]_{\mathbf{k}}\Big],\end{aligned} \tag{4.31}$$

$$\begin{aligned}Y[E_j]^{ab}_{(5)\,\mathbf{k}} = \frac{1}{\kappa\sqrt{2\left(\kappa^2-k_3^2\right)}}\Big[&k_3\left(\nabla^aY[E_j]^b_{(2)\,\mathbf{k}} + \nabla^bY[E_j]^a_{(2)\,\mathbf{k}}\right)\\ &-i\kappa^2\left(\hat{z}^aY[E_j]^b_{(2)\,\mathbf{k}} + \hat{z}^bY[E_j]^a_{(2)\,\mathbf{k}}\right)\Big]\,,\end{aligned} \tag{4.32}$$

so long as the vector harmonics $Y[E_j]^a_{(1)\,\mathbf{k}}$ and $Y[E_j]^a_{(2)\,\mathbf{k}}$ are well defined, and so long as $\kappa^2\neq 0$, $k_3^2\neq 0$, and $\kappa^2\neq k_3^2$. When $\kappa^2=0$, the tensor harmonics must be spatially constant tensor fields. Six linearly independent spatially constant tensor fields exist in the space E_1, so any orthonormal set can be used as $\kappa=0$ harmonics in that space. In the spaces $E_2,\ldots,E_5$ the only spatially constant tensor fields are g^{ab} and $g^{ab}-3\hat{z}^a\hat{z}^b$, so (suitably normalized) they are the only $\kappa=0$ tensor harmonics on those spaces. When $k_3=0$ and $\kappa\neq 0$ the tensor harmonics are independent of z, i.e. they depend only on x and y. In this case the expressions for the vector harmonics are given in section 3 and the expressions for the $A=0,\ldots,3$ and 5 tensor harmonics are given in eqs. (4.27)–(4.30) and (4.32). However the class $A=4$ tensor harmonics are not well defined in this case. Finally if $\kappa^2=k_3^2$ and $\kappa\neq 0$ the tensor harmonics are independent of x and y, i.e. the harmonics depend only on z. In this case the class $A=1$ and $A=2$ vector harmonics do not exist, so the only well defined tensor harmonics are the class $A=0$ and $A=1$ harmonics. It is straightforward to show that the tensor harmonics defined in eqs. (4.27)–(4.32) are eigenfunctions of the covariant Laplace operator with eigenvalue $-\kappa^2$, satisfy orthonormality conditions that are the analogs of those given in eq. (4.19), satisfy trace identities that are the analogs of those given in eqs. (4.20) and (4.21), and divergence identities that are the analogs of those given in eqs. (4.22)–(4.26).

The third (most general) approach to constructing tensor harmonics on these compact orientable flat spaces, E_j, is based on the fact that these spaces are quotients, $\mathbb{E}^3/\Gamma$, of Euclidean space $\mathbb{E}^3$ and an isometry group Γ. Therefore the problem of finding tensor harmonics on the E_j spaces is equivalent to finding the Γ-invariant tensor harmonics on $\mathbb{E}^3$. This can be done using the method developed in ref. [8] to derive expressions for the scalar harmonics. The *Tensor Action Lemma* and the *Tensor Invariance Lemma* described in the appendix to this paper provide the tools needed to construct linear combinations of the $\mathbb{E}^3$ tensor harmonics that are invariant under all the elements of the symmetry groups Γ. The tensor harmonics obtained using this method on the spaces $E_1,\ldots,E_5$ are more complicated than those derived using the first two approaches. So here we report only the tensor harmonics $Y[E_6]^{ab}_{(A)\mathbf{k}}$ with $A=0,\ldots,5$ obtained in this way for the otherwise intractable E_6 case:

$$\begin{aligned}
Y[E_6]^{ab}_{(A)\,\mathbf{k}} = \frac{1}{2}\Big[&Y[E_1]^{ab}_{(A)\,\mathbf{k}} + (-1)^{n_1-n_2}M_1[E_6]^a{}_cM_1[E_6]^b{}_dY[E_1]^{cd}_{(A)\,\mathbf{k}M_1[E_6]}\\
&+(-1)^{n_2-n_3}M_2[E_6]^a{}_cM_2[E_6]^b{}_dY[E_1]^{cd}_{(A)\,\mathbf{k}M_2[E_6]}\\
&+(-1)^{n_3-n_1}M_3[E_6]^a{}_cM_3[E_6]^b{}_dY[E_1]^{cd}_{(A)\,\mathbf{k}M_3[E_6]}\Big]\\
&\text{for}\quad \big(n_1,n_2\in\mathbb{Z}^+,n_3\in\mathbb{Z}\big)\quad\text{or}\quad\big(n_1=0,n_2,n_3\in\mathbb{Z}^+\big)\\
&\text{or}\quad\big(n_2=0,n_1,n_3\in\mathbb{Z}^+\big),
\end{aligned}\tag{4.33}$$

$$Y[E_6]^{ab}_{(A)\,(k_1,0,0)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]^{ab}_{(A)\,(k_1,0,0)} + Y[E_1]^{ab}_{(A)\,(-k_1,0,0)}\Big]\ \text{ for }\ \big(n_1\in 2\mathbb{Z}^+,n_2=n_3=0\big),\tag{4.34}$$

$$Y[E_6]^{ab}_{(A)\,(0,k_2,0)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]^{ab}_{(A)\,(0,k_2,0)} + Y[E_1]^{ab}_{(A)\,(0,-k_2,0)}\Big]\ \text{ for }\ \big(n_2\in 2\mathbb{Z}^+,n_1=n_3=0\big),\tag{4.35}$$

$$Y[E_6]^{ab}_{(A)\,(0,0,k_3)} = \frac{1}{\sqrt{2}}\Big[Y[E_1]^{ab}_{(A)\,(0,0,k_3)} + Y[E_1]^{ab}_{(A)\,(0,0,-k_3)}\Big]\ \text{ for }\ \big(n_3\in 2\mathbb{Z}^+,n_1=n_2=0\big).\tag{4.36}$$

where $Y[E_1]^{ab}_{(A)\,\mathbf{k}}$ are the tensor harmonics on E_1 described above. We note that the expressions in eqs. (4.33)–(4.36) for the $A=0,\ldots,3$ cases are equivalent to eqs. (4.2)–(4.5). It is straightforward to show that the tensor harmonics defined in eqs. (4.33)–(4.36) are eigenfunctions of the covariant Laplace operator with eigenvalue $-\kappa^2$, satisfy orthonormality conditions that are the analogs of those given in eq. (4.19), satisfy the trace conditions given in eqs. (4.20)–(4.21), and satisfy divergence identities that are the analogs of those given in eqs. (4.22)–(4.26). The proofs of these properties use the fact that the matrices $M^a{}_b$ that define the symmetries of these spaces preserve the structure of the metric: $g_{ab} = M^c{}_a M^d{}_b g_{cd}$ and $g^{ab} = M^a{}_c M^b{}_d g^{cd}$.

5 Summary and discussion

This paper introduces a uniform notation for the scalar, vector and tensor harmonics on the flat compact orientable three-manifolds $E_1, E_2, \ldots, E_6$. The $Y[E_j]_{\mathbf{k}}$ represent the scalar harmonics on the space E_j (for $j = 1, \ldots, 6$) with parameters $\mathbf{k} = k_a = (k_1, k_2, k_3)$. To enforce the appropriate periodicities, these parameters must be given by $k_a = 2\pi\left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3}\right)$ in the spaces E_1, E_2, E_3 and E_6, and $k_a = 2\pi\left(-\frac{n_2}{L_1}, \frac{2n_1-n_2}{\sqrt{3}L_2}, \frac{n_3}{L_3}\right)$ in the spaces E_4 and E_5, where the n_1, n_2 and n_3 are integers and L_1, L_2 and L_3 are the periodicity lengths in each dimension. (The lengths L_1 and L_2 must also be equal in the spaces E_3, E_4 and E_5 to preserve the symmetries.) Explicit expressions for these scalar harmonics, constructed originally in ref. [8], are summarized in section 2. The $Y[E_j]^a_{(A)\,\mathbf{k}}$ represent the three classes of vector harmonics, with $A = 0, 1, 2$, on the space E_j. Explicit expressions for these harmonics are constructed in section 3. Finally the $Y[E_j]^{ab}_{(A)\,\mathbf{k}}$ represent the six classes of symmetric second-rank tensor harmonics, with $A = 0, 1, \ldots, 5$, on the space E_j. Explicit expressions for these harmonics are constructed in section 4. All these harmonics satisfy a number of useful properties.

The scalar harmonics on the six compact orientable flat three-manifolds E_j (for $j = 1, 2, \ldots, 6$) are eigenfunctions of the covariant Laplace operator,

$$\nabla^b\nabla_b Y[E_j]_{\mathbf{k}} = -\kappa^2 Y[E_j]_{\mathbf{k}}, \tag{5.1}$$

and satisfy the orthonormality conditions

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} Y[E_j]_{\mathbf{k}} Y[E_j]^*_{\mathbf{k}'}\, dx\, dy\, dz = \delta_{n_1 n_1'}\,\delta_{n_2 n_2'}\,\delta_{n_3 n_3'}\,. \tag{5.2}$$

Therefore it is easy to represent any (square integrable) scalar field on these manifolds in terms of these basis functions:

$$f(\mathbf{x}) = \sum_{n_1}\sum_{n_2}\sum_{n_3} f_{\mathbf{k}} Y[E_j]_{\mathbf{k}}, \tag{5.3}$$

where the coefficients $f_{\mathbf{k}}$ are given by

$$f_{\mathbf{k}} = \int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} f(\mathbf{x})\, Y[E_j]^*_{\mathbf{k}}\, dx\, dy\, dz. \tag{5.4}$$

The vector harmonics on the six compact orientable flat three-manifolds E_j (for $j = 1, 2, \ldots, 6$) are eigenfunctions of the covariant Laplace operator,

$$\nabla^b\nabla_b Y[E_j]^a_{(A)\,\mathbf{k}} = -\kappa^2 Y[E_j]^a_{(A)\,\mathbf{k}}\,, \tag{5.5}$$

and satisfy the orthonormality conditions

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ab}Y[E_j]^a_{(A)\,\mathbf{k}}Y[E_j]^{b*}_{(B)\,\mathbf{k}'}\,dx\,dy\,dz = \delta_{AB}\;\delta_{n_1 n_1'}\,\delta_{n_2 n_2'}\,\delta_{n_3 n_3'}\,. \tag{5.6}$$

Therefore it is easy to represent any (square integrable) vector field on these manifolds in terms of these basis functions:

$$v^a(\mathbf{x}) = \sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{A} v_{(A)\,\mathbf{k}}Y[E_j]^a_{(A)\,\mathbf{k}}, \tag{5.7}$$

where the coefficients $v_{(A)\,\mathbf{k}}$ are given by

$$v_{(A)\,\mathbf{k}} = \int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ab}v^a(\mathbf{x})\,Y[E_j]^{b*}_{(A)\,\mathbf{k}}\,dx\,dy\,dz. \tag{5.8}$$

These vector harmonics also satisfy the divergence identities,

$$\nabla_a Y[E_j]^a_{(0)\,\mathbf{k}} = -\kappa Y[E_j]_{\mathbf{k}}, \tag{5.9}$$

$$\nabla_a Y[E_j]^a_{(1)\,\mathbf{k}} = \nabla_a Y[E_j]^a_{(2)\,\mathbf{k}} = 0, \tag{5.10}$$

on each of the six flat compact orientable three-manifolds. The vanishing divergences of the class $A = 1$ and $A = 2$ vector harmonics make them useful for constructing representations of the electromagnetic field.

The tensor harmonics on the six compact orientable flat three-manifolds E_j (for $j = 1, 2, \ldots, 6$) are eigenfunctions of the covariant Laplace operator,

$$\nabla^c\nabla_c Y[E_j]^{ab}_{(A)\,\mathbf{k}} = -\kappa^2 Y[E_j]^{ab}_{(A)\,\mathbf{k}}, \tag{5.11}$$

and satisfy the orthonormality conditions

$$\int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ac}g_{bd}Y[E_j]^{ab}_{(A)\,\mathbf{k}}Y[E_j]^{cd*}_{(B)\,\mathbf{k}'}\,dx\,dy\,dz = \delta_{AB}\;\delta_{n_1 n_1'}\,\delta_{n_2 n_2'}\,\delta_{n_3 n_3'}\,, \tag{5.12}$$

for $A = 0, \ldots, 5$ and $B = 0, \ldots, 5$. Therefore it is easy to represent any (square integrable) symmetric second-rank tensor field on these manifolds in terms of these basis functions:

$$t^{ab}(\mathbf{x}) = \sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{A} t_{(A)\,\mathbf{k}}Y[E_j]^{ab}_{(A)\,\mathbf{k}}, \tag{5.13}$$

where the coefficients $t_{(A)\,\mathbf{k}}$ are given by

$$t_{(A)\,\mathbf{k}} = \int_{-L_1/2}^{L_1/2}\int_{-L_2/2}^{L_2/2}\int_{-L_3/2}^{L_3/2} g_{ac}\,g_{bd}\,t^{ab}(\mathbf{x})\,Y[E_j]^{cd*}_{(A)\,\mathbf{k}}\,dx\,dy\,dz. \tag{5.14}$$

The traces of these tensor harmonics are given by,

$$g_{ab}Y[E_j]^{ab}_{(0)\,\mathbf{k}} = \sqrt{3}Y[E_j]_{\mathbf{k}}, \tag{5.15}$$

$$g_{ab}Y[E_j]^{ab}_{(A)\,\mathbf{k}} = 0, \quad \text{for} \quad A = 1, \ldots, 5, \tag{5.16}$$

while the divergences are given by

$$\nabla_a Y[E_j]^{ab}_{(0)\,\mathbf{k}} = \frac{\kappa}{\sqrt{3}} Y[E_j]^{b}_{(0)\,\mathbf{k}}, \tag{5.17}$$

$$\nabla_a Y[E_j]^{ab}_{(1)\,\mathbf{k}} = -\frac{2\kappa}{\sqrt{6}} Y[E_j]^{b}_{(0)\,\mathbf{k}}, \tag{5.18}$$

$$\nabla_a Y[E_j]^{ab}_{(2)\,\mathbf{k}} = -\frac{\kappa}{\sqrt{2}} Y[E_j]^{b}_{(1)\,\mathbf{k}}, \tag{5.19}$$

$$\nabla_a Y[E_j]^{ab}_{(3)\,\mathbf{k}} = -\frac{\kappa}{\sqrt{2}} Y[E_j]^{b}_{(2)\,\mathbf{k}}. \tag{5.20}$$

$$\nabla_a Y[E_j]^{ab}_{(4)\,\mathbf{k}} = \nabla_a Y[E_j]^{ab}_{(5)\,\mathbf{k}} = 0. \tag{5.21}$$

The vanishing traces and divergences of the class $A = 4$ and $A = 5$ tensor harmonics make them useful for constructing representations of the dynamical gravitational wave degrees of freedom of the gravitational field.

Cosmological observations provide measurements of physical properties, like the temperature of the cosmic microwave radiation, as functions on the two-sphere of our sky. The components of the three-dimensional scalar, vector and tensor harmonics used in the construction of a cosmological model would need to be projected onto such a two-sphere to facilitate comparisons with the observations. The analysis needed to do this projection has been carried out for the scalar harmonics on these flat three-dimensionsl manifolds in ref. [8]. Analogous projections of the vector and tensor harmonics onto the appropriate spin-weighted spherical harmonic bases could also be carried out, but we defer that analysis to a future paper.

A Vector and tensor action and invariance lemmas

All multi-connected three-dimensional flat spaces E_j are quotients, $\mathbb{E}^3/\Gamma$, of Euclidean space $\mathbb{E}^3$ by an isometry group Γ. The problem of finding the scalar, vector and tensor harmonics on $\mathbb{E}^3/\Gamma$ is equivalent to the problem of finding the Γ-invariant harmonics of $\mathbb{E}^3$. Two technical lemmas were developed in ref. [8] to facilitate the construction of the scalar harmonics on these spaces. The first of these, the *Action Lemma*, determines how an element of the isometry group Γ transforms one scalar harmonic on $\mathbb{E}^3$ into another. The second, the *Invariance Lemma*, constructs a harmonic that is invariant under the repeated action of any element of the isometry group. Together these lemmas were used in ref. [8] to derive the explicit expressions for the scalar harmonics summarized in section 2 of this paper. Those lemmas for scalar harmonics are generalized here to action and invariance lemmas for the vector and tensor harmonics on these flat spaces.

Every isometry $\gamma \in \Gamma$ on these manifolds consists of a reflection/rotation followed by a translation, i.e. they are transformations of the form $\mathbf{x}' = \mathbf{M}\cdot\mathbf{x} + \mathbf{T}$ where $\mathbf{M}$ is a unitary matrix and $\mathbf{T}$ is a vector, or in component notation $x'^a = M^{\,a}{}_b x^b + T^a$.

Lemma 1 (Vector Action Lemma)**.** *The natural action of an isometry $\gamma \in \Gamma$ of Euclidean space $\mathbb{E}^3$ takes a vector harmonic $\mathbf{u}Y_{\mathbf{k}}(\mathbf{x}) = \mathbf{u}\,e^{i\,\mathbf{k}\cdot\mathbf{x}}$ to another vector harmonic $(\mathbf{M}\cdot\mathbf{u})\,e^{i\,\mathbf{k}\cdot\mathbf{T}}Y_{\mathbf{kM}}(\mathbf{x})$, where $\mathbf{u}$ is any constant vector.*

Proof. The action of the isometry γ on $\mathbf{u}\,Y_{\mathbf{k}}$ is given by:

$$\begin{aligned}\mathbf{u}\,Y_{\mathbf{k}} \mapsto \gamma\left(\mathbf{u}\,e^{i\,\mathbf{k}\cdot\mathbf{x}}\right) &= \gamma(\mathbf{u})\,\gamma\left(e^{i\,\mathbf{k}\cdot\mathbf{x}}\right) = (\mathbf{M}\cdot\mathbf{u})\,e^{i\,\mathbf{k}\cdot(\mathbf{M}\cdot\mathbf{x}+\mathbf{T})}\\ &= (\mathbf{M}\cdot\mathbf{u})\,e^{i\mathbf{k}\cdot\mathbf{T}}e^{i\,\mathbf{kM}\cdot\mathbf{x}} = (\mathbf{M}\cdot\mathbf{u})\,e^{i\,\mathbf{k}\cdot\mathbf{T}}Y_{\mathbf{kM}}(\mathbf{x}),\end{aligned}$$

where we have used $\gamma(\mathbf{u}) = \mathbf{u}' = \mathbf{M}\cdot\mathbf{u}$. □

Lemma 2 (Vector Invariance Lemma). *If γ is an isometry of $\mathbb{E}^3$ with matrix part $\mathbf{M}$ and translational part $\mathbf{T}$, if $\mathbf{u}\, Y_{\mathbf{k}}$ is a vector harmonic on $\mathbb{E}^3$, and if n is the smallest positive integer such that $\mathbf{k} = \mathbf{k}\mathbf{M}^n$ (typically n is simply the order of the matrix $\mathbf{M}$), then the action of γ*

1. *preserves the n-dimensional space of harmonics spanned by $\{\mathbf{u}\, Y_{\mathbf{k}}, (\mathbf{M}\cdot\mathbf{u})\, Y_{\mathbf{kM}}, \cdots, (\mathbf{M}^{n-1}\cdot\mathbf{u})\, Y_{\mathbf{kM}^{n-1}}\}$ as a set, and*
2. *leaves invariant the harmonic, $a_0\, \mathbf{u}\, Y_{\mathbf{k}} + a_1\, (\mathbf{M}\cdot\mathbf{u})\, Y_{\mathbf{kM}} + \cdots + a_{n-1}\left(\mathbf{M}^{n-1}\cdot\mathbf{u}\right) Y_{\mathbf{kM}^{n-1}}$, if and only if $a_{j+1} = e^{i\,\mathbf{kM}^j\cdot\mathbf{T}} a_j$ for each j (mod n).*

Proof. Both parts are immediate corollaries of lemma 1. Specifically, the action of γ takes the linear combination

$$a_0\, \mathbf{u}\, Y_{\mathbf{k}} + a_1\, (\mathbf{M}\cdot\mathbf{u})\, Y_{\mathbf{kM}} + \cdots + a_{n-2}\left(\mathbf{M}^{n-2}\cdot\mathbf{u}\right) Y_{\mathbf{kM}^{n-2}} + a_{n-1}\left(\mathbf{M}^{n-1}\cdot\mathbf{u}\right) Y_{\mathbf{kM}^{n-1}}, \tag{A.1}$$

into

$$\begin{aligned} &a_0\, (\mathbf{M}\cdot\mathbf{u})\, e^{i\mathbf{k}\cdot\mathbf{T}} Y_{\mathbf{kM}} + a_1\left(\mathbf{M}^2\cdot\mathbf{u}\right) e^{i\,\mathbf{k}\cdot\mathbf{MT}} Y_{\mathbf{kM}^2} + \cdots \\ &\quad + a_{n-2}\left(\mathbf{M}^{n-1}\cdot\mathbf{u}\right) e^{i\,\mathbf{kM}^{n-2}\mathbf{T}} Y_{\mathbf{kM}^{n-1}} + a_{n-1}\mathbf{u}\, e^{i\,\mathbf{kM}^{n-1}\cdot\mathbf{T}} Y_{\mathbf{k}}. \end{aligned} \tag{A.2}$$

Therefore the n-dimensional subspace spanned by $\{\mathbf{u}\, Y_{\mathbf{k}}, (\mathbf{M}\cdot\mathbf{u})\, Y_{\mathbf{kM}}, \cdots, (\mathbf{M}^{n-1}\cdot\mathbf{u})\, Y_{\mathbf{kM}^{n-1}}\}$ is preserved as a set. Comparing the coefficients of the expressions in eqs. (A.1) and (A.2), it follows that they are identical if only and if $a_{j+1} = e^{i\,\mathbf{kM}^j\cdot\mathbf{T}} a_j$ for each j (mod n). □

Lemma 3 (Tensor Action Lemma). *The natural action of an isometry γ of Euclidean space $\mathbb{E}^3$ takes a tensor harmonic $\mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}} = \mathbf{u}\otimes\mathbf{v}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$ to another tensor harmonic $(\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\, e^{i\,\mathbf{k}\cdot\mathbf{T}} Y_{\mathbf{kM}}(\mathbf{x})$, where $\mathbf{u}$ and $\mathbf{v}$ are arbitrary constant vectors.*

Proof. The action of the isometry γ on $\mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}}$ is given by:

$$\begin{aligned} \mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}} \mapsto \gamma\,(\mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}}) &= \gamma\,(\mathbf{u})\otimes\gamma\,(\mathbf{v})\;\gamma\,(Y_{\mathbf{k}}) = (\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\, e^{i\,\mathbf{k}\cdot(\mathbf{M}\cdot\mathbf{x}+\mathbf{T})} \\ &= (\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\, e^{i\,\mathbf{k}\cdot\mathbf{T}} Y_{\mathbf{kM}}(\mathbf{x}), \end{aligned} \tag{A.3}$$

where we have used $\gamma\,(\mathbf{u}\otimes\mathbf{v}) = \mathbf{u}'\otimes\mathbf{v}' = (\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})$. □

Lemma 4 (Tensor Invariance Lemma). *If γ is an isometry of $\mathbb{E}^3$ with matrix part $\mathbf{M}$ and translational part $\mathbf{T}$, if $\mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}}$ is a tensor harmonic on $\mathbb{E}^3$, and if n is the smallest positive integer such that $\mathbf{k} = \mathbf{k}\mathbf{M}^n$ (typically n is simply the order of the matrix $\mathbf{M}$), then the action of γ*

1. *preserves the n-dimensional space of harmonics spanned by $\{\mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}}, (\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\, Y_{\mathbf{kM}}, \cdots, (\mathbf{M}^{n-1}\cdot\mathbf{u})\otimes(\mathbf{M}^{n-1}\cdot\mathbf{v})\, Y_{\mathbf{kM}^{n-1}}\}$ as a set, and*
2. *leaves invariant the harmonic, $a_0\, \mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}} + a_1\, (\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\, Y_{\mathbf{kM}} + \cdots + a_{n-1}\left(\mathbf{M}^{n-1}\cdot\mathbf{u}\right)\otimes\left(\mathbf{M}^{n-1}\cdot\mathbf{v}\right) Y_{\mathbf{kM}^{n-1}}$, if and only if $a_{j+1} = e^{i\,\mathbf{kM}^j\cdot\mathbf{T}} a_j$ for each j (mod n).*

Proof. Both parts are immediate corollaries of lemma 3. Specifically, the action of γ takes the linear combination

$$\begin{aligned} &a_0\, \mathbf{u}\otimes\mathbf{v}\, Y_{\mathbf{k}} + a_1\, (\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\, Y_{\mathbf{kM}} + \cdots + a_{n-2}\left(\mathbf{M}^{n-2}\cdot\mathbf{u}\right)\otimes\left(\mathbf{M}^{n-2}\cdot\mathbf{v}\right) Y_{\mathbf{kM}^{n-2}} \\ &\quad + a_{n-1}\left(\mathbf{M}^{n-1}\cdot\mathbf{u}\right)\otimes\left(\mathbf{M}^{n-1}\cdot\mathbf{v}\right) Y_{\mathbf{kM}^{n-1}}, \end{aligned} \tag{A.4}$$

into

$$
\begin{aligned}
&a_0\,(\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\,e^{i\mathbf{k}\cdot\mathbf{T}}Y_{\mathbf{kM}}+a_1\left(\mathbf{M}^2\cdot\mathbf{u}\right)\otimes\left(\mathbf{M}^2\cdot\mathbf{v}\right)e^{i\,\mathbf{k}\cdot\mathbf{MT}}Y_{\mathbf{kM}^2}+\cdots\\
&\quad+a_{n-2}\left(\mathbf{M}^{n-1}\cdot\mathbf{u}\right)\otimes\left(\mathbf{M}^{n-1}\cdot\mathbf{v}\right)e^{i\,\mathbf{kM}^{n-2}\mathbf{T}}Y_{\mathbf{kM}^{n-1}}+a_{n-1}\mathbf{u}\otimes\mathbf{v}\,e^{i\,\mathbf{kM}^{n-1}\cdot\mathbf{T}}Y_{\mathbf{k}}.
\end{aligned}
\tag{A.5}
$$

Therefore the n-dimensional subspace spanned by $\left\{\mathbf{u}\otimes\mathbf{v}\,Y_{\mathbf{k}},(\mathbf{M}\cdot\mathbf{u})\otimes(\mathbf{M}\cdot\mathbf{v})\,Y_{\mathbf{kM}},\cdots,\left(\mathbf{M}^{n-1}\cdot\mathbf{u}\right)\otimes\left(\mathbf{M}^{n-1}\cdot\mathbf{v}\right)Y_{\mathbf{kM}^{n-1}}\right\}$ is preserved as a set. Comparing the coefficients of the expressions in eqs. (A.4) and (A.5), it follows that they are identical if only and if $a_{j+1}=e^{i\,\mathbf{kM}^j\cdot\mathbf{T}}a_j$ for each j (mod n). □

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