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LIMITS ON THE GRAVITATIONAL REDSHIFT FROM NEUTRON STARS[1]

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ABSTRACT

Upper and lower limits on the gravitational redshift from the surface of nonrotating neutron stars of given mass are derived. For 1.4 $M_\odot$ stars these limits are $0.854 \geq z \geq 0.184$, when the equation of state of neutron star matter is assumed to be known for densities below nuclear density ($\rho_0 = 3 \times 10^{14}$ g cm^{-3}) and when the equation of state above this value is assumed to obey $0 \leq dp/d\rho \leq c^2$. Limits are also presented for a complete range of neutron star masses, and for a variety of choices of ρ_0 (the density below which the equation of state is assumed to be known). These limits are compared to redshifts deduced from γ-ray burst observations.

Subject headings: gravitation — relativity — stars: neutron

I. INTRODUCTION

The purpose of this paper is to investigate the range of gravitational redshifts which can be expected in the observation of emission lines originating near the surface of neutron stars. Recent observations of emission lines in the spectra of γ-ray burst events make such an investigation timely. Some of the observed lines have energies in the 0.40–0.46 MeV range and have been interpreted as gravitationally redshifted 0.511 MeV e^+e^- annihilation lines (see Mazets *et al.* 1981; Ramaty and Lingenfelter 1981; Teegarden and Cline 1980). The gravitational redshifts inferred from these observations lie in the range $0.1 < z < 0.3$. This range is consistent with the redshifts computed for neutron stars in the 1.2–1.4 $M_\odot$ range for a variety of models of the equation of state at supernuclear densities (see Arnett and Bowers 1977 or Lindblom and Detweiler 1983). In this paper, upper and lower limits on the gravitational redshift are derived which are independent of the details of the properties of matter above nuclear density.

The analysis used here to derive bounds on the gravitational redshifts from neutron stars is completely analogous to the methods developed by Hartle (1978) and others to derive bounds on the masses and moments of inertia for neutron stars. In this analysis one assumes that general relativity correctly describes the gravitational interaction at neutron star densities. One also assumes that the stars are nonrotating, spherical, and composed of fluid matter (that is, matter having isotropic stresses). At the highest densities, where the equation of state of matter is only poorly understood, one makes only minimal assumptions about the matter, i.e., $0 \leq dp/d\rho \leq c^2$. These conditions are consequences of microscopic stability and causality considerations (see Hiscock and Lindblom 1983). At lower densities, $\rho < \rho_0$, below about nuclear density, one assumes that the equation of state is known and is given by Baym, Pethick, and Sutherland (1971). Bounds are presented on the gravitational redshift for a range of assumptions for ρ_0, the density below which the equation of state is assumed to be known.

Upper limits on the gravitational redshift z from general relativistic stars have been known since the earliest investigations of the theory. Schwarzschild (1916) showed that incompressible fluid stars must satisfy $2M/R \leq \frac{8}{9}$, which is equivalent to bounding the redshift by $z \leq 2$. Buchdahl (1959) showed that the upper limit $z \leq 2$ applies also to compressible fluid stars. The upper limits presented here are considerably stronger. We show that the redshift must be bounded above by $z \lesssim 0.9$. This bound depends on the assumption of microscopic stability and causality of the matter above nuclear density; but the bound depends only very weakly on the cutoff density ρ_0. Lower limits on the gravitational redshift are also derived for the first time. These lower bounds depend sensitively on the mass of the neutron star and on ρ_0.

II. THE ANALYSIS

The limits on the gravitational redshift from neutron stars are derived here in very much the same way as Hartle (1978) and others derived limits on the masses and moments of inertia of neutron stars. One assumes that the neutron star is divided into two regions: (*a*) an outer envelope with densities below ρ_0 where the equation of state is assumed to be known; and (*b*) an inner core with densities above ρ_0. One assumes that Einstein's equations correctly describe the gravitational interaction throughout the neutron stars; thus

$$\frac{dp}{dr} = -(\rho + p)\frac{m + 4\pi r^3 p}{r(r - 2m)}, \tag{1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho . \tag{2}$$

In these equations p is the pressure of the neutron star matter, ρ is the density, and m is the mass contained within radius r. Once the mass m_0 and radius r_0 of the core are known, then equations (1) and (2) can be integrated using the known equation of state in the envelope region (Baym, Pethick, and Sutherland 1971) to determine the total mass, radius, and redshift of the star. Thus the total mass $M(m_0, r_0, \rho_0)$ and the gravitational redshift $z(m_0, r_0, \rho_0)$ are well defined functions of the core mass m_0, the core radius r_0, and the cutoff density ρ_0. It is straightforward to evaluate these functions numerically.

The next step in obtaining limits on the mass or redshift of neutron stars is to demonstrate, for given cutoff density ρ_0,

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that only a compact region of the (m_0, r_0)-plane is allowed for cores of neutron stars. Hartle (1978) summarizes the analysis needed to bound the allowed region of the (m_0, r_0)-plane. Once these bounds are established, it follows that the functions $M(m_0, r_0, \rho_0)$ and $z(m_0, r_0, \rho_0)$ will achieve maximum values on the allowed compact domain in (m_0, r_0). These maximum values can be evaluated numerically. They provide upper limits to the total mass and redshift of neutron stars.

Hartle (1978) describes two analyses which limit the allowable domain of the (m_0, r_0)-plane. The first analysis requires the following assumption about the properties of the matter in the core of the neutron star:

$$dp/d\rho \geq 0\ . \qquad (3)$$

This assumption is a necessary condition for the microscopic stability of the neutron star matter (see, e.g., Hiscock and Lindblom 1983). The bounds on the redshift which follow from the use of this assumption alone are referred to here as *weak bounds*. The second analysis which limits the admissible domain of the (m_0, r_0)-plane assumes, in addition to equation (3), the following:

$$dp/d\rho \leq c^2\ . \qquad (4)$$

The constant c is the speed of light. This assumption limits the equation of state in the core further by requiring that the sound speed in the core material be subluminal. Bounds on the redshift obtained by imposing equations (3) and (4) on the core are referred to here as *causal bounds*.

Hartle (1978) describes clearly the analysis which produces limits on the allowed domain in the (m_0, r_0)-plane based on the assumptions in equations (3) and (4). The interested reader is encouraged to consult that work and the references cited therein for the details of the analysis. Figure 1 illustrates the allowed domains for a cutoff density $\rho_0 = 3 \times 10^{14}$ g cm^{-3}. The entire shaded region represents allowed cores when the weak condition, equation (3), is imposed. The smaller darkly shaded region represents the allowed cores when the casual assumptions, equations (3) and (4), are imposed.

Once the allowed domain of the (m_0, r_0)-plane is determined, it is a straightforward numerical exercise to evaluate the functions $M(m_0, r_0, \rho_0)$ and $z(m_0, r_0, \rho_0)$ on the allowed region. The maximum values of these functions on the allowed regions can be determined in this way. Table 1 lists the results of such computations. Upper bounds on the mass and the redshift are given for several different values of the cutoff density ρ_0. The values of ρ_0 were chosen to bracket the density of normal nuclear matter. The physical processes involved in determining the equation of state below these densities are thought to be

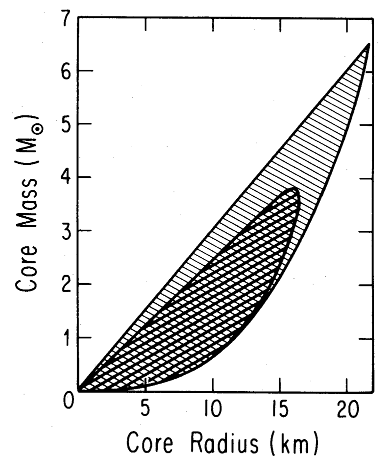

FIG. 1.—The shaded region shows the allowed domain for core masses and radii for a cutoff density of $\rho_0 = 3.0 \times 10^{14}$ g cm^{-3}. The entire shaded region is allowed when the weak assumption (eq. [3]) is imposed on the core, while only the darkly shaded region is allowed when the causal assumption (eq. [4]) is also imposed on the core.

well understood. Separate upper bounds based on the weak and the casual assumptions are given in Table 1. The upper bounds on the redshift are essentially independent of the value of ρ_0, while the upper bounds on the mass vary like $\rho_0^{-1/2}$.

A more useful set of bounds on the gravitational redshift may also be obtained from this analysis. The level surfaces of the function $M(m_0, r_0, \rho_0)$ define a set of curves in the (m_0, r_0)-plane for a given value of ρ_0. Let $\gamma_M(\lambda) = [m_0(\lambda), r_0(\lambda)]$ represent the curve (parameterized by λ) which includes all those stars having total mass M. When confined to the domain of the (m_0, r_0)-plane allowed for neutron star cores, the curves $\gamma_M(\lambda)$ are compact sets. Consequently the redshift function $z(m_0, r_0, \rho_0)$ evaluated along one of these curves $z[\gamma_M(\lambda), \rho_0]$ will achieve a maximum and minimum value. These extreme values of $z[\gamma_M(\lambda), \rho_0]$ give the allowed upper and lower bounds of the redshift from a neutron star of mass M. Figure 2 illustrates these bounds for $\rho_0 = 3 \times 10^{14}$ g cm^{-3}. The large shaded region shows the allowed redshifts when only the weak assumption, equation (3), is imposed on the neutron star cores. The smaller darkly shaded region shows the allowed redshifts when the causal assumptions, equations (3) and (4), are imposed on the core. Figure 2 reveals that the upper bounds are essentially independent of the neutron star mass. The upper bounds are also essentially independent of ρ_0, as one sees in Table 1. The lower limit on the gravitational redshift shown in Figure 2 applies to stars containing some matter at nuclear

TABLE 1

MAXIMUM REDSHIFTS AND MASSES FOR NEUTRON STARS

		CAUSAL BOUNDS		WEAK BOUNDS	
ρ_0 (g cm^{-3})	$p_0/\rho_0 c^2$	z_{max}	$M_{max}/M_\odot$	z_{max}	$M_{max}/M_\odot$
5.0×10^{14}	1.58×10^{-2}	0.858	3.02	1.909	5.09
4.0×10^{14}	1.26×10^{-2}	0.864	3.37	1.918	5.69
3.0×10^{14}	9.69×10^{-3}	0.869	3.88	1.927	6.57
2.0×10^{14}	7.13×10^{-3}	0.875	4.75	1.937	8.05
1.0×10^{14}	4.46×10^{-3}	0.883	6.71	1.948	11.38

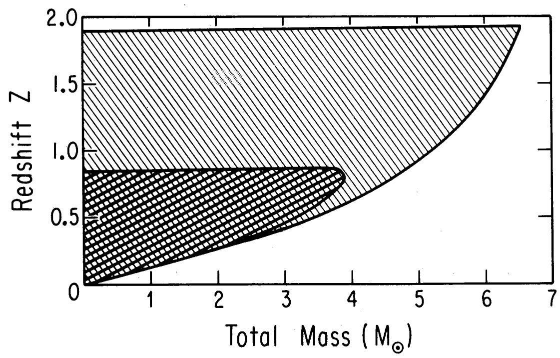

FIG. 2.—The shaded region represents the allowed redshifts for given total neutron star mass, based on a cutoff density of $\rho_0 = 3 \times 10^{14}$ g cm^{-3}. The large shaded region represents the limits based on the weak assumption (eq. [3]), while the smaller darkly shaded region represents the limits based also on the causal assumption (eq. [4]).

densities. Below 1 $M_\odot$ (the Chandrasekhar mass limit for the Baym-Pethick-Sutherland equation of state) there exists a second class of stellar models which do not contain any matter at nuclear densities: white dwarfs and planets. These low-density objects have redshifts which are smaller than 10^{-3}.

Tables 2 and 3 give numerical values for the weak upper and lower bounds in the astrophysically interesting mass range between 1 $M_\odot$ and 3 $M_\odot$. Tables 4 and 5 give numerical values for the casual upper and lower bounds in the 1 $M_\odot$ to 3 $M_\odot$ range. One sees from these tables that the lower bounds depend sensitively on the neutron star mass as well as the cutoff density ρ_0. The upper bounds, on the other hand, are relatively insensitive to both the mass and ρ_0.

III. DISCUSSION

It is instructive to compare the bounds on the gravitational redshift presented in the previous section with (*a*) redshifts computed for neutron star models based on "realistic" equations of state in the supernuclear regime and (*b*) observations of γ-ray burst spectra which have been interpreted as gravitationally redshifted emission lines.

A large number of neutron star models have been computed by Arnett and Bowers (1977) for a sample of 12 different "realistic" equations of state in the supernuclear density domain. Gravitational redshifts for these models have been tabulated by Lindblom and Detweiler (1983). In Figure 3 the redshifts from some of these models are compared to the redshift bounds computed here. The four solid curves, labeled B, D, M, and O, in Figure 3 are the mass-redshift relationships for four of these equation-of-state models. The dashed curves are the lower casual boundaries for various values of ρ_0. The labels B, D, M, and O refer to specific equation-of-state models which are defined in the above references. These four equations of state were selected to illustrate the range of models included in the Arnett and Bowers (1977) study. We see that the "realistic" models tend to have redshifts which lie closer to the

TABLE 2

WEAK LOWER BOUNDS ON THE GRAVITATIONAL REDSHIFT

$M/M_\odot$	ρ_0: 5.0×10^{14}	4.0×10^{14}	3.0×10^{14}	2.0×10^{14}	1.0×10^{14}
1.0	0.159	0.145	0.129	0.109	0.082
1.1	0.176	0.160	0.142	0.120	0.090
1.2	0.193	0.175	0.155	0.130	0.098
1.3	0.210	0.190	0.168	0.141	0.105
1.4	0.228	0.206	0.181	0.151	0.113
1.5	0.245	0.221	0.194	0.162	0.120
1.6	0.263	0.237	0.207	0.172	0.128
1.8	0.301	0.269	0.234	0.194	0.142
2.0	0.341	0.303	0.262	0.216	0.157
2.2	0.382	0.338	0.291	0.238	0.172
2.4	0.427	0.376	0.321	0.261	0.187
2.6	0.475	0.425	0.352	0.284	0.203
2.8	0.527	0.457	0.385	0.308	0.218
3.0	0.583	0.501	0.419	0.333	0.234

TABLE 3

WEAK UPPER BOUNDS ON THE GRAVITATIONAL REDSHIFT

$M/M_\odot$	ρ_0: 5.0×10^{14}	4.0×10^{14}	3.0×10^{14}	2.0×10^{14}	1.0×10^{14}
1.0	1.865	1.883	1.900	1.916	1.935
1.4	1.867	1.884	1.900	1.917	1.935
1.8	1.869	1.885	1.901	1.917	1.935
2.2	1.872	1.887	1.902	1.917	1.936
2.6	1.875	1.889	1.903	1.918	1.936
3.0	1.880	1.892	1.905	1.919	1.936

TABLE 4

CAUSAL LOWER BOUNDS ON THE GRAVITATIONAL REDSHIFT

	ρ_0				
$M/M_\odot$	5.0×10^{14}	4.0×10^{14}	3.0×10^{14}	2.0×10^{14}	1.0×10^{14}
1.0	0.162	0.148	0.131	0.111	0.083
1.1	0.179	0.163	0.144	0.121	0.091
1.2	0.197	0.178	0.157	0.132	0.099
1.3	0.215	0.194	0.171	0.143	0.106
1.4	0.234	0.210	0.184	0.154	0.114
1.5	0.253	0.227	0.198	0.165	0.122
1.6	0.273	0.244	0.212	0.176	0.129
1.8	0.314	0.279	0.241	0.198	0.145
2.0	0.360	0.317	0.271	0.221	0.160
2.2	0.410	0.357	0.303	0.245	0.176
2.4	0.467	0.401	0.337	0.270	0.192
2.6	0.533	0.450	0.373	0.295	0.208
2.8	0.615	0.506	0.412	0.322	0.224
3.0	0.753	0.571	0.455	0.351	0.241

TABLE 5

CAUSAL UPPER BOUNDS ON THE GRAVITATIONAL REDSHIFT

	ρ_0				
$M/M_\odot$	5.0×10^{14}	4.0×10^{14}	3.0×10^{14}	2.0×10^{14}	1.0×10^{14}
1.0	0.832	0.842	0.852	0.863	0.874
1.4	0.836	0.844	0.854	0.863	0.875
1.8	0.841	0.847	0.856	0.864	0.875
2.2	0.846	0.851	0.858	0.865	0.875
2.6	0.853	0.856	0.860	0.867	0.876
3.0	0.829	0.861	0.863	0.868	0.876

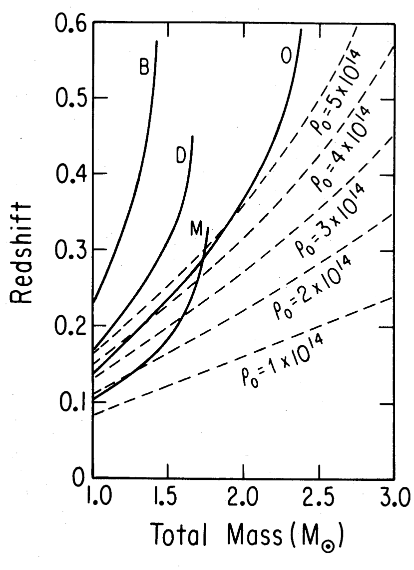

FIG. 3.—The dashed lines represent the causal lower bounds on the gravitational redshift for several different cutoff densities ρ_0. The solid curves (labeled B, D, M, and O) represent the mass redshift relationship for several "realistic" equations of state from the Arnett and Bowers (1977) survey of neutron star properties.

lower bounds than to the upper bounds. In fact, some of the "realistic" models violate the lower bounds for some values of ρ_0. These equation-of-state models do not all agree with the Baym, Pethick, and Sutherland (1971) equation of state used here to densities as high as 5×10^{14} g cm^{-3}. This suggests that some caution should be used in applying the lower bounds.

It is also interesting to compare these limits to the existing observations relevant to gravitational redshifts from neutron stars. Table 6 lists the dates of γ-ray burst events along with the energies of emission lines in the 0.4–0.5 MeV range. This table also contains a redshift which is based on the interpreta-

TABLE 6

OBSERVED γ-RAY EMISSION LINES

Date of Event	E (MeV)	$Z = \frac{0.551 - E}{E}$
1974 Jun 10[a]	0.41	0.25
1978 Sep 18[b]	0.40	0.28
1978 Nov 4[b]	0.40	0.28
1978 Nov 19[c]	0.42	0.22
1979 Jan 16[b]	0.42	0.22
1979 Mar 5[b]	0.43	0.19
1979 Apr 18[b]	0.45	0.14
1979 Jun 22[b]	0.46	0.11

[a] Ramaty and Lingenfelter 1981.
[b] Mazets *et al.* 1981.
[c] Teegarden amd Cline 1980.

tion of these lines as redshifted e^+e^- pair annihilation lines. The uncertainties associated with the energies of these lines are listed in the references, and are generally about 10%. From Table 6 one can see that the redshift is a rapidly varying function of the energy. An uncertainty of 10% in the energy can lead to an uncertainty of a factor of 2 or more in the computed redshift. Given this uncertainty, we see that the observed redshifts of these emission lines are consistent with the limits on the redshifts from typical 1.4 $M_\odot$ neutron stars derived in this paper.

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