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Stability and Causality in Dissipative Relativistic Fluids*

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The standard theory of relativistic dissipative fluid mechanics developed by Eckart contains several undesirable features: thermal and viscous fluctuations propagate acausally; there exist generic short wavelength secular instabilities; and there is not a well-posed initial value problem for rotating fluids. In this paper we examine whether the generalization of Eckart's theory developed by Israel has succeeded in eliminating these features. We first generalize Israel's theory to include the possibility of nonuniform equilibrium configurations. This generalization allows us to describe equilibrium configurations which may be rotating and self-gravitating such as neutron stars. We then evaluate the stability conditions for these fluids and compute the characteristic velocities at which perturbations propagate. Our main result is that if these fluids are stable, then the characteristic velocities are subluminal and the perturbations propagate via hyperbolic equations. Thus Israel's theory is causal for all stable fluids. In addition, there is no generic instability, and the initial value problem is well posed. In our opinion, for these reasons, Israel's theory should replace Eckart's as the standard theory of relativistic dissipative fluid mechanics.

I. INTRODUCTION

In this paper we investigate the phenomenological theory of dissipative relativistic fluid mechanics developed by Werner Israel [1]. Israel's theory was designed to overcome a number of unattractive features in the standard theory developed by Carl Eckart [2] in 1940. These difficulties are: (a) thermal conductivity and viscosity give rise to fluctuations which propagate acausally; (b) there exist generic short

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wavelength secular instabilities driven by the dissipation processes [3]; (c) there is not a well-posed initial value problem for rotating fluids in the standard theory. In this paper we investigate whether Israel's theory has succeeded in overcoming these difficulties.

The causal properties of Israel's theory have previously been studied by Stewart [4], and Israel and Stewart [5]. They computed the characteristic velocities of the perturbations about equilibrium and then used the relativistic kinetic theory to evaluate these velocities in the limit of a dilute gas. Their analysis confirms that Israel's theory is causal in this limit. In this paper, we confirm that Israel's theory behaves causally in a far wider range of circumstances.

We begin our analysis by constructing a generalization of Israel's theory in Section II, which is appropriate for the study of nonuniform equilibrium states such as stars. The generalized theory contains two additional new thermodynamic coefficients not included in Israel's theory. Our theory reduces to Israel's, however, for spatially homogeneous states.

Next, we investigate the conditions under which the equilibrium configurations of the generalization of Israel's theory are stable. The equilibrium configurations considered here include the possibilities of nonuniformities caused by rotation or gravitational fields. We study the stability of these equilibrium states by examining the behavior of small perturbations about equilibrium. We limit our investigation, however, to perturbations which decouple from the gravitational perturbations; we set the perturbations in the metric tensor (gravitational field) to zero. This is appropriate for fluids having negligible self-gravity, and for short wavelength perturbations even in self-gravitating fluids. We present in Section III a set of conditions on the thermodynamic potentials and their derivatives which guarantee the stability of the fluid to these perturbations. Our stability analysis is based on the construction of a monotonically decreasing energy functional for these perturbations.

As a byproduct of our stability analysis, we present a comprehensive discussion of relativistic thermodynamics. We present the complete set of Maxwell relations for the set of thermodynamic variables of interest in this problem. We also determine the complete set of thermodynamic derivatives which have definite sign as a consequence of our stability conditions. These definite signed derivatives include the specific heats, compressibilities, etc.

We evaluate the causal properties of Israel's theory in Section IV. We compute the characteristic velocities, and find that there are three classes of waves. There are two different longitudinal modes; one corresponds to normal sound and the other to second sound. There is, in addition, a transverse shear mode having two polarizations.

The major result of our analysis is to show the equivalence between the conditions for the causality of these characteristic velocities and the stability conditions for these fluids. We show that the stability conditions derived in Section III imply that the characteristic velocities are real and less than the speed of light. The stability conditions imply, in addition, that the equations for the perturbations form a hyperbolic system with a well-posed initial value problem. We then show that a result which is somewhat weaker than the converse is also true: if the characteristic velocities are subluminal, and if the perturbation equations are a hyperbolic system (in a particular sense which we will define later), then the stability conditions are necessarily satisfied. From these results we conclude that the theory of dissipative relativistic fluid mechanics developed by Israel has succeeded in eliminating the problems inherent in Eckart's theory. Thermal fluctuations propagate causally in any stable fluid in this theory. The generic short wavelength secular instability inherent in Eckart's theory does not exist in Israel's theory. The perturbations have a well-posed initial value problem, even for rotating fluids. Thus, Israel's phenomenological theory of dissipative fluid mechanics has many extremely attractive features. In particular, it is completely compatible with special and general relativity. Therefore, in our opinion, Israel's theory should replace the Eckart theory as the standard theory of relativistic dissipative fluid mechanics.

II. Müller–Israel Fluid Mechanics

In this section we review (and to some extent generalize) the theory of dissipative relativistic fluid mechanics developed by Israel [1]. The method used by Israel to generalize the standard (Eckart) relativistic theory [2] was developed earlier by Müller [6] in the context of nonrelativistic fluids. Their method generalizes the standard model of the entropy current for out of equilibrium systems, and then enforces the second law of thermodynamics in the simplest possible way. In Section II(a) we review their method in detail, and use it to construct a theory which includes the possibility of nonuniform equilibrium configurations (such as stars). Surprisingly perhaps, the nonuniform theory developed here must involve two additional thermodynamic coefficients not included in Israel's [1] theory.

Given the general theory constructed by the Müller–Israel technique, we proceed to systematically investigate its properties. We begin by determining the equilibrium states of such a system in Section II(b). The equilibrium states found here are identical to those for the standard relativistic theory [2]. Next, we investigate the equations of motion for small perturbations about any equilibrium configuration in Section II(c). We work here in the Eulerian framework and restrict our attention to the linearized equations of motion. We also make the simplifying assumption of ignoring perturbations in the gravitational field (the spacetime metric tensor). This assumption is appropriate whenever gravitational effects are unimportant, and also for short-length scale perturbations of self-gravitating equilibrium configurations.

(a) *The General Equations of Motion*

We begin by introducing the mathematical variables which are used to describe the state of a relativistic dissipative fluid. The stress energy tensor T^{ab} for such a system can be written quite generally as:

$$T^{ab} = \rho u^a u^b + (p + \tau)\, q^{ab} + u^a q^b + u^b q^a + \tau^{ab}. \tag{1}$$

In this expression u^a is the four-velocity of the particles in a fluid which has energy density ρ and pressure p (this is the so-called Eckart frame). The tensor q^{ab} is a projection tensor, related to u^a and the spacetime metric tensor g^{ab} by:

$$q^{ab} = g^{ab} + u^a u^b. \tag{2}$$

There are, in addition, three fields which describe the out-of-equilibrium properties of the fluid: q^a, τ, and τ^{ab}. The vector field q^a is the heat flow, while τ and τ^{ab} are stresses caused by viscosity in the fluid. These fields satisfy the constraints:

$$0 = u^a q_a = u^a \tau_{ab} = \tau_a^a = \tau_{ab} - \tau_{ba} = u^a q_{ab}. \tag{3}$$

The local conservation of energy and momentum in such a fluid is expressed by the equation

$$\nabla_a T^{ab} = 0, \tag{4}$$

where ∇_a is the covariant derivative compatible with the metric g_{ab}.

There are, in addition, a number of thermodynamic variables which play an important role in the theory. The number density of particles in the fluid n is taken, along with ρ, to be a fundamental thermodynamic variable which can (in principle) be measured by an observer comoving with the fluid. For the simple one-component fluids considered here, the number density satisfies the conservation law

$$\nabla_a(n u^a) = 0. \tag{5}$$

The entropy per particle s is not directly observable; however, it can be inferred for a particular type of fluid by using the first law of thermodynamics and making measurements of the fluid in various equilibrium configurations. The catalogue of values of s for various values of the observables ρ and n is called the equation of state:

$$s = s(\rho, n). \tag{6}$$

The first law of thermodynamics relates the derivatives of this function to other observable thermodynamic variables. Thus, the pressure p and temperature T are given by

$$\frac{1}{T} = n \left(\frac{\partial s}{\partial \rho}\right)_n, \tag{7}$$

$$p = -\rho - n^2 T \left(\frac{\partial s}{\partial n}\right)_\rho. \tag{8}$$

To complete the theory of dissipative fluid mechanics, we must also specify how the variables τ, τ^{ab}, and q^a are determined. The Müller–Israel theory and the standard Eckart theory base their choices for these variables on the need to implement the second law of thermodynamics within the theory. To see how this is done, we consider the vector field s^a which represents the current of entropy within the fluid. From this current, one can define a total entropy S by integrating over a spacelike surface Σ:

$$S(\Sigma) = \int_\Sigma s^a\, d^3x_a. \tag{9}$$

The second law of thermodynamics requires this total entropy to be a nondecreasing function of time (for isolated systems). Thus, if S is evaluated on any later spacelike surface Σ', the value of the entropy must not decrease:

$$S(\Sigma') - S(\Sigma) = \int \nabla_a s^a d^4x \geqslant 0. \tag{10}$$

For isolated systems the two surface integrals for S can be converted to a volume integral by Gauss' theorem as indicated in Eq. (10). For this inequality to hold for every surface Σ' to the future of Σ the following inequality must also hold:

$$\nabla_a s^a \geqslant 0. \tag{11}$$

The second law of thermodynamics is therefore equivalent in these theories to Eq. (11).

The next step in constructing a theory of dissipative fluid mechanics is to model the entropy current s^a. The standard theory of Eckart [2] uses the following expression for the entropy current

$$s^a = snu^a + q^a/T. \tag{12}$$

The first term, snu^a, represents the entropy carried along by the motion of the fluid. The second term, q^a/T, represents the entropy flux which occurs whenever there is heat flow. The divergence of this current can be evaluated by using the equations of motion for the fluid, Eqs. (4) and (5), with the result:

$$T\nabla_a s^a = -\{\tau\nabla_a u^a + q^a[T^{-1}\nabla_a T + u^b\nabla_b u_a] + \tau^{ab}\langle\nabla_a u_b\rangle\}. \tag{13}$$

The brackets $\langle\ \rangle$ which appear in Eq. (13) have the meaning:

$$\langle A_{ab}\rangle = \tfrac{1}{2}q_a^c q_b^d[A_{cd} + A_{dc} - \tfrac{2}{3}q_{cd}q^{ef}A_{ef}], \tag{14}$$

for any second rank tensor A_{ab}. The simplest way to ensure that Eq. (13) is consistent with the second law of thermodynamics, Eq. (11), is to require that τ, τ^{ab}, and q^a be given by:

$$\tau = -\zeta\nabla_a u^a, \tag{15}$$

$$q^a = -\kappa q^{ab}[\nabla_b T + Tu^c\nabla_c u_b], \tag{16}$$

$$\tau^{ab} = -2\eta\langle\nabla^a u^b\rangle. \tag{17}$$

The three thermodynamic functions ζ, η, κ must be positive; they may be identified as the bulk viscosity, the shear viscosity, and the thermal conductivity, respectively. These equations constitute the standard theory of dissipative fluid mechanics. With these expressions the divergence of the entropy current assumes a manifestly positive form:

$$\nabla_a s^a = \frac{\tau^2}{\zeta T} + \frac{q^a q_a}{\kappa T^2} + \frac{\tau^{ab}\tau_{ab}}{2\eta T} \geqslant 0. \tag{18}$$

Motivated by the noncausal propagation of fluctuations in the standard theory, Müller [6] and Israel [1] sought to generalize the theory in an attempt to overcome that difficulty. They proposed using a more complicated model for the entropy current than that given in Eq. (12). They point out, for example, that the expression for the entropy density, sn, used in Eq. (12), is really only expected to hold in an equilibrium state. The equation of state (which is used to find s for given values of ρ and n) is only supposed to accurately describe the state of the fluid when it is in equilibrium. Thus, argue Müller and Israel, one might expect the physical density of entropy to differ from sn by terms which go to zero for a fluid in an equilibrium state: i.e., τ, q^a, and τ^{ab}. They propose to generalize the expression for the entropy current, therefore, by adding a complete set of second order (in deviations away from equilibrium) terms. Their model for the entropy current has the form:

$$s^a = snu^a + q^a/T - \tfrac{1}{2}(\beta_0\tau^2 + \beta_1 q^b q_b + \beta_2 \tau_{bc}\tau^{bc})\, u^a/T + \alpha_0 \tau q^a/T + \alpha_1 \tau^a_b q^b/T. \tag{19}$$

The new thermodynamic coefficients β_0, β_1 and β_2 model the deviations of the physical entropy density from sn; and α_0 and α_1 model changes in the entropy current due to possible viscous-heat flux coupling. The divergence of this current is now computed with the aid of the equations of motion. The resulting expression is

$$\begin{aligned} T\nabla_a s^a = &-\tau\left[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a q^a - \gamma_0 T q^a \nabla_a\left(\frac{\alpha_0}{T}\right) + \frac{1}{2}\tau T\nabla_a\left(\frac{\beta_0}{T}u^a\right)\right] \\ &- q^a\left[\frac{1}{T}\nabla_a T + u^b\nabla_b u_a + \beta_1 u^b \nabla_b q_a - \alpha_0 \nabla_a \tau - \alpha_1 \nabla_b \tau^b_a \right. \\ &\left. - (1-\gamma_0)\,\tau T\nabla_a\left(\frac{\alpha_0}{T}\right) - (1-\gamma_1)\, T\tau^b_a \nabla_b\left(\frac{\alpha_1}{T}\right) + \frac{1}{2} T q_a \nabla_b\left(\frac{\beta_1}{T}u^b\right)\right] \\ &- \tau^{ab}\left\langle \nabla_a u_b + \beta_2 u^c \nabla_c \tau_{ab} - \alpha_1 \nabla_a q_b \right. \\ &\left. - \gamma_1 T q_a \nabla_b\left(\frac{\alpha_1}{T}\right) + \frac{1}{2} T\tau_{ab}\nabla_c\left(\frac{\beta_2}{T}u^c\right)\right\rangle. \end{aligned} \tag{20}$$

The simplest way to ensure that the second law of thermodynamics is satisfied for this theory is to constrain τ, q^a, and τ^{ab} by the following equations:

$$\tau = -\zeta \left[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a q^a - \gamma_0 T q^a \nabla_a \left(\frac{\alpha_0}{T} \right) + \frac{1}{2} \tau T \nabla_a \left(\frac{\beta_0}{T} u^a \right) \right], \quad (21)$$

$$q^a = -\kappa T q^{ab} \left[\frac{1}{T} \nabla_b T + u^c \nabla_c u_b + \beta_1 u^c \nabla_c q_b - \alpha_0 \nabla_b \tau - \alpha_1 \nabla_c \tau^c_b \right.$$
$$\left. - (1-\gamma_0)\, \tau T \nabla_b \left(\frac{\alpha_0}{T} \right) - (1-\gamma_1)\, T \tau^c_b \nabla_c \left(\frac{\alpha_1}{T} \right) + \frac{1}{2} T q_b \nabla_c \left(\frac{\beta_1}{T} u^c \right) \right], \quad (22)$$

$$\tau^{ab} = -2\eta \left\langle \nabla^a u^b + \beta_2 u^c \nabla_c \tau^{ab} - \alpha_1 \nabla^a q^b - \gamma_1 T q^a \nabla^b \left(\frac{\alpha_1}{T} \right) + \frac{1}{2} T \tau^{ab} \nabla_c \left(\frac{\beta_2}{T} u^c \right) \right\rangle. \quad (23)$$

With these conditions, the Müller–Israel entropy current satisfies Eq. (18). Thus the second law of thermodynamics is contained within this theory. The expressions for τ, q^a, and τ^{ab} given here in Eqs. (21)–(23) differ slightly from those given by Israel [1]. Our expressions contain several terms involving the gradients of the α's and β's which were neglected by Israel. Some of the terms, $q^{ab}\nabla_b(\alpha_i/T)$, are spatial gradients and cannot *a priori* be ignored for nonuniform situations such as stars. The other terms, $\nabla_a(\beta_i u^a/T)$, are time derivatives and will be neglected when we consider the equations for small perturbations from equilibrium in Section II(c). Also note that Eqs. (21)–(23) contain two new thermodynamic coefficients γ_0 and γ_1. These new coefficients were introduced because of the ambiguity involved in factoring the terms on the right hand side of Eq. (20) which involve the products τq^a and $\tau_{ab} q^b$. Since the magnitudes of the γ's are not known *a priori*, they could in principle be large. Thus, the terms involving the spatial gradients, $q^{ab}\nabla_b(\alpha_i/T)$, could be important even when the gradients themselves are quite small.

Equations (21)–(23) together with the conservation laws, Eqs. (4) and (5), form a complete system of equations for the dynamical variables (n, ρ, u^a, q^a, τ, and τ^{ab}) of this theory. When appropriate, the gravitational interaction can be included for these fluids by including Einstein's equation

$$G_{ab} = 8\pi T_{ab}, \quad (24)$$

for the spacetime metric tensor g_{ab}. Our aim in the remainder of this paper is to systematically study this theory. We are particularly interested in examining the stability conditions for these fluids; and we are interested in determining whether or not the Müller–Israel procedure has succeeded in producing a causal theory of dissipative fluids.

(b) *Equilibrium Configurations*

The first step in the process of understanding the properties of the new dissipative fluid equations is to determine the possible equilibrium configurations of these fluids.

In the equilibrium state, the entropy of the fluid must not change with time. Thus, the divergence of the entropy current must vanish for a fluid in equilibrium. This divergence is a sum of positive terms (see Eq. (18)), each of which must vanish in the equilibrium state. Thus, in equilibrium the viscous stresses and heat flow must vanish:

$$0 = \tau = \tau^{ab} = q^a. \tag{25}$$

These conditions may be used with Eqs. (21)–(23) to infer the additional properties of equilibrium fluids:

$$0 = \nabla_a u^a, \tag{26}$$

$$0 = \langle \nabla_a u_b \rangle, \tag{27}$$

$$0 = q^{ab}(\nabla_b T + T u^c \nabla_c u_b). \tag{28}$$

The conservation laws, Eqs. (4) and (5), yield the following additional information when these equilibrium conditions are imposed:

$$0 = u^a \nabla_a n, \tag{29}$$

$$0 = u^a \nabla_a \rho, \tag{30}$$

$$0 = q^{ab}[\nabla_b p + (\rho + p)\, u^c \nabla_c u_b]. \tag{31}$$

Equations (29) and (30) imply that all of the thermodynamic variables (s, T, p) must be constant along the integral curves of u^a, since each of these variables depends only upon ρ and n through the equation of state. This fact and Eqs. (26)–(28) result in the requirement that the vector field u^a/T satisfies Killing's equation:

$$\nabla_a(u_b/T) + \nabla_b(u_a/T) = 0. \tag{32}$$

The final equation, (Eq. (31)) is equivalent to the condition that a certain thermodynamic potential,

$$\Theta = \frac{\rho + p}{nT} - s, \tag{33}$$

has vanishing gradient. In general, the gradient of Θ has the form (use Eqs. (7) and (8)):

$$nT \nabla_a \Theta = \nabla_a p - \frac{\rho + p}{T} \nabla_a T. \tag{34}$$

This gradient vanishes whenever Eqs. (28)–(31) are satisfied.

To summarize, the requirement that the fluid be in an equilibrium state is equivalent to the requirements: (a) that u^a/T be a Killing vector field, (b) that the heat flow and viscous stresses vanish, and (c) that Θ have vanishing gradient. These conditions are the same as those found by Israel [1], and are the same as the conditions for the equilibrium states in the standard theory.

(c) *Perturbations about the Equilibrium State*

For the remainder of this paper we will limit our attention to the study of fluid states which are nearly in equilibrium. It could be argued that this is the only possible domain of validity of the Müller–Israel theory. The entropy current model for this theory could be viewed as a power series expansion in τ, q^a, and τ^{ab} which has been truncated at second order. Thus the theory would only be applicable when the neglected higher order terms are small, i.e., whenever τ, q^a, and τ^{ab} are small, so that the fluid is near equilibrium. In any case, the study of small perturbations about equilibrium will allow us to investiate the stability of these fluids; and we will be able to study whether or not these perturbations propagate causally.

We will analyze the perturbations about equilibrium in terms of the Eulerian framework to avoid the gauge ambiguities inherent in the Lagrangian approach [3, 7]. We denote by δQ the difference between the actual nonequilibrium value of a field Q at a given point of spacetime, and the value which Q has in a fiducial equilibrium state at the same spacetime point. Thus the quantities δn, $\delta\rho$, δu^a, $\delta\tau$, $\delta\tau^{ab}$, δq^a, δg_{ab} are the fields which describe the perturbations about some equilibrium state. The fields which do not include the prefix δ (n, ρ, u^a, g_{ab}, etc.) will henceforth refer to the fiducial equilibrium configuration, and they are assumed to satisfy the properties outlined in Section II(b). The equilibrium configurations considered here are not limited in any way, however, other than the constraints stated in Section II(b); in particular, these configurations could include strong gravitational fields and rapid rotation. We assume that the perturbation quantities δQ are small departures from equilibrium, so that we can adequately describe their evolution by using the linearized equations of motion. For simplicity, we also limit attention here to perturbations which leave the gravitational field fixed: $\delta g_{ab} = 0$. This approximation is relevant in any situation where gravity plays no significant role (special relativity) and also for short wavelength perturbations of any equilibrium state [3].

The equations of motion for the perturbation variables (δn, $\delta\rho$, δu^a, $\delta\tau$, δq^a, $\delta\tau^{ab}$) are obtained from the general equations of motion (Eqs. (4), (5), (21)–(23)) by linearizing these equations about the fiducial equilibrium state. These equations become:

$$0 = \nabla_a \delta T^{ab}, \tag{35}$$

$$0 = u^a \nabla_a \delta n + \nabla_a (n \delta u^a), \tag{36}$$

$$\delta\tau = -\zeta \left[\nabla_a \delta u^a + \beta_0 u^a \nabla_a \delta\tau - \alpha_0 \nabla_a \delta q^a - \gamma_0 T \delta q^a \nabla_a \left(\frac{\alpha_0}{T} \right) \right], \tag{37}$$

$$\begin{aligned} \delta q^a = -\kappa T q^{ab} \bigg[&\nabla_b \left(\frac{\delta T}{T} \right) + u^c \nabla_c \delta u_b + \delta u^c \nabla_c u_b + \beta_1 u^c \nabla_c \delta q_b - \alpha_0 \nabla_b \delta\tau \\ &- \alpha_1 \nabla_c \delta\tau^c_b - (1 - \gamma_0)\, T \delta\tau \nabla_b \left(\frac{\alpha_0}{T} \right) - (1 - \gamma_1)\, T \delta\tau^c_b \nabla_c \left(\frac{\alpha_1}{T} \right) \bigg], \end{aligned} \tag{38}$$

$$\delta\tau^{ab} = -2\eta \left\langle \nabla^a \delta u^b + \delta u^a u^c \nabla_c u^b + \beta_2 u^c \nabla_c \delta\tau^{ab} - \alpha_1 \nabla^a \delta q^b - \gamma_1 T \delta q^a \nabla^b \left(\frac{\alpha_1}{T} \right) \right\rangle. \tag{39}$$

The perturbed stress-energy tensor which appears in these equations is related to the standard variables by

$$\delta T^{ab} = (\rho + p)(\delta u^a u^b + u^a \delta u^b) + \delta\rho u^a u^b + (\delta p + \delta\tau)\, q^{ab} + u^a \delta q^b + u^b \delta q^a + \delta\tau^{ab}. \tag{40}$$

The derivative ∇_a which appears in these equations is the covariant derivative of the background equilibrium spacetime. Spacetime indices are raised and lowered with the background metric tensor; e.g., $\delta u_a = g_{ab}\delta u^b$. The perturbation variables also satisfy a number of constraints which follow from Eq. (3):

$$0 = u_a \delta q^a = \delta\tau^{ab} - \langle \delta\tau^{ab} \rangle = u_a \delta u^a. \tag{41}$$

Finally, the perturbed thermodynamic variables satisfy the following relationships because of the first law of thermodynamics:

$$\delta\rho = nT\delta s + \frac{\rho + p}{n}\,\delta n, \tag{42}$$

$$\delta p = nT\delta\Theta + \frac{\rho + p}{T}\,\delta T. \tag{43}$$

This concludes our general discussion of the Müller–Israel fluid mechanics. We have introduced the system of equations needed to study these fluids in a wide range of physical situations which may include strong gravitational fields and rapid rotation. Our aim in the remainder of this paper will be to determine the conditions under which the fluid motions represented by these equations are stable and causal.

III. Stability Conditions

In this section we determine the conditions under which the equilibrium configurations of Müller–Israel fluids are in fact stable equilibria. The notion of stability used here is formulated in terms of the properties of the solutions to the first order perturbation equations. If all solutions to the perturbation equations with regular initial data are bounded functions of time, then the equilibrium configuration is considered stable. Small departures from the original equilibrium state remain small in this case. If on the other hand there exist solutions to the perturbation equations which become unbounded in time, then the equilibrium is considered unstable. Small departures from such an equilibrium state can evolve to become very large departures.

In practice, the perturbation Eqs. (35)–(39) are extremely complicated. It is in general very difficult to determine the conditions which guarantee that the solutions to such a system of equations remain bounded. Under some circumstances, however, there exists a so-called energy functional associated with the equations of motion, which makes it relatively easy to determine the stability criteria. An energy functional is a monotonically decreasing function of time which depends quadratically on the perturbation variables. If such a functional is non-negative for all possible values of the perturbation variables, then it would suggest that the equilibrium is stable. In that case, the energy is a decreasing function of time which is bounded below by zero. If, on the other hand, perturbations having negative energy exist, then the energy is unbounded below and could evolve towards negative infinity. The existence of such negative energy perturbations suggests the equilibrium is not stable. To rigorously establish the relationship between the sign of the energy functional and stability, detailed mathematical analysis of the evolution equations for the perturbations is required.

In the sections that follow, we will present an analysis of the stability of the equilibrium configurations of a Müller–Israel fluid which is based on an energy functional. In Section III(a) we present an energy functional for these fluids which is quadratic and monotonically decreasing with time. We factor this energy in Section III(b) to determine the conditions under which this energy is positive. The conditions for positive energy reduce to a number of inequalities involving the thermodynamic variables (and their derivatives) of the equilibrium configuration, as well as the Müller–Israel coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1,$ and β_2. We use these positive energy conditions in Section III(c) to infer other associated thermodynamic inequalities such as the positivity of the specific heats, etc. We postpone until the Appendix the proofs of the theorems which establish the relationship between the positivity of the energy and the stability of the fluid. We show there that the positivity of the energy is a sufficient condition to guarantee the stability of the fluid, and we prove a result which falls only slightly short of showing that the positivity of the energy is also a necessary condition for stability.

(a) *An Energy Functional*

A monotonically decreasing energy functional which is quadratic in the perturbation fields can be introduced for Müller–Israel fluids in terms of an energy current E^a defined by the expression:

$$\begin{aligned} TE^a = {} & \delta T^a_b \delta u^b - \frac{1}{2}(\rho + p)\, u^a \delta u^b \delta u_b \\ & + \frac{1}{2}(\rho + p)^{-1}\left[\left(\frac{\partial \rho}{\partial p}\right)_s (\delta p)^2 + \left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta (\delta s)^2\right] u^a \\ & + \frac{\delta T}{T}\delta q^a + \frac{1}{2}\left[\beta_0(\delta\tau)^2 + \beta_1 \delta q^b \delta q_b + \beta_2 \delta\tau^{bc}\delta\tau_{bc}\right] u^a \\ & - \alpha_0 \delta\tau\delta q^a - \alpha_1 \delta\tau^a_b \delta q^b. \end{aligned} \tag{44}$$

The total energy associated with a spacelike surface Σ (i.e., the energy at one instant of time) is given by the integral of the current E^a over the surface:

$$E(\Sigma) = \int_\Sigma E^a d^3x_a. \quad (45)$$

From an argument analogous to that given for the entropy of the fluid in Section II(a), this energy will be a decreasing function of time (for fluids with compact spatial support) as long as the divergence of the energy current is negative. The divergence of E^a can be computed in a straightforward manner from Eq. (44). The resulting expression can be simplified by the use of the perturbation Eqs. (35)–(39). The final expression has the form:

$$\nabla_a E^a = -\left[\frac{(\delta\tau)^2}{T\zeta} + \frac{\delta q^a \delta q_a}{\kappa T^2} + \frac{\partial\tau^{ab}\delta\tau_{ab}}{2\eta T}\right] \leqslant 0. \quad (46)$$

Therefore, the energy functional $E(\Sigma)$ is a monotonically decreasing function of time. The positivity of this functional is consequently a useful indication of stability.

Before we turn to our investigation of the properties of E, it is perhaps appropriate to comment briefly on the method by which we located an appropriate energy current. There was, unfortunately, no elegant derivation which one might hope to apply to other situations. Our "derivation" was in fact based on a series of modifications and generalizations of previously existing results for similar physical situations. To create the expression for the energy current given by Eq. (44) we modified somewhat the energy functional for the standard Eckart fluid mechanics which was developed by Lindblom and Hiscock [3]. That energy was in turn developed by modifying expressions for perfect fluid energies developed by Friedman and Schutz [7, 8]. Those perfect fluid energies were derived by using Noether's theorem and the Lagrangian which exists for the adiabatic perfect fluid perturbation equations.

(b) *The Conditions for Positive Energy*

The energy functional E defined in Eqs. (44) and (45) is a complicated quadratic form in the perturbation variables (δn, $\delta\rho$, δu^a, $\delta\tau$, $\delta\tau^{ab}$, δq^a). Our aim in this section is to determine the conditions under which this functional is positive. To accomplish this, we rewrite E in a factored form which makes the conditions for positivity evident.

Let the vector n^a be the future directed unit normal to the spacelike surface Σ upon which E is defined. Associated with n^a are a number of useful tensors. The vector λ^a is the velocity of observers moving along n^a relative to the fluid:

$$\lambda^a = (\delta^a_b + u^a u_b)\, n^b / u_c n^c. \quad (47)$$

It is straightforward to show that the norm, $\lambda^2 = \lambda_a \lambda^a$, has the following bounds,

$$0 \leqslant \lambda^2 \leqslant 1, \tag{48}$$

no matter which spacelike surface is used. The projection tensor γ_{ab}, defined by

$$\gamma_{ab} = g_{ab} + u_a u_b - \lambda^{-2} \lambda_a \lambda_b, \tag{49}$$

will also be useful.

The expression for the energy E can be rewritten in terms of an energy density e, defined by

$$e = TE^a n_a / u^b n_b. \tag{50}$$

In terms of this density, the energy E has the form:

$$E(\Sigma) = \int_\Sigma e \frac{u^a}{T} d^3 x_a. \tag{51}$$

The energy E will be positive for all possible values of the perturbation functions if and only if the density e is also a positive functional of the perturbation functions at each point in the fluid. It would be trivial to determine the conditions under which e was positive, if one could represent e in the following factored form:

$$e = \frac{1}{2} \sum_{A=1}^{N} \Omega_A (\delta Z_A)^2. \tag{52}$$

In this expression the δZ_A represent certain linearly independent combinations of the perturbation functions, and the Ω_A are certain functions of the thermodynamic variables. The condition that e be positive would then be equivalent to the condition that each of the Ω_A be positive. It is straightforward (but tedious) to show that the energy density e does indeed have this factored form when the Ω_A and δZ_A are defined as follows:

$$\Omega_1 = (\rho + p)^{-1} \left(\frac{\partial p}{\partial \rho}\right)_s, \tag{53}$$

$$\Omega_2 = (\rho + p)^{-1} \left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta, \tag{54}$$

$$\Omega_3 = (\rho + p) \left[1 - \lambda^2 \left(\frac{\partial p}{\partial \rho}\right)_s\right] - \left[\frac{1}{\beta_0} + \frac{2}{3\beta_2} + \frac{K^2}{\Omega_6}\right] \lambda^2, \tag{55}$$

$$\Omega_4 = (\rho + p) - \frac{2\beta_2 + (\beta_1 + 2\alpha_1)\lambda^2}{2\beta_1\beta_2 - \alpha_1^2 \lambda^2}, \tag{56}$$

$$\Omega_5 = \beta_0, \tag{57}$$

$$\Omega_6 = \frac{\beta_1}{\lambda^2} - \left[\frac{\alpha_0^2}{\beta_0} + \frac{2\alpha_1^2}{3\beta_2} + \frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_n\right], \tag{58}$$

$$\Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2}\lambda^2, \tag{59}$$

$$\Omega_8 = \beta_2, \tag{60}$$

$$\delta Z_1 = \delta p + (\rho + p)\left(\frac{\partial p}{\partial \rho}\right)_s \left[\lambda_a \delta u^a + \frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_s \lambda_a \delta q^a\right], \tag{61}$$

$$\delta Z_2 = \delta s + \frac{1}{T}(\rho + p)\left(\frac{\partial T}{\partial \rho}\right)_p \left(\frac{\partial s}{\partial p}\right)_\Theta \lambda_a \delta q^a, \tag{62}$$

$$\delta Z_3 = \lambda^{-1}\lambda_a \delta u^a, \tag{63}$$

$$\delta Z_4^a = \gamma_b^a \delta u^b, \tag{64}$$

$$\delta Z_5 = \delta\tau + \frac{1}{\beta_0}\lambda_a \delta u^a - \frac{\alpha_0}{\beta_0}\lambda_a \delta q^a, \tag{65}$$

$$\delta Z_6 = \lambda_a \delta q^a + \frac{K}{\Omega_6}\lambda_a \delta u^a, \tag{66}$$

$$\delta Z_7^a = \gamma_b^a \delta q^b + \frac{2\beta_2 + \alpha_1\lambda^2}{2\beta_1\beta_2 - \alpha_1^2\lambda^2}\gamma_b^a \delta u^b, \tag{67}$$

$$\delta Z_8^{ab} = \delta\tau^{ab} + \frac{1}{\beta_2}\langle\lambda^a \delta u^b\rangle - \frac{\alpha_1}{\beta_2}\langle\lambda^a \delta q^b\rangle. \tag{68}$$

In these equations the expression K is defined as

$$K = \frac{1}{\lambda^2} + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\left(\frac{\partial T}{\partial n}\right)_s. \tag{69}$$

The linear independence of the δZ_A is easily verified.

The positivity of the energy functional E is therefore equivalent to the positivity of the eight coefficients Ω_A:

$$\Omega_A \geqslant 0. \tag{70}$$

A number of these coefficients depend on the choice of spacelike surface Σ because of their dependence on the parameter λ^2. There is enough freedom in the choice of surface at any given point in the star, so that the parameter λ^2 can take on any value between zero and one. Thus the positivity of the Ω_A must be imposed for all values of λ^2 between zero and one to assure the positivity of the energy on every possible spacelike surface. The analysis given in Section IV reveals that the most restrictive of these conditions occurs when $\lambda^2 = 1$; thus the conditions $\Omega_A(\lambda^2 = 1) > 0$ imply $\Omega_A > 0$ for all $\lambda^2 < 1$.

The thermodynamic constraints which arise from the positivity of the Ω_A are in many cases quite complex. We will explore a number of the consequences of these conditions in the remainder of the paper. It is appropriate to comment briefly, however, on some of their more obvious features here. The square of the adiabatic sound speed, $(\partial p/\partial \rho)_s$, is bounded between zero and one (the speed of light) as a consequence of the positivity of Ω_1 and Ω_3. The positivity of Ω_2 is equivalent to the relativistic Schwarzschild condition for stability against convection [9] in these equilibrium models. The three conditions Ω_5, Ω_7, and Ω_8 guarantee that the three new thermodynamic potentials β_0, β_1, and β_2 must be positive. This confirms the expectation that the value of the physical entropy density is smaller than the equilibrium value sn [1]. These potentials (β_0, β_1, and $(3/2)\beta_2$) are also bounded below by $(\rho + p)^{-1}$ as a consequence of the positivity of Ω_3. In later sections we will explore more fully the consequences and implications of the positivity of the Ω_A.

(c) *Thermodynamic Inequalities*

The purpose of this section is to explore in some detail the thermodynamics of a relativistic fluid. We are particularly interested in determining the set of thermodynamic derivatives which have definite sign as a consequence of stability. We also catalogue here the Maxwell relations for these thermodynamic variables.

Two of the conditions for the positivity of the energy functional (which we derived in the previous section) are the simple thermodynamic inequalities:

$$\left(\frac{\partial p}{\partial \rho}\right)_s \geqslant 0, \tag{71}$$

$$\left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta \geqslant 0. \tag{72}$$

There are in addition six thermodynamic derivatives which have definite sign as a consequence of the first law of thermodynamics. These six derivatives are given by:

$$\left(\frac{\partial \rho}{\partial s}\right)_n = nT \geqslant 0, \tag{73}$$

$$\left(\frac{\partial \rho}{\partial n}\right)_s = \frac{\rho + p}{n} \geqslant 0, \tag{74}$$

$$\left(\frac{\partial s}{\partial n}\right)_\rho = -\frac{\rho + p}{n^2 T} \leqslant 0, \tag{75}$$

$$\left(\frac{\partial p}{\partial \Theta}\right)_T = nT \geqslant 0, \tag{76}$$

$$\left(\frac{\partial p}{\partial T}\right)_\Theta = \frac{\rho + p}{T} \geqslant 0, \tag{77}$$

$$\left(\frac{\partial \Theta}{\partial T}\right)_p = -\frac{\rho + p}{nT^2} \leqslant 0, \tag{78}$$

(see Eqs. (7), (8), and (34)).

There are 64 different partial derivatives which can be formed from the fundamental set of variables ρ, n, s, p, T, Θ. These 64 derivatives are not all independent, but are related in complex ways by the Maxwell relations and by the chain rule. Our aim is to use the relations which exist among the partial derivatives to determine which additional derivatives must have definite sign as a consequence of Eqs. (71)–(78). It will be helpful to first establish the set of Maxwell relations for these variables.

The Maxwell relations are the integrability conditions for the first law of thermodynamics. These relations are often useful tools for manipulating and simplifying thermodynamic expressions. We list these relationships here for future reference:

$$\left(\frac{\partial \rho}{\partial s}\right)_n = n^2 \left(\frac{\partial T}{\partial n}\right)_s \tag{79}$$

$$\left(\frac{\partial \rho}{\partial \Theta}\right)_T = T^2 \left(\frac{\partial n}{\partial T}\right)_\Theta \tag{80}$$

$$\left(\frac{\partial n}{\partial T}\right)_p = n^2 \left(\frac{\partial s}{\partial p}\right)_T \tag{81}$$

$$\left(\frac{\partial T}{\partial n}\right)_\rho = T^2 \left(\frac{\partial \Theta}{\partial \rho}\right)_n \tag{82}$$

$$\left(\frac{\partial p}{\partial s}\right)_\rho = -n^2 T^2 \left(\frac{\partial \Theta}{\partial \rho}\right)_s \tag{83}$$

$$\left(\frac{\partial \rho}{\partial \Theta}\right)_p = -n^2 T^2 \left(\frac{\partial s}{\partial p}\right)_\Theta \tag{84}$$

$$\left(\frac{\partial n}{\partial s}\right)_p = -n^2 \left(\frac{\partial T}{\partial p}\right)_s \tag{85}$$

$$\left(\frac{\partial T}{\partial \Theta}\right)_\rho = -T^2 \left(\frac{\partial n}{\partial p}\right)_\Theta \tag{86}$$

$$\left(\frac{\partial p}{\partial T}\right)_n = -n^2 \left(\frac{\partial s}{\partial n}\right)_T \tag{87}$$

$$\left(\frac{\partial \rho}{\partial n}\right)_T = -T^2 \left(\frac{\partial \Theta}{\partial T}\right)_n \tag{88}$$

$$\left(\frac{\partial \rho}{\partial s}\right)_p = n^2T^2\left(\frac{\partial \Theta}{\partial p}\right)_s \tag{89}$$

$$\left(\frac{\partial p}{\partial \Theta}\right)_\rho = n^2T^2\left(\frac{\partial s}{\partial \rho}\right)_\Theta. \tag{90}$$

The first six of these equations result directly from the integrability conditions necessary for the first law. For example, Eq. (79) results from the identity $\partial^2\rho/\partial n\partial s = \partial^2\rho/\partial s\partial n$ and the expressions in Eqs. (73) and (74). The remaining six equations are alternate forms which were obtained by applying the chain rule to the derivatives contained in the first six expressions.

We are now prepared to determine the additional thermodynamic derivatives which have definite sign as a consequence of Eqs. (71)–(78). There are a total of eleven additional partial derivatives with definite sign. These include the specific heats $T(\partial s/\partial T)_p$ and $T(\partial s/\partial T)_n$, the compressibilities $n^{-1}(\partial n/\partial p)_s$ and $n^{-1}(\partial n/\partial p)_T$, as well as a number of less familiar derivatives. For each of these eleven derivatives, we use the chain rule and Maxwell relations to express the derivative as a sum of definite signed quantities:

$$\left(\frac{\partial p}{\partial n}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s\left(\frac{\partial \rho}{\partial n}\right)_s \geqslant 0, \tag{91}$$

$$\left(\frac{\partial \rho}{\partial p}\right)_\Theta = \left(\frac{\partial \rho}{\partial p}\right)_s + \left(\frac{\partial \rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_\Theta\left(\frac{\partial s}{\partial p}\right)_\Theta^2 \geqslant 0, \tag{92}$$

$$\left(\frac{\partial \rho}{\partial T}\right)_\Theta = \left(\frac{\partial \rho}{\partial p}\right)_\Theta\left(\frac{\partial p}{\partial T}\right)_\Theta \geqslant 0, \tag{93}$$

$$\left(\frac{\partial \Theta}{\partial s}\right)_p = -\frac{1}{n^2T^2}\left(\frac{\partial \rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_\Theta \leqslant 0, \tag{94}$$

$$\left(\frac{\partial T}{\partial s}\right)_p = \left(\frac{\partial T}{\partial \Theta}\right)_p\left(\frac{\partial \Theta}{\partial s}\right)_p \geqslant 0, \tag{95}$$

$$\left(\frac{\partial T}{\partial s}\right)_n = \left(\frac{\partial T}{\partial s}\right)_p + n^2\left(\frac{\partial n}{\partial p}\right)_s\left(\frac{\partial T}{\partial n}\right)_s^2 \geqslant 0, \tag{96}$$

$$\left(\frac{\partial T}{\partial p}\right)_n = \left(\frac{\partial T}{\partial s}\right)_n\left(\frac{\partial s}{\partial p}\right)_n \geqslant 0, \tag{97}$$

$$\left(\frac{\partial \Theta}{\partial s}\right)_\rho = \left(\frac{\partial \Theta}{\partial s}\right)_p - \frac{1}{n^2T^2}\left(\frac{\partial \rho}{\partial p}\right)_s\left(\frac{\partial p}{\partial s}\right)_\rho^2 \leqslant 0, \tag{98}$$

$$\left(\frac{\partial \Theta}{\partial n}\right)_\rho = \left(\frac{\partial s}{\partial n}\right)_\rho\left(\frac{\partial \Theta}{\partial s}\right)_\rho \geqslant 0, \tag{99}$$

$$\left(\frac{\partial n}{\partial p}\right)_T = \left(\frac{\partial n}{\partial p}\right)_s + \frac{1}{n^2}\left(\frac{\partial s}{\partial T}\right)_p \left(\frac{\partial n}{\partial s}\right)_p^2 \geqslant 0, \tag{100}$$

$$\left(\frac{\partial \Theta}{\partial n}\right)_T = \left(\frac{\partial \Theta}{\partial p}\right)_T \left(\frac{\partial p}{\partial n}\right)_T \geqslant 0. \tag{101}$$

Thus there are a total of eighteen partial derivatives with definite sign out of the possible set of 64. To conclude that this is the complete set of definite signed derivatives is a more difficult proposition. The outlines of the following argument were suggested by Robert Geroch. There are only two independent thermodynamic variables; the remaining variables can be determined from the two independent ones by the equation of state and first law of thermodynamics (see Eqs. (6)–(8), and (33), for example). The gradients of the thermodynamic variables are therefore elements of a two dimensional vector space. Figure 1 shows possible configurations of these vectors. These diagrams contain within them information about the signs of all the possible partial derivatives. Consider three thermodynamic variables W, X, Y. Their gradients are related by the expression:

$$\mathbf{d}W = \left(\frac{\partial W}{\partial X}\right)_Y \mathbf{d}X + \left(\frac{\partial W}{\partial Y}\right)_X \mathbf{d}Y. \tag{102}$$

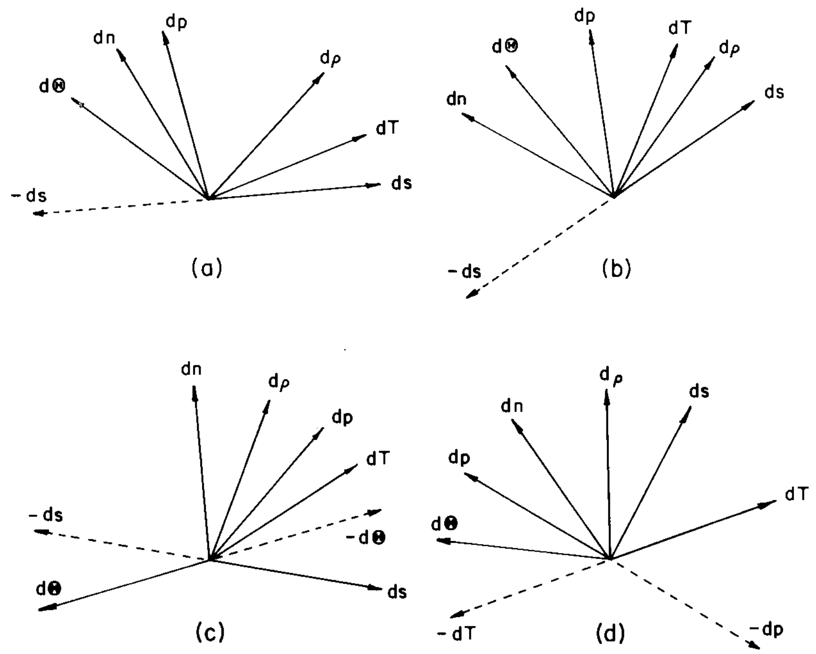

FIG. 1. These vector diagrams represent all possible thermodynamic systems which are consistent with the stability conditions, the Maxwell relations, and the first law of thermodynamics.

If we knew for some reason that $(\partial W/\partial X)_Y \geqslant 0$, then it would follow that the vectors $\mathbf{d}W$ and $\mathbf{d}X$ must lie on the same side of the vector $\mathbf{d}Y$. In other words, the vector $\mathbf{d}Y$ defines a direction which divides the two dimensional space into two half planes; when $(\partial W/\partial X)_Y \geqslant 0$ then the two vectors $\mathbf{d}W$ and $\mathbf{d}X$ must lie in the same half plane. The forms of the first law of thermodynamics are simply constraints, then, among these gradients:

$$\mathbf{d}\rho = nT\,\mathbf{d}s + \frac{\rho + p}{n}\,\mathbf{d}n, \tag{103}$$

$$\mathbf{d}p = nT\,\mathbf{d}\Theta + \frac{\rho + p}{T}\,\mathbf{d}T. \tag{104}$$

These equations limit, therefore, the possible set of vector diagrams which can represent real thermodynamic systems. We can add to these constraints the Maxwell relations (Eqs. (79)–(90)) and the positive energy conditions (Eqs. (71)–(72)). We have written down every distinct vector diagram which is consistent with these conditions. The four resulting diagrams are given in Fig. 1. Any one of these four diagrams represents a possible thermodynamics, since all the constraints were imposed on each diagram.

We can read the signs of all the partial derivatives from each of the diagrams in Fig. 1. For example, $(\partial\theta/\partial n)_s \geqslant 0$ in diagram (a), since the vectors $\mathbf{d}\theta$ and $\mathbf{d}n$ are in the same half plane with respect to the vector $\mathbf{d}s$. It is straightforward then to check that the only partial derivatives which have the same sign in all four possible diagrams are the eighteen derivatives already discussed. This establishes that we have identified all the definite signed partial derivatives in this system.

IV. Causality

The motivation for extending the standard Eckart theory of relativistic fluid mechanics was the desire to obtain a theory in which all perturbations propagated causally. Has the Müller–Israel procedure succeeded in producing a truly causal theory? To investigate this question, Stewart [4], and Israel and Stewart [5] have computed the characteristic velocities for (essentially) the system of perturbation Eqs. (35)–(39). The expressions which they give for these characteristic velocities are extremely complicated functions of the equilibrium thermodynamic variables and the coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1$, and β_2. It is not possible to determine by inspection whether or not these velocities are less than the speed of light, or even if they are all real. Israel and Stewart carried their investigations further by limiting consideration to the dilute gas limit where relativistic kinetic theory could be used to obtain explicit expressions for the thermodynamic variables as well as the α's and β's. In this limit, they were able to conclude that the characteristic velocities for these equations were less than the speed of light.

In this section we investigate the causal properties of the perturbations of a Müller–Israel fluid under a far wider range of circumstances than those considered previously [4]–[5]. We present in Section IV(a) our computation of the characteristic velocities for the system of perturbation Eqs. (35)–(39). We find that perturbations can propagate with three distinct characteristic velocities in these fluids. In Section IV(b) we investigate the conditions under which these characteristic velocities are causal. We find that any fluid which has stable equilibrium configurations (i.e., which satisfy Eq. (70)) will necessarily have real characteristic velocities which are less than the speed of light. We also show that the perturbation equations are a symmetric hyperbolic system. The existence and uniqueness of solutions to the Cauchy problem for the perturbation equations is consequently guaranteed. Finally, we show in Section IV(c) that a fluid which has causal characteristic velocities and whose propagation equations form a hyperbolic system (in a particular sense defined later) must in fact also satisfy the stability criteria. Thus, causality and hyperbolicity are essentially equivalent to stability in these fluids.

(a) *The Characteristic Velocities and Hyperbolicity*

The system of equations for the perturbations of a Müller–Israel fluid, Eqs. (35)–(39), have the following general form:

$$a^{A\,d}_{B}\nabla_d Y^B + b^A_B Y^B = 0. \tag{105}$$

In this expression, Y^B represents the list of fourteen fields which make up the perturbations to these fluids, i.e., δn, $\delta\rho$, δu^a, $\delta\tau$, δq^a, $\delta\tau^{ab}$. The index B runs over these fourteen fields, while the index A runs over the fourteen equations of motion for the perturbation variables. The matrices a^{Ad}_B and b^A_B are functions of the unperturbed equilibrium fluid configurations.

A three-dimensional surface is called a characteristic surface for these equations if the initial values of the fields Y^B cannot be freely specified on that surface. The level surfaces of a function ϕ which satisfies the equation:

$$\det(a^{A\,d}_{B}\nabla_d\phi) = 0 \tag{106}$$

is such a characteristic surface (see, e.g., Courant and Hilbert [10], p. 170). Discontinuities in the initial data propagate along these characteristic surfaces. The characteristic velocities are the slopes of these characteristic surfaces.

To solve the characteristic Eq. (106), we make a particular choice of coordinates. Consider coordinates x^0, x^1, x^2, and x^3 chosen so that at some point r in the fluid they are orthonormal and comoving. Thus they satisfy the following conditions at r:

$$g^{ab}\partial_a\partial_b = -(\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2, \tag{107}$$

$$u^a\partial_a = \partial_0, \tag{108}$$

and

$$\phi = \phi(x^0, x^1). \tag{109}$$

In this coordinate system, Eq. (106) has the simplified form:

$$0 = \det(v a_B^{A0} - a_B^{A1}), \tag{110}$$

where v is the characteristic velocity defined by

$$v = -\partial_0 \phi / \partial_1 \phi. \tag{111}$$

These characteristic velocities play the role of eigenvalues in Eq. (110). The form of the characteristic matrix simplifies further when one chooses the following set of fourteen perturbation variables:

$$Y^B = \{T\delta\theta, \delta T/T, \delta\tau, \delta u^1, \delta q^1, \delta\tau^{11}, \delta u^2, \delta q^2, \delta\tau^{21}, \delta u^3, \delta q^3, \delta\tau^{31}, \delta\tau^{22} - \delta\tau^{33}, \delta\tau^{23}\}. \tag{112}$$

With this choice of variables the characteristic matrix block diagonalizes as follows:

$$v\mathbf{a}^0 - \mathbf{a}^1 = \begin{pmatrix} \mathbf{Q} & 0 & 0 & 0 \\ 0 & \mathbf{R} & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & \mathbf{S} \end{pmatrix}. \tag{113}$$

The matrices **Q**, **R**, and **S** are defined as follows:

$$\mathbf{Q} = \begin{bmatrix} \frac{v}{T}\left(\frac{\partial n}{\partial \Theta}\right)_T & \frac{v}{T}\left(\frac{\partial \rho}{\partial \Theta}\right)_T & 0 & -n & 0 & 0 \\ \frac{v}{T}\left(\frac{\partial \rho}{\partial \Theta}\right)_T & vT\left(\frac{\partial \rho}{\partial T}\right)_\Theta & 0 & -(\rho + p) & -1 & 0 \\ 0 & 0 & \beta_0 v & -1 & \alpha_0 & 0 \\ -n & -(\rho + p) & -1 & v(\rho + p) & v & -1 \\ 0 & -1 & \alpha_0 & v & \beta_1 v & \alpha_1 \\ 0 & 0 & 0 & -1 & \alpha_1 & \frac{3}{2}\beta_2 v \end{bmatrix}, \tag{114}$$

$$\mathbf{R} = \begin{pmatrix} v(\rho + p) & v & -1 \\ v & \beta_1 v & \alpha_1 \\ -1 & \alpha_1 & 2\beta_2 v \end{pmatrix}, \tag{115}$$

and

$$\mathbf{S} = \begin{pmatrix} 2\beta_2 v & 0 \\ 0 & \beta_2 v \end{pmatrix}. \tag{116}$$

The determinant of the characteristic matrix is simply the product of the determinants of these blocks:

$$\det(v\mathbf{a}^0 - \mathbf{a}^1) = \det \mathbf{Q}(\det \mathbf{R})^2 \det \mathbf{S}. \tag{117}$$

The expressions for the determinants of these blocks are given by

$$\det \mathbf{Q} = \frac{3}{2} v^2 [Av^4 + Bv^2 + C] \left[\left(\frac{\partial n}{\partial \Theta}\right)_T \left(\frac{\partial \rho}{\partial T}\right)_\Theta - \left(\frac{\partial n}{\partial T}\right)_\Theta \left(\frac{\partial \rho}{\partial \theta}\right)_T \right], \tag{118}$$

$$\det \mathbf{R} = v\{2\beta_2[\beta_1(\rho + p) - 1]\, v^2 - [(\rho + p)\,\alpha_1^2 + 2\alpha_1 + \beta_1]\}, \tag{119}$$

$$\det \mathbf{S} = 2(\beta_2 v)^2. \tag{120}$$

The functions A, B, and C which appear in Eq. (118) are defined by

$$A = \beta_0\beta_2[\beta_1(\rho + p) - 1], \tag{121}$$

$$B = -(\rho + p)\, D - \beta_1 E - 2F, \tag{122}$$

$$C = (DE - F^2)/\beta_0\beta_2; \tag{123}$$

and D, E, and F are defined by

$$D = \beta_0\beta_2 \left[\frac{\alpha_0^2}{\beta_0} + \frac{2\alpha_1^2}{3\beta_2} + \frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_n\right], \tag{124}$$

$$E = \beta_0\beta_2 \left[(\rho + p)\left(\frac{\partial p}{\partial \rho}\right)_s + \frac{1}{\beta_0} + \frac{2}{3\beta_2}\right], \tag{125}$$

$$F = \beta_0\beta_2 \left[\frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\left(\frac{\partial T}{\partial n}\right)_s\right]. \tag{126}$$

The roots of the characteristic Eq. (110) are simply the collection of roots obtained by setting the determinants of each of the matrix blocks individually to zero. The two characteristic velocities corresponding to the zeros of det **S** are both zero. The matrix **R** has one characteristic velocity which is zero, and two nonvanishing characteristic velocities given by:

$$v_T^2 = \frac{(\rho + p)\,\alpha_1^2 + 2\alpha_1 + \beta_1}{2\beta_2[\beta_1(\rho + p) - 1]}. \tag{127}$$

These roots are referred to as transverse velocities, since the matrix **R** involves the components of the perturbation variables which are tangent to the characteristic surfaces (and therefore transverse to the direction of propagation of a disturbance). The matrix **Q** has two characteristic velocities which are zero, and four nonzero characteristic velocities which are the roots of the quartic equation,

$$Av_L^4 + Bv_L^2 + C = 0. \tag{128}$$

The roots of this equation are referred to as the longitudinal velocities. We expect one pair of these velocities to correspond roughly to the propagation of sound in these fluids, while the other pair of velocities correspond roughly to the propagation of temperature perturbations (second sound).

The system of equations for the perturbations of a Müller–Israel fluid (Eq. (105)) is called a symmetric system because the matrices $\mathbf{a}^d$ are all symmetric matrices (see Eq. (113)). Such a system is further known as a symmetric hyperbolic system (see Courant and Hilbert [10], p. 593) if some linear combination of the $\mathbf{a}^d$ is (positive) definite. Symmetric hyperbolic systems have well-posed initial value (Cauchy) problems. The usual definition of hyperbolic systems imposes conditions (such as reality) on the characteristic velocities. The usual definition does not allow, however, the possibility of multiple characteristics having the same velocities. Since this situation always occurs in the equations for the fluid perturbations, we must use the definition of hyperbolicity given above.

It is interesting to investigate the conditions under which the equations for the perturbations (Eq. (105)) are a symmetric hyperbolic system. To do this we must determine the conditions under which there exists a (positive) definite linear combination of the matrices $\mathbf{a}^d$. Any positive definite matrix has the property that any square submatrix, whose diagonal coincides with the original diagonal, is also positive definite. Thus, in particular, the diagonal elements themselves must be positive definite. Only the matrix $\mathbf{a}^0$ has nonzero diagonal elements; so these elements must be positive definite if any positive definite combination of the $\mathbf{a}^d$ exists. These diagonal elements will be positive whenever the following set of inequalities are satisfied:

$$\left(\frac{\partial n}{\partial \Theta}\right)_T > 0, \tag{129}$$

$$\left(\frac{\partial \rho}{\partial T}\right)_\Theta > 0, \tag{130}$$

$$\beta_0 > 0, \tag{131}$$

$$\beta_1 > 0, \tag{132}$$

$$\beta_2 > 0. \tag{133}$$

The matrix $\mathbf{a}^0$ also includes some 2×2 diagonal blocks. The positions of these 2×2 blocks in $\mathbf{a}^0$ are such that no other matrix has nonzero elements in those locations. Thus each of these 2×2 matrices must be positive definite if any positive definite linear combination of the $\mathbf{a}^d$ is to exist. These matrix blocks will be positive definite if Eqs. (129)–(132) are satisfied and if their determinants are positive:

$$\beta_1(\rho + p) - 1 > 0, \tag{134}$$

$$\left(\frac{\partial \rho}{\partial T}\right)_n > 0. \tag{135}$$

The inequalities given in Eqs. (129)–(135) are necessary for the existence of a positive definite linear combination of the $\mathbf{a}^d$, as we have argued. These conditions are also sufficient for the existence of such a linear combination. When Eqs. (129)–(135) are satisfied, then the matrix $\mathbf{a}^0$ is itself positive definite. We conclude, therefore, that equations (105) for the perturbations of a Müller–Israel fluid form a symmetric hyperbolic system if Eqs. (129)–(135) are satisfied. These conditions are necessary and sufficient for the representation of the perturbation equations outlined in Eqs. (112)–(116) to be symmetric hyperbolic. By choosing other dependent variables and taking linear combinations of the given equations, it is possible to produce a symmetric hyperbolic form of the equations under somewhat weaker conditions (the reality of the characteristic velocities) than Eqs. (129)–(135). In the following sections we show that the stability of a Müller–Israel fluid is equivalent to the requirement that the characteristic velocities be subluminal and the requirement that the equations be a symmetric hyperbolic system in the somewhat restrictive sense of Eqs. (129)–(135).

(b) *Stability Implies Causality and Hyperbolicity*

We are now prepared to show that the perturbations of a Müller–Israel fluid propagate causally by a hyperbolic system of equations in any fluid which satisfies the conditions for stability, Eq. (70). We assume that the stability inequalities are strictly satisfied: $\Omega_A > 0$. We begin by considering the velocity of the transverse modes defined in Eq. (127). It will be helpful to recall from Eqs. (53)–(60) some special cases of the stability criteria $\Omega_A(\lambda) > 0$:

$$\Omega_4(0)\,\Omega_7(0) = \beta_1(\rho + p) - 1 > 0, \tag{136}$$

$$\Omega_4(1) = (\rho + p) - \frac{2(\beta_2 + \alpha_1) + \beta_1}{2\beta_1\beta_2 - \alpha_1^2} > 0, \tag{137}$$

$$\Omega_7(1) = \beta_1 - \frac{\alpha_1^2}{2\beta_2} > 0, \tag{138}$$

$$\Omega_8 = \beta_2 > 0. \tag{139}$$

Using these expressions, it is straightforward to verify that

$$1 - v_T^2 = \frac{\Omega_4(1)\,\Omega_7(1)}{\Omega_4(0)\,\Omega_7(0)} > 0, \tag{140}$$

and in addition that:

$$v_T^2 = \frac{1}{2\beta_2}\left[\frac{\rho + p}{\Omega_4(0)\,\Omega_7(0)}\left(\alpha_1 + \frac{1}{\rho + p}\right)^2 + \frac{1}{\rho + p}\right] > 0. \tag{141}$$

These two inequalities, (140) and (141), place the desired upper and lower bounds on the transverse velocity:

$$0 < v_T^2 < 1. \tag{142}$$

Consequently, this velocity is real and less than the speed of light (one in our units).

The situation is more complicated for the longitudinal velocities. The squares of these velocities $x = v_L^2$ are the roots of the quadratic polynomial,

$$P(x) = Ax^2 + Bx + C, \tag{143}$$

where A, B, and C were defined in Eqs. (121)–(123). The squares of these velocities will be real, therefore, if and only if the discriminant of this polynomial is non-negative. The following expression for this discriminant can be verified using Eqs. (121)–(126):

$$B^2 - 4AC = \frac{1}{\beta_1(\rho+p)} \{[(\rho+p)\,D + \beta_1 E + 2\beta_1(\rho+p)\,F]^2 + [\beta_1(\rho+p) - 1][(\rho+p)\,D - \beta_1 E]^2\} \geqslant 0. \tag{144}$$

Thus, the squares of the longitudinal velocities are real if the stability conditions are satisfied.

We will determine the locations of the zeros of the quadratic polynomial $P(x)$ by a geometrical argument. The coefficient of the quadratic term, A, is positive when the stability conditions are satisfied. Therefore, the parabola $P(x)$ is positive for sufficiently large x. We will demonstrate that the zeros of $P(x)$ must be larger than $x = 0$ by showing that $P(0) > 0$ and that $dP(0)/dx < 0$. Similarly we will demonstrate that those zeros are smaller than $x = 1$ by showing that $P(1) > 0$ and that $dP(1)/dx > 0$. The following expressions can be verified from the definitions of A, B, and C:

$$P(0) = C = \frac{2}{3}(\alpha_1 - \alpha_0)^2 + \frac{2}{3}\beta_0 n \left(\frac{\partial p}{\partial n}\right)_s \left[\alpha_1 + \frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_s\right]^2 + \beta_0\beta_2 \frac{1}{T^2}\left(\frac{\partial T}{\partial s}\right)_p \left(\frac{\partial p}{\partial n}\right)_s + \beta_2 n \left(\frac{\partial p}{\partial n}\right)_s \left[\alpha_0 + \frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_s\right]^2 + \left[\beta_2 + \frac{2}{3}\beta_0\right]\frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_p > 0, \tag{145}$$

$$\frac{dP(0)}{dx} = B = -\beta_2(\rho+p)\left[\alpha_0 + \frac{1}{\rho+p}\right]^2 - \frac{2}{3}\beta_0(\rho+p)\left[\alpha_1 + \frac{1}{\rho+p}\right]^2 - \left[\beta_1 - \frac{1}{\rho+p}\right]\left[\beta_2 + \frac{2}{3}\beta_0 + \beta_0\beta_2(\rho+p)\left(\frac{\partial p}{\partial \rho}\right)_s\right] + \beta_0\beta_2\left(\frac{\partial \Theta}{\partial s}\right)_\rho < 0, \tag{146}$$

$$P(1) = A + B + C = \beta_0 \beta_2 \Omega_3(1)\, \Omega_6(1) > 0, \tag{147}$$

$$\frac{dP(1)}{dx} = 2A + B = \beta_0 \beta_1 \beta_2 \Omega_3(1) + \frac{\beta_0 \beta_2}{\beta_1 \Omega_6(1)} [\beta_1 K(1) - \Omega_6(1)]^2 + \frac{\beta_0 \beta_2}{\beta_1} [\beta_1(\rho + p) - 1]\, \Omega_6(1) > 0. \tag{148}$$

The function $K(\lambda)$ which appears in these expressions is defined in Eq. (69). The indicated inequalities are straightforward consequences of the stability conditions, Eq. (70), and the associated thermodynamic inequalities Eqs. (91)–(101). Therefore the longitudinal velocities v_L^2, the roots of $P(x)$, must lie in the range:

$$0 < v_L^2 < 1. \tag{149}$$

We have shown that all of the characteristic velocities are real and less than the speed of light. The perturbations of a Müller–Israel fluid propagate causally therefore in any fluid having stable equilibrium configurations.

The system of equations for the perturbations of a Müller–Israel fluid is a symmetric hyperbolic system when the conditions given in Eqs. (129)–(135) are satisfied. These conditions are a straightforward consequence of the stability conditions, in particular Eqs. (57), (93), (97), (101), (136), and (139). Therefore, we have shown that the perturbations of a Müller–Israel fluid propagate causally from a well-posed initial value problem as long as the stability criteria ($\Omega_A > 0$) are satisfied for this fluid.

(c) *Causality and Hyperbolicity Imply Stability*

In this final section we show that if the perturbations for a Müller–Israel fluid have characteristic velocities which are less than the speed of light and if the equations which govern the evolution of these perturbations are a symmetric hyperbolic system (in the somewhat restrictive sense of the conditions outlined in Eqs. (129)–(135)), then the fluid must also satisfy the stability conditions (Eq. (70)).

In Section IV(b) we showed that the characteristic velocities will be less than the speed of light as long as the following inequalities are satisfied:

$$1 - v_T^2 = \frac{\Omega_4(1)\, \Omega_7(1)}{\Omega_4(0)\, \Omega_7(0)} > 0, \tag{150}$$

$$P(1) = \beta_0 \beta_2 \Omega_3(1)\, \Omega_6(1) > 0, \tag{151}$$

and

$$\frac{dP(1)}{dx} = \beta_0 \beta_1 \beta_2 \Omega_3(1) + \frac{\beta_0 \beta_2}{\beta_1 \Omega_6(1)} [\beta_1 K(1) - \Omega_6(1)]^2 + \frac{\beta_0 \beta_2}{\beta_1} [\beta_1(\rho + p) - 1]\, \Omega_6(1) > 0. \tag{152}$$

We also require that the perturbation equations be a hyperbolic system (in the somewhat restrictive sense discussed above) by requiring that Eqs. (129)–(135) be satisfied. From this set of conditions (Eqs. (129)–(135) and (150)–(152)), we now show that the stability criteria $\Omega_A > 0$ follow.

Let us begin with the stability criteria (Ω_1 and Ω_2) which involve the thermodynamic derivatives. We wish to show that these expressions are positive whenever the conditions for hyperbolicity are imposed: i.e., that $(\partial n/\partial \Theta)_T$, $(\partial \rho/\partial T)_\Theta$, and $(\partial \rho/\partial T)_n$ are positive from Eqs. (129)–(135). The following identities verify the desired results:

$$\Omega_1^{-1} = (\rho + p)\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{1}{T}\left[n^2T^2\left(\frac{\partial \Theta}{\partial n}\right)_T + \frac{1}{n^2}\left(\frac{\partial p}{\partial s}\right)_n^2\left(\frac{\partial \rho}{\partial T}\right)_n\right] > 0, \tag{153}$$

$$\Omega_2 = (\rho + p)^{-1}\left(\frac{\partial \rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_\Theta = \frac{n^4T^2}{(\rho + p)}\left(\frac{\partial T}{\partial \rho}\right)_\Theta\left(\frac{\partial \rho}{\partial p}\right)_s$$
$$\times\left[\left(\frac{\partial \Theta}{\partial n}\right)_T + \frac{1}{T^2}\left(\frac{\partial p}{\partial n}\right)_T^2\left(\frac{\partial T}{\partial \rho}\right)_n\right] > 0. \tag{154}$$

The positivity of the stability conditions Ω_5 and Ω_8 are immediate consequences of the hyperbolicity of the perturbation equations [see Eqs. (131)–(133)]:

$$\Omega_5 = \beta_0 > 0, \tag{155}$$

$$\Omega_8 = \beta_2 > 0. \tag{156}$$

We turn next to the conditions which follow from the requirement that the longitudinal velocities v_L are causal: Eqs. (151) and (152). Given that β_0 and β_2 are positive, Eq. (151) implies that $\Omega_3(1)$ and $\Omega_6(1)$ are both nonzero and both have the same sign. This fact, the conditions on β_1 from Eqs. (132)–(134), and Eq. (152) imply that both $\Omega_3(1)$ and $\Omega_6(1)$ must be positive. These conditions are sufficient to guarantee the positivity of $\Omega_6(\lambda)$ for all relevant values of λ since

$$\Omega_6(\lambda) \geqslant \Omega_6(1) > 0 \tag{157}$$

for $\lambda^2 < 1$. The positivity of Ω_7 is also assured whenever Ω_6 is positive, and when Eqs. (153)–(156) are satisfied:

$$\Omega_7(\lambda) \geqslant \Omega_7(1) = \Omega_6(1) + \frac{\alpha_0^2}{\beta_0} + \frac{\alpha_1^2}{6\beta_2} + \frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_n > 0, \tag{158}$$

for $\lambda^2 < 1$.

The positivity of the final stability condition Ω_4 is a consequence of the causality of the transverse velocity v_T. The identity,

$$\Omega_4(0) = \frac{1}{\beta_1}\left[\beta_1(\rho + p) - 1\right] > 0, \tag{159}$$

verifies that $\Omega_4(0)$ must be positive when β_1 satisfies the conditions for hyperbolicity, Eqs. (132)–(134). The positivity of $\Omega_4(1)$ is then guaranteed by Eq. (158) and the causality condition Eq. (150).

The stability conditions $\Omega_3(\lambda)$ and $\Omega_4(\lambda)$ depend on λ in complicated ways. To determine the point on the interval $\lambda^2 < 1$ where these expressions are minimized, we compute the derivatives:

$$\begin{aligned}\frac{d\Omega_3}{d(\lambda^2)} = &-\frac{1}{\varepsilon_2}\left\{\left[\frac{\beta_1}{\beta_1-\lambda^2\varepsilon_2}\right]^2 - 1\right\}\left[\varepsilon_1+\frac{\varepsilon_2}{\beta_1}\right]^2 \\ &-\frac{1}{\beta_1^2}\left\{\frac{1}{\beta_0}(\alpha_0+\beta_1)^2+\frac{2}{3\beta_2}(\alpha_1+\beta_1)^2+\frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_p\right. \\ &\left.+(\rho+p)\left(\frac{\partial p}{\partial \rho}\right)_s\left[\beta_1-\frac{1}{n}\left(\frac{\partial p}{\partial \rho}\right)_n\left(\frac{\partial n}{\partial p}\right)_s\right]^2\right\} \leqslant 0,\end{aligned} \tag{160}$$

$$\frac{d\Omega_4}{d(\lambda^2)} = -2\beta_2\frac{(\alpha_1+\beta_1)^2}{(2\beta_1\beta_2-\alpha_1^2\lambda^2)^2} \leqslant 0. \tag{161}$$

In these expressions ε_1 and ε_2 are defined by:

$$\varepsilon_1 = K(\lambda) - \frac{1}{\lambda^2} = \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\left(\frac{\partial T}{\partial n}\right)_s, \tag{162}$$

$$\varepsilon_2 = \frac{\beta_1}{\lambda^2} - \Omega_6 = \frac{\alpha_0^2}{\beta_0} + \frac{2\alpha_1^2}{3\beta_2} + \frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_n \geqslant 0. \tag{163}$$

Therefore the minima of Ω_3 and Ω_4 occur at $\lambda^2 = 1$. Consequently, since Ω_3 and Ω_4 are positive for $\lambda^2 = 1$, they must also be positive for all $\lambda^2 < 1$:

$$\Omega_3(\lambda) \geqslant \Omega_3(1) > 0, \tag{164}$$

$$\Omega_4(\lambda) \geqslant \Omega_4(1) > 0. \tag{165}$$

We have shown, therefore, that all of the stability criteria $\Omega_A(\lambda) > 0$ are a consequence of the causality conditions and the condition that the perturbation equations are a hyperbolic system.

Appendix

The purpose of this appendix is to establish relationships between the stability of a Müller–Israel fluid and the positivity of the energy functional developed in Section III. We prove two results which establish these relationships. The first demonstrates that the positivity of the energy functional is a sufficient condition for stability.

PROPOSITION A. *The perturbations of a Müller–Israel fluid will not grow without bound (as measured by a square integral norm) if the conditions for the positivity of the energy, $\Omega_A > 0$, are satisfied.*

The second result falls somewhat short of demonstrating that the energy positivity conditions are necessary conditions for stability as well:

PROPOSITION B. *If the thermodynamic inequalities*

$$0 < \left(\frac{\partial p}{\partial \rho}\right)_s < 1, \tag{A1}$$

and

$$\left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta > 0 \tag{A2}$$

are satisfied, and if (at least) one of the conditions for the positivity of the energy is violated, then there exist perturbations which grow without bound.

We believe that the energy positivity conditions are in fact strictly necessary for stability. A somewhat nonrigorous proof of the general proposition can be given which is completely analogous to the proof given by Lindblom [11] for Newtonian dissipative fluids. We prefer to present here a more rigorous proof of the less general Proposition B. This result does demonstrate that the less familiar of the conditions $Q_A > 0$ are in fact necessary for stability when the familiar conditions (A1) and (A2) are satisfied.

Proof of Proposition A. Consider the linear transformation which relates the perturbation variables (δn, $\delta\rho$, δu^a, δq^a, $\delta\tau$, $\delta\tau^{ab}$) to the variables δZ_A (see Eqs. (61)–(68)). This transformation is one to one and onto. This follows from the fact that when all of the δZ_A are zero, then all of the perturbation variables (δn, $\delta\rho$, δu^a, δq^a, $\delta\tau$, $\delta\tau^{ab}$) must also be zero. The positivity of the energy conditions $Q_A > 0$, and the fact that the energy is a monotonically decreasing function of time, guarantee that the energy is bounded below by zero and above by the initial value of the energy. This guarantees the boundedness of the δZ_A (in terms of a square integral norm). Since the transformation between the δZ_A and the perturbation variables (δn, $\delta\rho$, δu^a, δq^a, $\delta\tau$, $\delta\tau^{ab}$) is one to one and onto, the boundedness of the δZ_A implies that the perturbation variables must also be bounded.

Proof of Proposition B. It is more convenient to demonstrate the contrapositive form of this proposition. Assume that the thermodynamic inequalities (A1) and (A2) are satisfied and that there are no solutions to the perturbation equations which grow without bound. We will show that this implies that the energy positivity conditions $\Omega_A > 0$ are necessarily satisfied. When the solutions to the perturbation equations are all bounded, then the evolution of the energy functional must also be bounded. The time derivative of the energy functional must evolve to zero in this case. The integral of the divergence of the energy current (see Eq. (46)) evolves to zero in this case also. This implies that the square integral norms of $\delta\tau$, δq^a, and $\delta\tau^{ab}$ evolve to zero also. When we evaluate the form of the energy in this limit, we find that

$$E(\Sigma) = \frac{1}{2}\int_{\Sigma}\Big\{(\rho + p)\,\gamma_{ab}\,\delta u^a \delta u^b + (\rho + p)^{-1}\left(\frac{\partial\rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta (\delta s)^2 + (\rho + p)^{-1}\left(\frac{\partial\rho}{\partial p}\right)_s \left[\delta p + (\rho + p)\left(\frac{\partial p}{\partial\rho}\right)_s \delta u^a \lambda_a\right]^2 + (\rho + p)\left[1 - \left(\frac{\partial p}{\partial\rho}\right)_s \lambda^2\right]\lambda^{-2}(\delta u^a \lambda_a)^2\Big\}\frac{u^c}{T}\,d\Sigma_c. \tag{A3}$$

This form of the energy is non-negative when the thermodynamic inequalities in (A1) and (A2) are satisfied. Therefore, when the solutions to the perturbation equations are bounded (that is, when the fluid is stable) the energy decreases toward a non-negative value. It follows that the energy of every possible initial perturbation must also be nonnegative. Thus the energy functional is positive definite, and all of the conditions $\Omega_A > 0$ must be satisfied.

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Note added in proof. Since this paper was written, we have become aware of a paper by Israel and Stewart (*Ann. Phys.* **118** (1979), 341) which derives causal dissipative hydrodynamical equations including the possibility of nonuniform, gravitating, and rotating equilibrium states. The theory considered by them is somewhat more general than the one which we present in Eqs. (21)–(23) of this paper. They point out that a number of different terms can be added to the equations of motion without changing the crucial entropy generation equation. One can add to Eq. (21) terms of the form

$$-[V_{ab}\tau^{ab} + W_a q^a],$$

to Eq. (22) the terms

$$\kappa T q^a{}_b[\tau W^b - X^{bcd}\tau_{cd} - Y^{[ac]}q_c],$$

and to Eq. (23) the terms

$$2\eta\langle\tau V^{ab} + X^{cab}q_c - Z^{[ac]}\tau_c{}^b\rangle.$$

In these expressions the tensors $V^{(ab)}$, W^a, $X^{a(bc)}$, $Y^{[ab]}$, and $Z^{[ab]}$ are arbitrary functions of the dynamical variables $(n, \rho, u^a, q^a, \tau, \tau^{ab})$.

The results of the present paper do not depend in any way upon the addition of terms of this form. The characteristic velocities for perturbations about equilibrium are the same for the general theory including the extra terms as those given by us in Eqs. (127) and (128) for the theory without them. This comes about because the linearized contributions from these additional terms depend on δq^a, $\delta\tau$, and $\delta\tau^{ab}$ but not on their derivatives. Consequently they make no contributions to the matrix $a^A{}_B{}^d$ in Eq. (105) or to the characteristic Eq. (106). The stability criteria given by us in Eq. (70) are also the correct ones for the theory containing these extra terms. This follows because the time derivative of the energy functional $E(\Sigma)$ given in Eqs. (44)–(46) is unchanged by the addition of these extra terms in the equations of motion for the fluid variables. Thus, the characteristic velocities and the stability criteria are both unchanged by the addition of these extra terms to the equations.

If one wished to study in detail the evolution of some particular fluid motion, these extra terms in the equations of motion would of course influence the behavior of the fluid. It is then necessary to determine more precisely the forms of the tensors $V^{\langle ab \rangle}$, etc. We suggest that the following two principles be applied to constrain the possible choices: (a) to preserve the linearity of the equations of motion in the deviations from equilibrium (q^a, τ, τ^{ab}), the tensors should be chosen to depend only on n, ρ, and u^a; (b) to prevent the equations of motion from developing the noncausal behavior found in the Eckart theory, the tensors should contain at most first derivatives of n, ρ, and u^a, and these should appear linearly to ensure the hyperbolicity of the full equations. These two principles severely restrict the possible choices of the tensors $V^{\langle ab \rangle}$, etc. We find only the following possibilities:

$$
\begin{aligned}
V^{ab} &= \varepsilon_1 \langle \nabla^a u^b \rangle, \\
W^a &= \varepsilon_2 q^{ab} \nabla_b \Theta + \varepsilon_3 [u^b \nabla_b u^a + T^{-1} \nabla^a T] + \delta_1 \eta^{abcd} u_b \nabla_c u^d, \\
X^{abc} &= \varepsilon_4 q^{a\langle b} \nabla^{c\rangle} \Theta + \varepsilon_5 [u^d \nabla_d u^{\langle c} q^{b\rangle a} + T^{-1} q^{a\langle b} \nabla^{c\rangle} T] \\
&\quad + \delta_2 q^{a\langle b} \eta^{c\rangle def} u_d \nabla_e u_f, \\
Y^{ab} &= \gamma_2 q^a{}_c q^b{}_d \nabla^{[c} u^{d]}, \\
Z^{ab} &= \gamma_3 q^a{}_c q^b{}_d \nabla^{[c} u^{d]}.
\end{aligned}
$$

The coefficients γ_i, δ_i, and ε_i are arbitrary functions of n and ρ. The terms multiplying the coefficients ε_i vanish in equilibrium, and thus should not be included in a first order theory. The terms multiplying the coefficients δ_i contain the totally antisymmetric tensor η^{abcd}. These terms will consequently have odd parity, and probably do not contribute to a macroscopic theory. Israel and Stewart (1979) evaluate these coefficients using kinetic theory for an ideal gas. They find the ε's and the δ's to be zero while the γ's are nonzero.

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