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ON THE ADIABATIC PULSATIONS OF ACCRETION DISKS AND ROTATING STARS

JAMES R. IPSER
Department of Physics, University of Florida, Gainesville, FL 32611
AND
LEE LINDBLOM
Department of Physics, Montana State University, Bozeman, MT 59717
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ABSTRACT

The adiabatic pulsations of rotating Newtonian fluids, including nonbarotropic configurations, are represented completely in terms of two scalar potentials. The second-order equations that determine these potentials are more convenient to solve numerically than the standard Lagrangian-displacement form of the equations. A variational principle is derived for these equations from which estimates of the frequencies of the modes may be obtained. This analysis generalizes previous work by describing the adiabatic pulsations of fluids having any (including nonbarotropic) equation of state, any (including differential) rotation law, and self-gravitation. It is applicable, therefore, to the pulsations of rotating stars or thick accretion disks.

Subject headings: hydrodynamics — stars: accretion — stars: pulsation — stars: rotation

1. INTRODUCTION

The pulsations of rotating fluid configurations is a complex problem of considerable physical and mathematical interest. Such pulsations could in principle be observed directly in rotating stars or accretion disks. In addition, these pulsations determine the stability of rotating fluid objects. Thus, their effects might be observable indirectly through the limits that any instability would place on the physically allowed set of equilibrium configurations.

In this paper we explore some of the interesting mathematical aspects of this pulsation problem. The traditional formulation of this problem is in terms of the Lagrangian-displacement vector field (see Lynden-Bell & Ostriker 1967). This vector field satisfies a complicated differential-integral equation. To our knowledge it has been solved numerically only for the special case of the axisymmetric pulsations of rotating stars (Clement 1981). The pulsations of rotating fluids can also be formulated in terms of two scalar potentials. This alternative formulation was given by Poincaré (1885) for the special case of uniform-density fluids, and by Ipser & Managan (1985) and Managan (1985) for the more general case of rotating fluids having barotropic equations of state. These perturbation potentials satisfy a coupled system of second-order partial-differential equations. These equations are convenient to solve numerically and have in fact been solved for a number of the nonaxisymmetric f-modes of rotating stars (Ipser & Lindblom 1989, 1990).

In this paper we extend the two-potential formalism to include the adiabatic pulsations of rotating stars having arbitrary (possibly nonbarotropic) equations of state. Section 2 presents a derivation of the system of second-order partial-differential equations satisfied by these potentials. While these equations are somewhat more complicated than those for the barotropic case, they have the same basic mathematical structure. They are now general enough, however, to describe the g-modes in addition to the f- and p-modes of rotating fluids. In § 3 we present a variational principle from which the frequencies of these modes may be estimated. The analogous variational principle for the barotropic case has been found to be a valuable tool for estimating the frequencies of the f- and p-modes.

2. THE TWO-POTENTIAL FORMALISM

The evolution of a Newtonian fluid is determined by the standard laws of hydrodynamics:

$$\partial_t \rho + \nabla_a(\rho v^a) = 0 , \tag{1}$$

$$\rho(\partial_t v^a + v^b \nabla_b v^a) = -\nabla^a p + \rho \nabla^a \Phi , \tag{2}$$

$$\nabla^a \nabla_a \Phi = -4\pi G\rho . \tag{3}$$

In these equations the mass density and pressure are represented by the scalars ρ and p, respectively. We assume that there exists an equation of state that relates these to the other thermodynamic variables, e.g., the temperature T and the entropy per unit mass s. We do *not* assume, however, that this equation of state is barotropic: the presence p need *not* be a function of the density ρ alone. Also, in these equations the velocity field is represented by v^a, the gravational field by Φ, and Newton's gravitation constant by G. The derivative operator ∂_t is the partial derivative with respect to time, while ∇_a is the standard Euclidean covariant derivative (i.e., partial derivatives in Cartesian coordinates). Tensor indices are raised and lowered with the Euclidean metric g_{ab} (the identity matrix in Cartesian coordinates) and its inverse g^{ab}.

In this paper we are concerned with the dynamics of small fluctuations about the stationary and axisymmetric equilibrium states of these fluids. We assume that the fluid velocity is purely axial in the equilibrium state, i.e., that $v^a = \Omega\phi^a$ where ϕ^a is a rotation

vector field which satisfies Killing's equation, $\nabla_a \phi_b + \nabla_b \phi_a = 0$. By taking the equilibrium velocity field to have this form, we are assuming that any meridional circulation may be treated as a small perturbation to the overall rotation motion of the fluid. Under these conditions equation (1) is satisfied identically, while equation (2) reduces to

$$\frac{1}{2}\Omega^2 \nabla_a \varpi^2 = \frac{\nabla_a p}{\rho} - \nabla_a \Phi \,, \tag{4}$$

where ϖ is the standard cylindrical radial coordinate. Since ρ is not necessarily a function of p alone, equation (4) does not imply that the angular velocity Ω is a function of ϖ alone.

In order to make precise our discussion of the small fluctuations about equilibrium, we introduce a smooth one-parameter family of solutions to equations (1)–(3): $\rho = \rho(\mu)$, $v^a = v^a(\mu)$, and $\Phi = \Phi(\mu)$. We take the solution with $\mu = 0$ to be one of the stationary and axisymmetric equilibrium solutions. The Eulerian perturbations of the fluid variables are defined, then, as the derivatives of these quantities with respect to μ evaluated at $\mu = 0$: $\delta\rho = d\rho/d\mu$, $\delta v^a = dv^a/d\mu$, and $\delta\Phi = d\Phi/d\mu$. These Eulerian perturbations are the tangent to this one parameter family of solutions, and thus measure the linear behavior of these solutions near equilibrium. The Lagrangian displacement ξ^a is defined as a potential for the velocity perturbation:

$$\delta v^a = \partial_t \xi^a + v^b \nabla_b \xi^a - \xi^b \nabla_b v^a \,. \tag{5}$$

Finally, we define the Lagrangian perturbations $\Delta\rho = \delta\rho + \xi^a \nabla_a \rho$ and $\Delta p = \delta p + \xi^a \nabla_a p$, where $\nabla_a \rho$ and $\nabla_a p$ are evaluated in the $\mu = 0$ equilibrium solution.

The evolution equations for the Eulerian perturbations are obtained by differentiating equations (1)–(3) with respect to μ, and evaluating the results at $\mu = 0$. The resulting expressions are

$$\partial_t \delta\rho + v^a \nabla_a \delta\rho + \nabla_a(\rho \delta v^a) = 0 \,, \tag{6}$$

$$\partial_t \delta v^a + v^b \nabla_b \delta v^a + \delta v^b \nabla_b v^a = -\frac{\nabla^a \delta p}{\rho} + \frac{\delta\rho \nabla^a p}{\rho^2} + \nabla^a \delta\Phi \,, \tag{7}$$

$$\nabla^a \nabla_a \delta\Phi = -4\pi G \delta\rho \,. \tag{8}$$

Quantities not preceeded by δ in these equations, and those that follow, are assumed to be evaluated in the $\mu = 0$ equilibrium solution.

Equations (6)–(8) would determine the evolution of the Eulerian perturbations if the relationship between the perturbed pressure δp and the other fluid variables were known. In this paper we fix this relationship by assuming that the Lagrangian change in the pressure is proportional to the Lagrangian change in the density,

$$\Delta p = \frac{p\Gamma}{\rho} \Delta\rho \,. \tag{9}$$

We assume that Γ is a time independent and axisymmetric, but otherwise arbitrary, function on the equilibrium fluid configuration. We refer to these perturbations as *adiabatic*, even though they are more general than the usual $\Delta s = 0$ perturbations. We refer to the function Γ as the adiabatic index, even though it need not satisfy the usual relationship $p\Gamma = \rho(\partial p/\partial\rho)_s$.

The traditional analysis of these perturbation equations (6)–(9) first solves equation (6) for $\delta\rho$ in terms of the Lagrangian displacement. The result is

$$\delta\rho = -\nabla_a(\rho \xi^a) \tag{10}$$

(see for example Lynden-Bell & Ostriker 1967). The other Eulerian quantities, δv^a and δp, may also be expressed in terms of ξ^a using equations (5) and (9), respectively. With these substitutions, equation (7) becomes a second-order dynamical equation for the evolution of ξ^a.

We adopt a different approach. We limit our attention to the normal-mode solutions of equations (6)–(9), i.e., those having time dependence $e^{i\omega t}$ and azimuthal angular dependence $e^{im\phi}$. The constant ω is the frequency of the mode; and, m is an integer. For these normal modes, equation (5) becomes an algebraic relationship between ξ^a and δv^a:

$$\xi_a = -i\left(\frac{g_{ab}}{\sigma} - \frac{i\phi_a \nabla_b \Omega}{\sigma^2}\right)\delta v^b \,, \tag{11}$$

where $\sigma = \omega + m\Omega$. With this relationship, equation (9) can be used to express $\delta\rho$ in terms of δp and δv^a:

$$\delta\rho = \frac{\rho}{p\Gamma}\delta p + \frac{i\rho^2}{\sigma}\delta v^a A_a \,, \tag{12}$$

where

$$A_a = \frac{\nabla_a \rho}{\rho^2} - \frac{\nabla_a p}{\rho p \Gamma} \,. \tag{13}$$

We note that $A_a = 0$ for barotropic fluids having the adiabatic index $p\Gamma = \rho(dp/d\rho)$. It will not be zero, however, for more general

fluids.[1] Eliminating $\delta\rho$ from equation (7), we obtain the following representation of the perturbed Euler equation for the normal modes,

$$\left(i\sigma g_{ab} + 2\nabla_b v_a - \phi_a \nabla_b \Omega - \frac{i}{\sigma}\nabla_a p A_b\right)\delta v^b = -\nabla_a \delta U - \rho(\delta U + \delta\Phi)A_a , \tag{14}$$

where $\delta U = \delta p/\rho - \delta\Phi$. This form of equation (7) is algebraic in δv^a! It is the generalization of the expression derived by Ipser & Managan (1985) and Managan (1985) for the case $A_a = 0$.

Equation (14) can be used to eliminate δv^a from the problem whenever the tensor

$$Q^{-1}_{ab} = \sigma g_{ab} - 2i\nabla_b v_a + i\phi_a \nabla_b \Omega - \frac{1}{\sigma}\nabla_a p A_b \tag{15}$$

can be inverted. This is always possible, unless its determinant vanishes. The determinant of $(Q^{-1})^a_b$ is a coordinate invariant which may be expressed as

$$\det (Q^{-1})^a_b = \frac{1}{\sigma}(\sigma^4 - \sigma^2 A^a \nabla_a p - 2\sigma^2 \Omega^a \omega_a + 2A^a \omega_a \Omega^b \nabla_b p) \equiv \frac{\sigma^3}{\lambda} , \tag{16}$$

where $\omega^a = (\nabla \times v)^a = \epsilon^{abc}\nabla_b v_c$ is the vorticity of the fluid, and $\Omega^a = \Omega z^a$ is the angular velocity vector. The tensor Q^{-1}_{ab} may be inverted, then, except when the frequency of the mode coincides with a root of the quartic polynomial in equation (16). We note that $n^2 = A^a \nabla_a p$ is the generalization to rotating fluids of the Brunt-Väisälä frequency. For nonrotating fluids this determinant, equation (16), vanishes only when $\omega = 0$ or $\omega^2 = n^2$.

When Q^{-1}_{ab} does have an inverse, it may be expressed in the form

$$Q^{ab} = \frac{\lambda}{\sigma^3}\left[(\sigma^2 - A^c \nabla_c p)g^{ab} - 2\omega^a \Omega^b + i\sigma\phi^a \nabla^b \Omega - 2i\sigma\Omega\nabla^a \phi^b + \nabla^a p A^b + \frac{i}{\sigma}\phi^a \phi^{bc}\nabla_c p A^d \omega_d - \frac{2i}{\sigma}\phi^b \phi^{ac} A_c \Omega^d \nabla_d p\right], \tag{17}$$

where $\phi^{ab} = \epsilon^{abc}\phi_c/\phi^d \phi_d$. This expression for Q^{ab} reduces to the one given by Ipser & Lindblom (1990) for the case $A^a = 0$. In cylindrical coordinates (z, ϖ, ϕ) the component of Q^{ab} are

$$Q^{ab} = \frac{\lambda}{\sigma^3}\begin{Bmatrix} \sigma^2 - A^\varpi \partial_\varpi p - 2\Omega\omega^z & A^\varpi \partial_z p & -\dfrac{2i\Omega}{\varpi\sigma} A^\varpi \partial_z p \\ A^z \partial_\varpi p - 2\Omega\omega^\varpi & \sigma^2 - A^z \partial_z p & -\dfrac{2i\Omega}{\varpi\sigma}(\sigma^2 - A^z \partial_z p) \\ -\dfrac{i\sigma}{\varpi}\omega^\varpi + \dfrac{i}{\varpi\sigma}A^c \omega_c \partial_\varpi p & \dfrac{i\sigma}{\varpi}\omega^z - \dfrac{i}{\varpi\sigma}A^c \omega_c \partial_z p & \dfrac{1}{\varpi^2}(\sigma^2 - A^c \partial_c p) \end{Bmatrix} . \tag{18}$$

Many of these results, e.g., equations (16) and (17), are most easily derived first in their cylindrical-coordinate representations and then re-expressed covariantly.

When Q^{-1}_{ab} can be inverted, all of the fluid-perturbation variables can be expressed uniquely in terms of the two scalar potentials δU and $\delta\Phi$. In particular δv^a may be expressed as

$$\delta v^a = iQ^{ab}\nabla_b \delta U + i\rho(\delta U + \delta\Phi)Q^{ab}A_b . \tag{19}$$

And using equations (12) and (19), $\delta\rho$ may be expressed as

$$\delta\rho = \Psi_1(\delta U + \delta\Phi) - \frac{\rho^2}{\sigma}A_a Q^{ab}\nabla_b \delta U , \tag{20}$$

where

$$\Psi_1 = \frac{\rho^2}{p\Gamma} - \frac{\rho^3}{\sigma}A_a Q^{ab}A_b . \tag{21}$$

The potentials δU and $\delta\Phi$ are determined by the remaining fluid equations (6) and (8). These become a system of second-order equations for δU and $\delta\Phi$ alone when the relationships given in equations (19) and (20) are used:

$$\nabla_a(\rho Q^{ab}\nabla_b \delta U) + \Psi_3 \delta U = -\rho^2 Q^{ab}A_b \nabla_a \delta\Phi - \Psi_2 \delta\Phi , \tag{22}$$

$$\nabla^a \nabla_a \delta\Phi + 4\pi G\Psi_1 \delta\Phi = \frac{4\pi G\rho^2}{\sigma}A_a Q^{ab}\nabla_b \delta U - 4\pi G\Psi_1 \delta U , \tag{23}$$

[1] Since the equilibrium solutions are axisymmetric, A_a will in general be a two-dimensional vector field. Thus, there exist scalar fields R and S such that $A_a = R\nabla_a S$. Since $\nabla_a \rho = \rho^2 R\nabla_a S + (\rho/p\Gamma)\nabla_a p$, we may regard ρ as a function of p and S. Therefore, quite generally, $p\Gamma = \rho(\partial p/\partial\rho)_S$. For these adiabatic perturbations it also follows, quite generally, that $\Delta S = 0$. Thus, these perturbations always have the formal structure of the traditional adiabatic perturbations, even though S may not be the physical entropy of the fluid.

where

$$\Psi_2 = \sigma\Psi_1 + \nabla_a(\rho^2 Q^{ab} A_b)\,, \tag{24}$$

$$\Psi_3 = \Psi_2 - \frac{m\lambda\rho^2}{\sigma^3}\left(\sigma A^a\nabla_a\Omega + \frac{4\sigma\Omega}{\varpi}A^a\nabla_a\varpi + \frac{1}{\sigma}A_a\phi^{ab}\nabla_b pA^c\omega_c\right), \tag{25}$$

The scalars Ψ_1, Ψ_2, and Ψ_3 are real, while the vectors $Q^{ab}A_b$ and $Q^{ba}A_b$ have real z and ϖ components but imaginary ϕ components. It follows that equations (23) and (24) are real equations for $\delta U(z, \varpi)$ and $\delta\Phi(z, \varpi)$ where $\delta U = e^{im\phi}\,\delta U(z, \varpi)$ and $\delta\Phi = e^{im\phi}\,\delta\Phi(z, \varpi)$. These equations generalize to arbitrary adiabatic pulsations of rotating fluids the equations derived by Poincaré (1985) for the uniform density case, and by Ipser & Managan (1985) and Managan (1985) for the barotropic case.

3. VARIATIONAL PRINCIPLES

The pulsation frequencies of rotating barotropic fluids can be estimated using a variational principle (Managan 1986; Ipser & Lindblom 1990). Here, we extend this variational principle to include arbitrary adiabatic pulsations of rotating fluids.

Introduce the potential $\delta V = \delta U/\sigma$. The pulsation equations (23) and (24) can be thought of, then, as determining the potentials δV and $\delta\Phi$. Thus, in slightly different notation, these equations become

$$\mathscr{L}_{\delta V}(\delta V, \delta\Phi) \equiv \nabla_a(\rho H^{ab}\nabla_b\delta V) + \sigma\Psi_3\,\delta V + \frac{\rho^2}{\sigma}H^{ab}A_b\nabla_a\delta\Phi + \Psi_2\,\delta\Phi = 0\,, \tag{26}$$

$$\mathscr{L}_{\delta\Phi}(\delta V, \delta\Phi) \equiv \frac{\nabla_a\nabla^a\,\delta\Phi}{4\pi G} + \Psi_1\,\delta\Phi - \frac{\rho^2}{\sigma}A_a H^{ab}\nabla_b\delta V + \sigma\Psi_1\,\delta V = 0\,, \tag{27}$$

where

$$H^{ab} = \sigma Q^{ab} - iQ^{ac}\nabla_c\Omega\phi^b\,. \tag{28}$$

The tensor H^{ab} that appears in these equations is Hermitian for real values of the pulsation frequency: $H^{ab} = H^{*ba}$, where the asterisk denotes complex conjugation. To see this, use the expression for Q^{ab} in equation (17) and the equilibrium structure equation (4). In particular the identities

$$2(\omega^a\Omega^b - \omega^b\Omega^a) = A^b\nabla^a p - A^a\nabla^b p\,, \tag{29}$$

$$\phi^{ab}\nabla_b pA^c\omega_c - 2\phi^{ab}A_b\Omega^c\nabla_c p + \nabla^a\Omega A^b\nabla_b p + 2\omega^a\Omega^b\nabla_b\Omega - \nabla^a pA^b\nabla_b\Omega = 0\,, \tag{30}$$

that are satisfied by the equilibrium solutions are needed.

Now let δV and $\delta\bar{V}$ be two independent potentials each having angular dependence $e^{im\phi}$. Assume that $\delta\Phi = \delta\Phi(\delta V)$ and $\delta\bar{\Phi} = \delta\bar{\Phi}(\delta\bar{V})$ are the solutions of equation (27) for the given potentials δV and $\delta\bar{V}$, respectively. Also assume that the outer boundary of the fluid (i.e., the $p = 0$ surface of the star or accretion disk) is a smooth compact surface of finite area. And finally, assume (probably unnecessarily) that $\rho = 0$ on this outer boundary. For such equilibrium configurations, and for a given value of the pulsation frequency ω, define the inner product of these potentials as

$$\mathscr{H}_\omega(\delta\bar{V}, \delta V) = \int \delta\bar{V}^*\mathscr{L}_{\delta V}[\delta V, \delta\Phi(\delta V)]\,d^3x\,, \tag{31}$$

where $\mathscr{L}_{\delta V}(\delta V, \delta\Phi)$ is defined in equation (26). It follows that

$$\mathscr{H}_\omega(\delta\bar{V}, \delta V) = \int\left(-\rho H^{ab}\nabla_a\delta\bar{V}^*\nabla_b\delta V + \sigma\Psi_3\,\delta\bar{V}^*\,\delta V + \frac{\nabla^a\,\delta\bar{\Phi}^*\nabla_a\delta\Phi}{4\pi G} - \Psi_1\,\delta\bar{\Phi}^*\,\delta\Phi\right)d^3x\,. \tag{32}$$

Thus, for real values of the frequency ω this inner product is Hermitian, $\mathscr{H}_\omega(\delta\bar{V}, \delta V) = \mathscr{H}^*_\omega(\delta V, \delta\bar{V})$, since H^{ab} is Hermitian and Ψ_1 and Ψ_3 are real.

Now consider the expression

$$\mathscr{H}_\omega(\delta V, \delta V) = 0 \tag{33}$$

as an equation for the frequency, ω, in terms of the potential δV. Since $\mathscr{H}_\omega(\delta V, \delta V)$ is Hermitian, this equation yields a value for ω that is stationary with respect to infinitesimal variations of δV precisely when δV is the normal-mode eigenfunction with ω the corresponding eigenvalue. Thus, equation (33) is a variational principle for the real pulsation frequencies of these rotating fluid configurations.

It is interesting that there is a second more general variational principle from which eigenfrequencies of equations (26) and (27) may be obtained. Introduce a "vector" potential δX whose components are δV and $\delta\Phi$, respectively:

$$\delta X = \begin{pmatrix}\delta V\\ \delta\Phi\end{pmatrix}. \tag{34}$$

A "vector" operator $\mathscr{L}(\delta X)$ can also be defined whose components are the left sides of equations (26) and (27), respectively:

$$\mathscr{L}(\delta X) = \begin{bmatrix} \mathscr{L}_{\delta V}(\delta V, \delta\Phi) \\ \mathscr{L}_{\delta\Phi}(\delta V, \delta\Phi) \end{bmatrix} . \tag{35}$$

And in anology with equation (31) an inner product $\hat{\mathscr{H}}_\omega(\delta\bar{X}, \delta X)$ can be defined as

$$\hat{\mathscr{H}}_\omega(\delta\bar{X}, \delta X) = \int \delta\bar{X}^* \cdot \mathscr{L}(\delta X)\, d^3x \ . \tag{36}$$

It follows that this inner product is also Hermitian for real values of the frequency, for

$$\hat{\mathscr{H}}_\omega(\delta\bar{X}, \delta X) = \int \left[-\rho H^{ab} \nabla_a \delta\bar{V}^* \nabla_b \delta V + \sigma \Psi_3 \delta\bar{V}^* \delta V + \frac{\nabla^a \delta\bar{\Phi}^* \nabla_a \delta\Phi}{4\pi G} - \Psi_1 \delta\bar{\Phi}^* \delta\Phi + \delta\bar{\Phi}^* \mathscr{L}_{\delta\Phi}(\delta V, \delta\Phi) + \delta\Phi \mathscr{L}^*_{\delta\Phi}(\delta\bar{V}, \delta\bar{\Phi}) \right] d^3x \ . \tag{37}$$

This form of this inner product was suggested by Cutler's (1991) representation of these equations for the barotropic uniform-angular-velocity case. This inner product reduces to $\mathscr{H}_\omega(\delta\bar{V}, \delta V)$ when $\delta\Phi$ is taken to be the solution to equation (27) for given δV.

The frequencies defined as the roots of

$$\hat{\mathscr{H}}_\omega(\delta X, \delta X) = 0 \ , \tag{38}$$

are stationary with respect to infinitesimal variations of δX whenever δX is a normal-mode eigenfunction with ω the corresponding eigenfrequency. This second form of the variational principle is more symmetric with respect to the roles played by the potentials δV and $\delta\Phi$. Thus, it could be regarded as the more natural one. However, the first form of the variational principle is probably the more useful tool for making explicit estimates of the frequencies of rotating stars and accretion disks. These estimates are made by taking as trial eigenfunctions some explicit $\delta X(\lambda_i)$ that depend on a number of parameters λ_i. The values of these parameters are chosen so that equation (38) is stationary with respect to small changes in each of these parameters (the Ritz method). Then the frequency is estimated as the root of equation (38) using this trial eigenfunction. Since the first form of the variational principle, equation (33), involves only the single potential δV (the second potential $\delta\Phi$ having been eliminated "analytically"), estimates of the frequencies for a given level of accuracy can be obtained using a smaller number of parameters in this case. The second form of the variational principle, equation (38), may well have interesting formal applications, however. It would also be of some interest to determine how the efficiencies of the two-potential variational principles compare to those based on the standard Lagrangian-displacement formalism as used for example by Clement (1989).

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