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THE RADIAL OSCILLATIONS OF NEUTRON STARS[1]

EDWARD N. GLASS AND LEE LINDBLOM
Institute of Theoretical Physics, Department of Physics, Stanford University
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ABSTRACT

We continue our investigation of the influence of the equation of state of matter (in the nuclear density regime) on the dynamical behavior of neutron stars. The properties of the two lowest frequency radial oscillation modes of neutron stars, constructed from 13 equations of state, are presented. We find that the frequencies of both radial oscillation modes studied here are generally higher than the frequency of the corresponding quadrupole f mode.

Subject headings: dense matter — equation of state — stars: neutron — stars: pulsation

I. INTRODUCTION

In this paper we continue the study of the oscillation modes of realistic neutron star models begun in the accompanying paper (Lindblom and Detweiler 1983). Since neutron stars involve both strong gravitational fields and matter at supernuclear densities, these objects offer a unique opportunity to explore a domain of physics untested in the laboratory. General relativity is used to describe the gravitational interaction in neutron stars, and we explore the effects of a number of different theories of ultradense matter on the oscillations of neutron stars.

An increasing body of observational data is being accumulated which appears to be associated with oscillating neutron stars. Some γ-ray and X-ray burst phenomena have been associated with neutron stars (see, e.g., Ramaty *et al.* 1980; Oda 1981). Also, observations of pulsars have revealed quasi-periodic subpulses which several authors (see, e.g., Boriakoff 1976; Van Horn 1980) have suggested might be identified with oscillations of the underlying neutron star. Such observations may eventually yield enough information to allow us to constrain our ignorance of the physics of high density matter and strong gravitational fields in these objects.

This paper explores the influence of the structure of matter at supernuclear densities on the radial pulsations of neutron stars. We compute the frequencies and eigenfunctions of the two lowest frequency radial oscillation modes for neutron stars constructed from 13 equations of state. The equation of state models used in this paper are identical to the models used by Lindblom and Detweiler (1983), and they are essentially the same as those used by Arnett and Bowers (1977) in their survey of the equilibrium structure of neutron stars. The equation of state models will be referred to here by the letters assigned to these models in Table 1 of Lindblom and Detweiler (1983) (e.g., model A, model B, etc.). A detailed description of these equation of state models is given in the accompanying paper (Lindblom and Detweiler 1983) and will not be repeated here.

The adiabatic index γ for these computations is taken to have the "equilibrium" value which is related to the equation of state by $\gamma = p^{-1}(\rho + p)(dp/d\rho)$. This expression is only correct for sufficiently low frequency oscillations (see, e.g., Meltzer and Thorne 1966; Chanmugan 1977). In practice, the actual adiabatic index does not differ significantly from the equilibrium value used here for densities above $\sim 10^{13}$ g cm^{-3}. For the neutron stars of primary interest here ($M \geq M_\odot$) the majority of the matter in the stars has a density well above 10^{13} g cm^{-3}. Consequently, the simple form of the adiabatic index used here does not have much effect on the computed frequencies.

We evaluate the oscillation frequencies for these neutron star models by explicitly integrating the radial perturbation equation. Our computations were performed in roughly the same manner as described by Bardeen, Thorne, and Meltzer (1966), Meltzer and Thorne (1966), and Glass and Harpaz (1983). A description of the details of our computations is given in the Appendix. In § II we present the detailed numerical results of our computations for neutron stars constructed from the 13 equation of state models. In § III we compare our computations to the few observational data which appear to be related to oscillating neutron stars.

An interesting feature of the results of our computations is the relationship between the frequencies of these radial oscillations and the quadrupole oscillation frequencies computed by Lindblom and Detweiler (1983).

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We had expected the fundamental radial frequency to be lower (for the same neutron star) than the fundamental quadrupole frequency. This was the case for the Harrison-Wheeler equation of state (our model H) models computed by Meltzer and Thorne (1966) and Thorne (1969). Every other equation of state considered by us, however, had some neutron star models with fundamental radial frequencies higher than their quadrupole frequencies. This curious result appears to be a manifestation of the stiffness of the nuclear matter equation of state. Totally incompressible fluid models have finite quadrupole oscillation frequencies, while their radial modes do not exist at all, having "infinite" frequencies. It is not hard to imagine, therefore, a star composed of compressible (but very stiff) matter whose radial frequency is higher than its quadrupole frequency (we thank Steven Detweiler for suggesting this explanation to us). This heuristic analysis is confirmed by the computations of Chandrasekhar and Lebovitz (1964). They found in the pulsations of Newtonian polytropes that small values of the adiabatic index, $\gamma \lesssim 1.6$, resulted in the fundamental radial mode having a lower frequency than the quadrupole mode. For stiffer equations of state, $\gamma \gtrsim 1.6$, the quadrupole mode had a lower frequency. The single exception to the general pattern of higher radial frequencies in our results came from the Harrison-Wheeler equation of state. Since this is the softest of the equations of state studied by us, it too is consistent with this general picture.

Figure 1 illustrates the ordering of the frequencies of the normal modes for one particular equation of state (our model L). Near the maximum neutron star mass (and, although not shown here, also near the minimum mass) the fundamental radial frequency approaches zero. Near these points the quadrupole mode therefore has higher frequency than the fundamental radial mode. Except near these special points, however, the quadrupole frequency is lower than the lowest radial frequency by about a factor of 2.

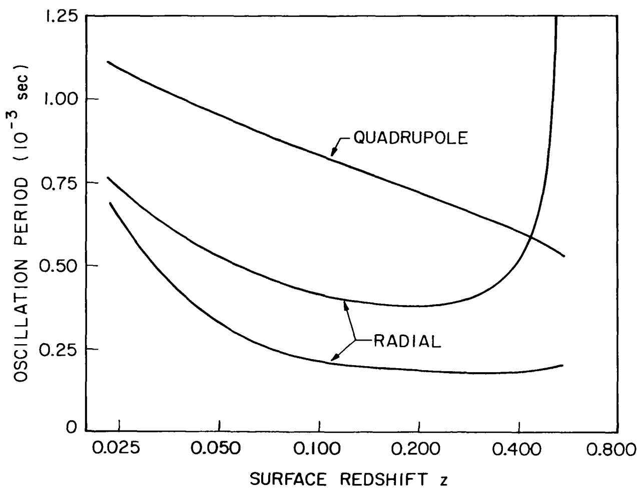

FIG. 1.—Oscillation periods for the two lowest radial modes and the lowest quadrupole mode are illustrated as a function of surface redshift for neutron stars constructed from equation of state L. This figure shows that the lowest frequency mode is not necessarily the radial mode.

II. RESULTS

The numerical results of our computations are presented in Tables 1–13 and Figures 2–5. Since the parameters describing these equilibrium stellar models have been extensively cataloged elsewhere (see Arnett and Bowers 1977; Lindblom and Detweiler 1983), we list here only the central density ρ_c in units of 10^{15} g cm^{-3} for each model. Tables 1–13 also list the oscillation period T (ms), the energy contained in the oscillations $E(10^{53}$ ergs), and a parameter d (which describes the radial eigenfunction) for each mode. The subscripts

TABLE 1

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state A)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
3.3880	0.7512	4.55	1.0	0.1369	15.97	−1.0
3.0000	0.5670	6.74	1.0	0.1355	13.37	−1.1
2.3440	0.4244	8.46	1.1	0.1337	8.49	−1.4
1.9950	0.3821	7.95	1.2	0.1333	5.65	−1.6
1.7780	0.3629	6.71	1.3	0.1334	3.26	−1.9
1.6980	0.3574	6.30	1.4	0.1335	2.74	−2.1
1.5850	0.3511	5.82	1.4	0.1339	2.31	−2.2
1.5140	0.3481	5.39	1.4	0.1342	1.92	−2.3
1.2590	0.3436	4.19	1.5	0.1369	1.20	−2.6
1.0000	0.3541	1.52	2.0	0.1467	0.07	−9.3
0.8913	0.3652	0.94	2.1	0.1557	0.03	−14.7
0.8000	0.3790	0.51	2.4	0.1679	0.01	−27.3
0.7080	0.3990	0.23	2.8	0.1860	0.00	−53.1

TABLE 2

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state B)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
6.2000	1.2162	0.03	0.9	0.1079	9.81	−1.3
5.0120	0.4475	5.55	1.0	0.1057	5.65	−1.9
3.9810	0.3198	7.89	1.1	0.1058	3.05	−3.0
3.3880	0.2772	7.63	1.2	0.1079	2.17	−3.8
3.0200	0.2576	6.56	1.4	0.1105	1.62	−4.6
3.0000	0.2567	9.07	1.2	0.1106	3.83	−3.0
2.6300	0.2420	6.15	1.4	0.1147	2.01	−4.1
1.9950	0.2354	1.93	2.6	0.1272	0.42	−7.8
1.2590	0.3076	0.37	4.4	0.1622	0.03	−18.5
1.0000	0.3629	0.16	4.1	0.1861	0.00	−36.7
0.8913	0.3850	0.17	3.1	0.1939	0.01	−21.4
0.8000	0.3993	0.12	3.2	0.2010	0.00	−32.2

TABLE 3

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state C)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
3.0000	2.0276	0.93	0.9	0.1611	15.04	−1.1
2.5120	0.8059	5.37	0.9	0.1599	13.94	−1.2
2.2390	0.6553	7.30	0.9	0.1597	11.61	−1.3
1.9950	0.5662	8.59	1.0	0.1597	9.44	−1.4
1.7780	0.5067	9.33	1.0	0.1599	7.77	−1.5
1.5850	0.4655	8.84	1.1	0.1605	4.95	−1.8
1.4130	0.4361	9.89	1.1	0.1614	6.54	−1.5
1.2590	0.4156	7.64	1.2	0.1632	3.01	−2.2
1.1220	0.4029	6.35	1.3	0.1659	1.97	−2.6
1.0000	0.3965	5.12	1.4	0.1700	1.33	−3.0
0.8000	0.3994	2.70	1.7	0.1822	0.37	−4.8
0.6000	0.4256	0.71	2.6	0.2078	0.01	−18.2
0.5000	0.4539	0.54	2.5	0.2238	0.02	−13.6

TABLE 4

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state D)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
3.9810	0.6335	0.18	1.3	0.1530	6.03	−1.7
3.5480	0.5817	1.66	1.2	0.1509	14.70	−1.0
3.0000	0.5412	3.49	1.2	0.1483	12.16	−1.1
2.5120	0.5141	5.24	1.2	0.1470	8.70	−1.2
2.2390	0.4922	6.70	1.0	0.1470	7.90	−1.2
1.9950	0.4613	8.09	0.9	0.1467	6.30	−1.2
1.7780	0.4305	9.10	0.8	0.1455	5.10	−1.2
1.5480	0.3930	9.46	0.8	0.1437	3.45	−1.5
1.4130	0.3556	8.91	0.9	0.1420	2.51	−1.9
1.2590	0.3291	6.23	1.3	0.1418	1.00	−3.3
1.1220	0.3202	4.48	1.6	0.1452	0.52	−4.8
1.0000	0.3181	2.43	2.0	0.1511	0.11	−10.0
0.8000	0.3245	1.11	2.4	0.1667	0.04	−13.2
0.6000	0.3617	0.14	5.2	0.2042	0.00	−94.0

TABLE 5

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state E)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
3.0700	0.5894	8.17	0.9	0.1316	18.29	−1.0
2.8180	0.5049	10.35	1.0	0.1302	17.21	−1.1
2.5120	0.4344	12.00	1.0	0.1287	13.26	−1.2
1.7780	0.3319	10.88	1.2	0.1265	4.17	−1.9
1.5850	0.3148	8.60	1.3	0.1271	1.95	−2.7
1.4130	0.3028	7.23	1.3	0.1287	1.34	−3.1
1.2590	0.2950	4.98	1.5	0.1315	0.52	−4.6
1.0000	0.2904	1.87	2.1	0.1426	0.07	−11.7
0.7943	0.3008	0.54	3.0	0.1631	0.01	−24.7
0.6310	0.3327	0.06	6.9	0.1978	0.00	−185.3

TABLE 6

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state F)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
5.0120	2.2222	0.81	1.1	0.1327	10.84	−1.4
4.8500	1.3258	1.42	1.1	0.1326	10.76	−1.4
4.6770	1.0274	1.99	1.1	0.1324	9.84	−1.5
4.5600	0.9148	2.38	1.1	0.1324	9.97	−1.4
4.4670	0.8492	2.65	1.2	0.1324	9.63	−1.5
4.2660	0.7486	3.21	1.2	0.1324	9.14	−1.5
3.5480	0.5730	4.66	1.2	0.1340	6.65	−1.7
2.8180	0.4980	5.40	1.2	0.1387	4.36	−1.8
2.5120	0.4748	5.55	1.1	0.1415	3.34	−1.9
2.2390	0.4565	5.59	1.0	0.1439	2.67	−2.0
1.9950	0.4392	5.49	1.0	0.1457	2.11	−2.1
1.7780	0.4214	5.28	1.0	0.1472	1.60	−2.2
1.4130	0.3885	4.40	1.1	0.1495	0.72	−2.9
1.2590	0.3740	3.71	1.2	0.1509	0.40	−3.7
1.1220	0.3623	2.89	1.3	0.1531	0.18	−5.3
1.0000	0.3541	2.14	1.5	0.1560	0.08	−8.0
0.7943	0.3474	0.85	2.1	0.1666	0.01	−29.3
0.6310	0.3556	0.20	3.8	0.1954	0.00	−235.8
0.5012	0.3823	0.02	10.2	0.2682	0.00	−695.8
0.3981	0.4469	0.00	102.9	0.4003	0.00	−359.2
0.3162	0.7338	0.00	1500.3	0.5505	0.00	−333.3

on each parameter give the number of nodes contained in that particular mode. Thus, T_0, E_0, and d_0 refer to the fundamental radial mode, while T_1, E_1, and d_1 refer to the mode having one node. The details involved with computing these quantities are described in the Appendix.

The energy contained in the oscillations is defined by equation (A13). This energy has been normalized to correspond to a radial pulsation whose surface amplitude is equal to the radius of the unperturbed star R. To scale the tabulated energies to correspond to smaller amplitude oscillations, one must multiply them by the factor $(\xi/R)^2$, where ξ is the desired amplitude of the fluid motion for the surface of the star. The parameter d (defined in eq. [A14]) is a measure of the nonlinearity of the radial displacement eigenfunction. Specifically, it measures the ratio of the amplitude of the fluid motion at the surface of the star to the amplitude of the motion of the fluid near the center of the star.

The accuracy of our computations was tested by varying the spacing of the radial grid on which the stellar model and radial perturbation functions were determined. These tests indicate that the oscillation periods T were determined to better than one part in 10^4, while the parameters E and d which depend on the radial displacement eigenfunction were determined to a few parts in 100.

Figures 2–5 present in graphical form the oscillation periods for the neutron stars constructed from the 13 equation of state models. Figures 2 and 3 give the

TABLE 7

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state G)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
6.3100	0.6869	1.44	1.0	0.1072	7.31	−1.7
6.0420	0.5758	2.52	1.0	0.1070	6.77	−1.8
5.2320	0.4068	5.21	1.0	0.1066	4.78	−2.1
4.5030	0.3299	6.86	1.1	0.1062	3.73	−2.4
4.1610	0.3052	6.83	1.2	0.1063	2.71	−3.0
3.8290	0.2863	6.87	1.3	0.1069	2.39	−3.4
3.4980	0.2721	6.38	1.4	0.1084	1.87	−4.2
3.1980	0.2625	5.39	1.5	0.1109	1.27	−5.3
2.9120	0.2558	4.82	1.6	0.1144	1.16	−5.6
2.6130	0.2504	3.83	1.7	0.1190	0.86	−6.0
2.3760	0.2469	7.30	1.2	0.1232	3.91	−2.6
2.2390	0.2454	1.95	2.3	0.1255	0.24	−10.1
1.9950	0.2444	1.88	2.3	0.1299	0.40	−7.1
1.7780	0.2485	1.20	2.9	0.1354	0.21	−8.9
1.5850	0.2620	0.71	3.8	0.1431	0.09	−12.8
1.4130	0.2832	0.44	4.5	0.1525	0.03	−19.1
1.2590	0.3077	0.41	4.1	0.1620	0.04	−15.5
1.1220	0.3363	0.26	4.1	0.1739	0.01	−21.5
1.0000	0.3629	0.16	4.1	0.1861	0.00	−36.7
0.7943	0.4001	0.12	3.1	0.2012	0.00	−30.1

TABLE 8

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state H)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
3.0000	0.8152	0.26	1.2	0.2201	0.03	−10.3
1.0000	0.8747	0.12	1.9	0.3387	0.00	−52.2
0.3000	1.2596	0.01	5.0	0.7804	0.00	−318.9

TABLE 9

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state I)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
1.9950	1.1677	4.31	0.9	0.1866	62.90	−0.6
1.7780	0.8165	11.05	0.9	0.1816	58.07	−0.7
1.5850	0.6752	16.43	1.0	0.1769	56.14	−0.7
1.4130	0.5941	20.16	1.0	0.1728	48.03	−0.8
1.2590	0.5408	22.39	1.0	0.1699	39.00	−0.8
1.1220	0.5023	23.38	1.0	0.1681	31.41	−0.9
1.0000	0.4705	22.92	1.0	0.1668	23.35	−1.0
0.9000	0.4429	20.84	1.0	0.1651	14.35	−1.1
0.7943	0.4168	19.58	1.0	0.1630	11.07	−1.1
0.6310	0.3832	11.42	1.1	0.1598	1.43	−2.6
0.5700	0.3727	8.81	1.1	0.1586	0.52	−3.8
0.5012	0.3618	3.48	1.6	...	...	...
0.4500	0.3539	2.98	1.5	...	...	...
0.3981	0.3458	1.33	1.9	...	...	...
0.3162	0.3284	0.01	15.6	...	...	...

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TABLE 10

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state L)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
1.4130	3.5881	5.71	0.6	0.2041	46.57	−0.5
1.2590	1.2012	13.28	0.6	0.1995	39.71	−0.6
1.1220	0.8623	19.32	0.6	0.1953	32.62	−0.7
1.0000	0.6994	23.56	0.7	0.1915	25.87	−0.8
0.8913	0.5755	26.00	0.7	0.1871	18.51	−1.0
0.7943	0.4997	24.91	0.8	0.1836	12.15	−1.2
0.6310	0.4077	15.29	1.2	0.1792	3.21	−2.4
0.6000	0.3982	13.05	1.4	0.1793	2.29	−2.9
0.5012	0.3808	6.59	1.9	0.1837	0.64	−5.3
0.5000	0.3807	6.51	1.9	0.1838	0.62	−5.4
0.4000	0.3844	2.06	2.9	0.1970	0.08	−12.3
0.3981	0.3849	1.98	2.9	0.1974	0.08	−12.6
0.3181	0.4311	0.29	5.6	0.2253	0.00	−138.0
0.3000	0.4522	0.16	6.7	0.2417	0.00	−590.6
0.2512	0.5453	0.02	13.1	0.3551	0.00	−3775.4
0.2239	0.6350	0.00	25.6	0.4841	0.00	−3832.0
0.2000	0.7565	0.00	120.5	0.6907	0.00	−1431.0
0.1995	0.7597	0.00	128.1	0.6961	0.00	−1361.4

TABLE 11

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state M)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
3.1620	0.7436	0.99	1.0	0.1888	4.15	−3.0
2.2390	0.5385	3.51	1.2	0.1997	3.68	−3.2
1.7780	0.5041	3.77	1.5	0.2094	3.24	−3.4
1.5850	0.5021	3.68	1.6	0.2154	3.03	−3.4
1.2590	0.5177	3.29	1.7	0.2295	2.58	−3.4
1.0000	0.5497	2.85	1.6	0.2453	2.16	−3.0
0.7943	0.5867	2.52	1.5	0.2599	1.72	−2.5
0.7080	0.5963	2.45	1.4	0.2636	1.55	−2.3
0.6310	0.6032	2.38	1.4	0.2665	1.34	−2.2
0.5623	0.6071	2.30	1.3	0.2681	1.09	−2.2
0.5012	0.6078	2.21	1.3	0.2682	0.82	−2.2
0.3981	0.6006	1.91	1.3	0.2643	0.31	−3.0
0.3162	0.5928	0.93	1.8	0.2660	0.01	−14.7
0.2512	0.6066	0.24	2.9	0.3039	0.00	−524.4
0.1995	0.6456	0.02	7.4	0.4339	0.00	−2337.3
0.1585	0.7291	0.00	100.2	0.6736	0.00	−888.7

TABLE 12

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state N)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
1.0000	0.6123	25.62	0.6	0.1704	28.39	−0.7
0.9000	0.4916	26.31	0.7	0.1646	16.25	−0.9
0.8000	0.4197	24.71	0.8	0.1596	10.18	−1.2
0.7000	0.3722	18.07	1.0	0.1557	3.37	−2.0
0.6000	0.3395	10.90	1.2	0.1548	0.84	−4.0
0.5500	0.3272	7.48	1.4	0.1567	0.36	−6.2
0.5000	0.3170	3.80	1.9	0.1618	0.08	−13.9
0.4500	0.3084	1.59	2.6	0.1712	0.02	−23.7
0.4000	0.3082	0.31	5.2	0.1887	0.00	−67.0

TABLE 13

RADIAL OSCILLATIONS OF NEUTRON STARS (equation of state O)

ρ_c (10^{15} g cm^{-3})	T_0 (10^{-3} s)	E_0 (10^{53} ergs)	d_0	T_1 (10^{-3} s)	E_1 (10^{53} ergs)	d_1
2.0000	2.2908	1.92	0.7	0.1790	62.94	−0.5
1.7800	0.9719	9.81	0.7	0.1738	60.90	−0.5
1.5000	0.6400	18.87	0.8	0.1660	49.72	−0.6
1.5290	0.6628	17.50	0.8	0.1669	45.34	−0.6
1.0000	0.4050	22.55	1.0	0.1483	13.88	−1.1
0.8000	0.3565	14.48	1.2	0.1431	2.23	−2.6
0.7499	0.3465	11.93	1.2	0.1434	1.15	−3.5
0.6683	0.3308	8.55	1.2	0.1465	0.49	−5.0
0.6390	0.3248	6.08	1.4	0.1491	0.18	−8.1
0.6310	0.3231	5.56	1.4	0.1499	0.14	−9.1
0.6000	0.3161	4.11	1.5	0.1533	0.08	−12.1
0.5623	0.3071	2.91	1.6	0.1581	0.06	−13.8
0.5300	0.2993	1.93	1.8	0.1634	0.04	−16.2
0.5000	0.2903	0.88	2.6	0.1703	0.01	−31.8
0.4750	0.2887	0.39	3.8	0.1788	0.00	−60.6
0.4500	0.2852	0.32	4.1	0.1796	0.00	−38.9

periods of the fundamental mode, while Figures 4 and 5 give the periods of the mode with one node. The figures have been split in two to improve the clarity of presentation. Figures 2 and 4 depict the periods for equation of state models A, C, F–L, while Figures 3 and 5 depict the periods for equation of state models B, D, E, M–O. The periods are graphed here versus the surface redshift z of each stellar model (the redshift measured at infinity for a photon emitted at the surface of the neutron star). This parameter is, in principle, directly observable. The

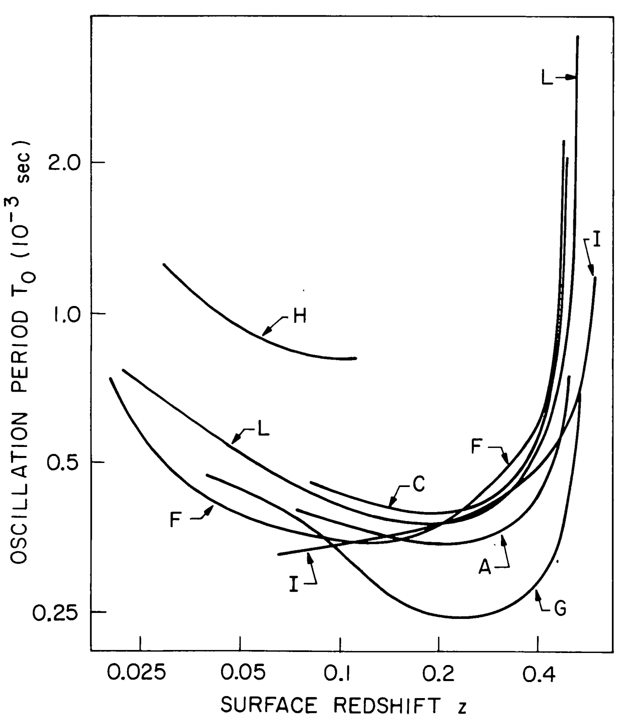

FIG. 2.—Oscillation period T_0 of the radial mode with no nodes is illustrated as a function of surface redshift z for neutron stars constructed from equations of state A, C, F–L.

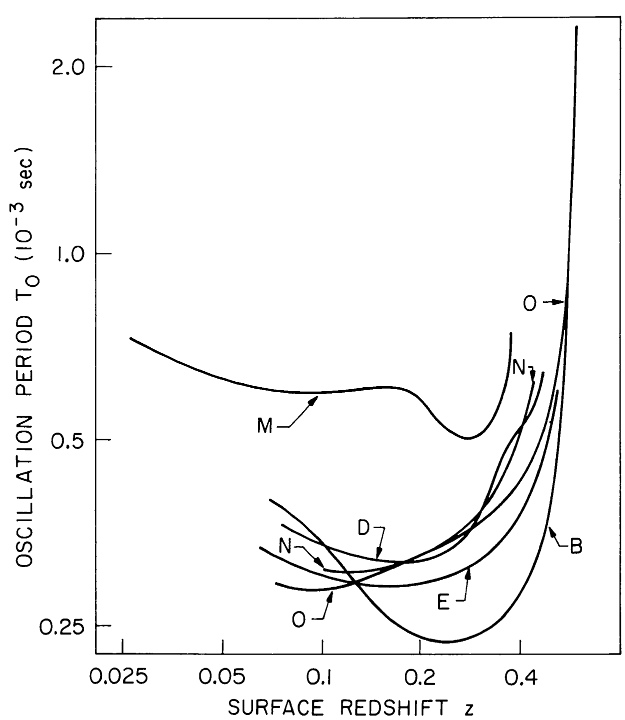

FIG. 3.—Oscillation period T_0 of the radial mode with no nodes is illustrated as a function of surface redshift z for neutron stars constructed from equations of state B, D, E, M–O.

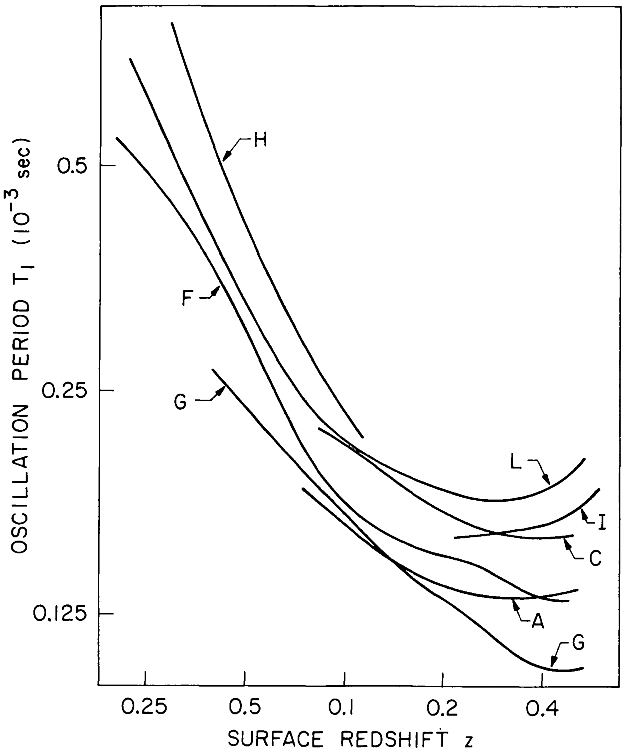

FIG. 4.—Oscillation period T_1 of the radial mode with one node is illustrated as a function of surface redshift z for neutron stars constructed from equations of state A, C, F–L.

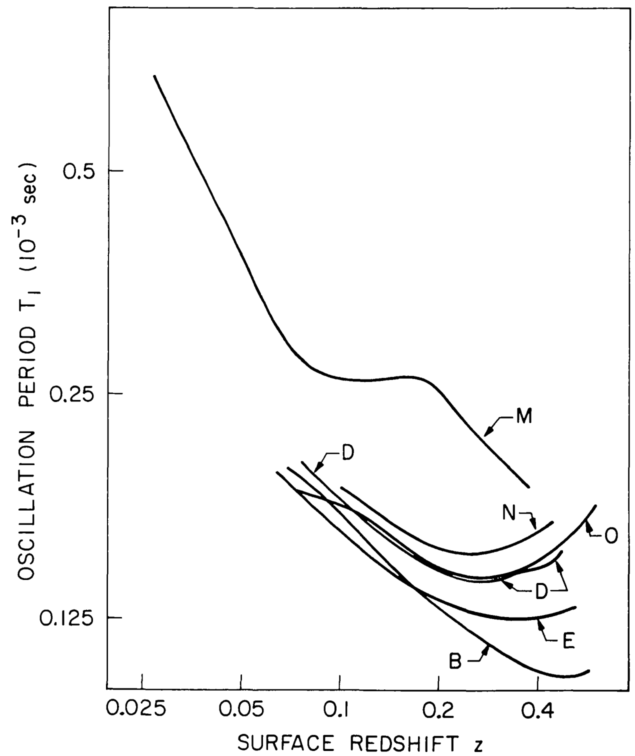

FIG. 5.—Oscillation period T_1 of the radial mode with one node is illustrated as a function of surface redshift z for neutron stars constructed from equations of state B, D, E, M–O.

relationship between z and other equilibrium parameters can be deduced from the tables and graphs given in Lindblom and Detweiler (1983).

The qualitative features of the periods graphed in Figures 2–5 are consistent with our expectations. The periods of the fundamental mode, T_0, are longer than the corresponding periods of the mode with one node, T_1. The periods of the fundamental mode become very large as the mass of the star approaches the maximum neutron star mass. Presumably these periods also become large near the minimum neutron star mass. Our graphs do not reveal this, however, because we did not explore the low mass neutron stars (partly because our equilibrium approximation for the adiabatic index is not good for low mass neutron stars and partly because our numerical techniques appeared to become unstable for low mass neutron stars). One surprising feature of our results, however, was that the fundamental radial mode was not the lowest frequency mode. As illustrated for equation of state L in Figure 1, the period of the quadrupole mode is longer than the period of the fundamental radial mode for a substantial number of neutron star models. This behavior was found for each equation of state studied here, except the Harrison-Wheeler equation of state (model H).

The curves in Figures 2–5 representing the periods of radial oscillation of neutron stars show a great deal more structure than do the curves in Lindblom and Detweiler (1983) representing the quadrupole oscillation frequencies of the same set of stellar models. The radial pulsations depend more sensitively on the equation of state than do the nonradial pulsations. The diversity exhibited in the shapes of the radial pulsation periods in Figures 2–5 is apparently a manifestation of this sensitivity. The dependence of these radial modes on the equation of state appears to be sufficiently complicated that qualitative statements such as "the softer the equation of state, the lower the frequency" do not appear to be true for these modes. The sensitive dependence of these modes on the ultra-high density equation of state (which produces the complicated and unaesthetic curves of Figs. 2–5) may ultimately provide the best way for us to investigate the interior structure of neutron stars.

III. DISCUSSION

At present the observations which may be related to oscillating neutron stars are of two types: X-ray and γ-ray burst phenomena, and the quasi-periodic micropulses from pulsars. Neither type of observation has yet produced data which can be convincingly attributed to neutron star pulsations, however.

The study of the micropulse structure of pulsar emission has yielded a number of periodicities with frequencies which are approximately consistent with neutron star oscillation frequencies. Boriakoff (1976) first reported a 0.9 ms periodicity in the pulsar PSR 2016+28. Additional periodicities with periods in this range have subsequently been reported in other sources by a number of authors: Cordes (1976), Cordes and Hankins (1977), Hankins and Boriakoff (1978), and Soglasnov *et al.* (1981). These periodicities are only correlated for periods of ~10 ms (see Cordes 1976). If these periodicities are associated with oscillating neutron stars, therefore, the neutron stars must be damped by some mechanism with time scales of ~10 ms. The quadrupole oscillation calculations presented by Lindblom and Detweiler (1983) confirm that gravitational radiation is too inefficient to damp the nonradial modes on this time scale, and that the oscillation periods of the nonradial modes are too short to account for a 0.9 ms period. As we have seen in the present work, the oscillation periods of the radial modes are, in general, *shorter* than the quadrupole periods. The fundamental radial mode will have a period as long as 0.9 ms only if the mass of the neutron star is very nearly equal to the maximum neutron star mass. For the equations of state considered here by us, the mass had to be within 1.5% of the maximum mass for all cases, and generally had to be within 0.6%. Thus, the mass must be very carefully fine tuned to produce periods as long as 0.9 ms. It seems unlikely to us, therefore, that a convincing case could be made for associating this periodicity with a radial neutron star oscillation (even if a suitable 10 ms damping mechanism were found).

The case for associating γ-ray burst phenomena (especially the 1979 March 5 event) with neutron stars has been reviewed in Lindblom and Detweiler (1983). Attempts to identify periodicities with ~1 ms time scales in the emission of these objects has not been successful to date (see, e.g., Weisskopf *et al.* 1981). Thus, none of the observed features of these objects has yet been associated with the radial oscillations of a neutron star.

APPENDIX

COMPUTING THE RADIAL OSCILLATIONS

In this appendix we discuss the details of the numerical calculation of the adiabatic radial oscillations of general relativistic neutron star models. The equation of motion for small radial oscillations of relativistic stellar models was first derived by Chandrasekhar (1964). We determine the eigenfrequencies of this system by explicitly integrating the radial perturbation equation. Neutron star pulsation frequencies were first determined in this way by Meltzer and Thorne (1966). The algorithm used here for integrating the equation and locating the eigenfrequencies is a tridiagonal matrix method used by Glass and Harpaz (1983) in their study of the stability of relativistic polytropes.

I. THE EQUILIBRIUM MODELS

The static spherical geometry which describes an equilibrium stellar model has a metric tensor of the form:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{A1}$$

The metric function $\lambda(r)$ is often replaced by the mass function defined by

$$m(r) = \tfrac{1}{2}r(1-e^{-\lambda}). \tag{A2}$$

Einstein's equations, which relate the curvature of this geometry to the energy density ρ and pressure p of the fluid in the star, are equivalent to the system of equations:

$$\frac{dm}{dr} = 4\pi r^2\rho, \tag{A3}$$

$$\frac{dp}{dr} = -\frac{\rho+p}{r(r-2m)}(m+4\pi r^3 p), \tag{A4}$$

$$\frac{d\nu}{dr} = -2(\rho+p)^{-1}\frac{dp}{dr}. \tag{A5}$$

This system of equations must be supplemented by an equation of state of the form $\rho = \rho(p)$. The equations can then be integrated from $r = 0$ in the usual way (see, e.g., Lindblom and Detweiler 1983) once the central density is specified.

In our numerical calculations we use Hamming's predictor-corrector-modifier algorithm (see Lambert 1973) started by Ralston's (1962) minimal error Runge-Kutta algorithm. The radial perturbation equation seems to be more sensitive to the low density portion of the stellar structure than was the quadrupole equation. To improve the accuracy in this region we halved the step size each time the pressure decreased by a factor of 10. The numerical integration was terminated when the pressure became negative. The zero pressure surface was located by three-point extrapolation from the last three values of the pressure. The values of the functions describing the equilibrium model (m, p, and ν) were saved for use in the radial perturbation equation on a uniform radial grid containing ~ 1000 points. The accuracy of the equilibrium model portion of our code is considerably better than the few parts per 10^4 obtained by Lindblom and Detweiler (1983).

The adiabatic index γ for these computations was determined directly from the equation of state by the relation

$$\gamma = \frac{\rho + p}{p}\frac{dp}{d\rho}. \tag{A6}$$

The restoring force for radial perturbations comes entirely from the compressibility of the fluid. The quadrupole perturbations, however, also use gravity as a restoring force. Consequently, the radial pulsations are far more sensitive to the adiabatic index γ than are the quadrupole pulsations. The function $\gamma(p)$ varies considerably in the "neutron drip" density region (10^{11}–10^{13} g cm^{-3}). Instead of numerically differentiating the equation of state table to determine γ in this region, we use the tabulated values of γ given by Baym, Pethick, and Sutherland (1972) for the entire range of validity of their equation of state. The values of γ were not generally available in tabulated form for the nuclear density portion of the equation of state, however. Since γ is generally smoother in this region, we were able to adequately determine it for each entry in the equation of state table by using a simple difference formula to compute the derivative. The values of γ for pressures intermediate between these tabulated ones are obtained by interpolation.

The radial perturbation equation also depends on the derivative of the adiabatic index:

$$\gamma' = d\gamma/dp. \tag{A7}$$

These were computed for each entry in the equation of state table by differencing the tabulated γ values described above. Intermediate values of γ' are again determined by interpolation.

II. THE RADIAL PULSATIONS

Let us consider perturbing the equilibrium stellar models described above by moving each particle of fluid in a purely radial direction. We denote by $\delta r(r,t)$ the time-dependent radial displacement of the particle of fluid located at position r in the unperturbed model. Since we are looking for normal modes, we assume δr has harmonic time dependence

$$\delta r(r,t) = X(r)e^{i\omega t}. \tag{A8}$$

Chandrasekhar (1964) used the perturbed Einstein equations to derive the differential equation for the radial displacement function $X(r)$. This equation is given by

$$Y\frac{d^2X}{dr^2} + \left(\frac{dY}{dr} - Z + 4\pi r\gamma p e^{\lambda} - \frac{1}{2}\frac{d\nu}{dr}\right)\frac{dX}{dr} + \left[\frac{1}{2}\left(\frac{d\nu}{dr}\right)^2 + \frac{2m}{r^3}e^{\lambda} - \frac{dZ}{dr} - 4\pi(\rho+p)Zre^{\lambda} + \omega^2 e^{\lambda-\nu}\right]X = 0, \tag{A9}$$

where

$$Y(r) = \gamma p/(\rho + p), \tag{A10}$$

$$Z(r) = Y\left(-\frac{2}{r} + \frac{1}{2}\frac{d\nu}{dr}\right). \tag{A11}$$

The functions $\rho(r)$, $p(r)$, $\lambda(r)$, $m(r)$, etc., are the equilibrium stellar model functions discussed above. In addition to equation (A9), two boundary conditions must be specified to determine the function $X(r)$. The fluid at the center of the star is assumed to remain at rest; thus, $X(0) = 0$ is one boundary condition. The secondary boundary condition requires the perturbed pressure to vanish on the perturbed boundary of the star. This condition is equivalent to the requirement that the Lagrangian change in the pressure,

$$\Delta p = - e^{\nu/2} r^{-2} \gamma p \frac{d}{dr}(r^2 e^{-\nu/2} X), \tag{A12}$$

vanish at the surface of the star. These two boundary conditions and the finiteness of $X(r)$ and $dX(r)/dr$ everywhere uniquely determine the eigenvalues ω^2 and the corresponding radial eigenfunctions $X(r)$.

The second-order differential equation (A9) is converted to a system of finite difference equations for numerical evaluation. The differencing scheme can be chosen to make the resulting system of equations into a tridiagonal matrix eigenvalue problem. The details involved in converting equation (A9) to tridiagonal matrix form are given in Glass and Harpaz (1983) and will not be repeated here. Once equation (A9) has been converted to tridiagonal matrix form, it is straightforward to compute the eigenvalues ω^2 and eigenfunctions $X(r)$ (see, e.g., Wilkinson 1965). The determinant of a tridiagonal matrix can be computed quickly and easily. We locate the eigenfrequencies therefore by computing the characteristic polynomial of the tridiagonal matrix and searching for zeroes. We then determine the eigenfunction $X(r)$ corresponding to the eigenvalue ω^2 and refine its accuracy using the techniques outlined in Wilkinson (1965). We check to ensure that the fundamental radial eigenfunction has no zeroes while the next lowest frequency mode has exactly one zero. The eigenvalues and eigenvectors of the 1000×1000 tridiagonal matrix used to represent the radial perturbation equation are easily determined to very great accuracy (typically better than one part in 10^9). Unfortunately, the eigenvalues of the system of difference equations do not agree with the eigenvalues of the differential equation to this degree of accuracy. By varying the number of radial grid points we estimate the accuracy of our published eigenfrequencies to be about one part in 10^4. The corresponding eigenfunctions appear to be determined to about one part in 100.

The eigenfunctions are used by us to compute the energy associated with the stellar pulsations. This energy can be derived from the Lagrangian formulation of equation (A9) given by Chandrasekhar (1964). The expression for the energy is given by

$$E = 4\pi \int_0^R \frac{1}{2}(\rho + p) e^{(\nu+\lambda)/2} r^2 \left\{ \left[\omega^2 e^{\lambda-\nu} + \frac{Z^2}{Y} - \frac{1}{4}\left(\frac{d\nu}{dr}\right)\left(\frac{d\nu}{dr} + \frac{8}{r}\right) + 8\pi e^{\lambda} p \right] \psi^2 + \frac{d\psi}{dr}\left[Y \frac{d\psi}{dr} - 2\psi Z \right] \right\} dr, \tag{A13}$$

where $\psi = r^2 X e^{-\nu/2}$. We use a simple trapezoidal rule to compute this integral. Our eigenfunctions are normalized so that $X(R) = R$, where R is the total radius of the star. Thus, to scale the energy for oscillations with smaller amplitude, the energy reported in the table must be multiplied by $(\xi/R)^2$, where ξ is the desired amplitude of the radial fluid displacement at the surface of the star.

We have not tabulated the entire radial eigenfunction $X(r)$ for each mode of each neutron star model considered. We have included in the tables, however, one parameter which describes the nonlinearity of X:

$$d = \lim_{r \to 0} \frac{rX(R)}{RX(r)}. \tag{A14}$$

Thus, d measures the amount of fluid displacement at the surface of the star relative to the amount of displacement near $r = 0$.

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EDWARD N. GLASS: Department of Physics, University of Windsor, Windsor, Ontario N9B 3P4, Canada

LEE LINDBLOM: Enrico Fermi Institute, University of Chicago, 5640 S. Ellis Avenue, Chicago, IL 60637