THE ASTROPHYSICAL JOURNAL, 398:569–573, 1992 October 20

DETERMINING THE NUCLEAR EQUATION OF STATE FROM NEUTRON-STAR MASSES AND RADII

LEE LINDBLOM
Department of Physics, Montana State University, Bozeman, MT 59717
Received 1992 January 29; accepted 1992 April 24

ABSTRACT

A method is developed for determining the nuclear equation of state directly from a knowledge of the masses and radii of neutron stars. This analysis assumes only that equilibrium neutron-star matter has the stress-energy tensor of an isotropic fluid with a barotropic equation of state, and that general relativity describes a neutron star's internal gravitational field. We present numerical examples which illustrate how well this method will determine the equation of state when the appropriate observational data become available.

Subject headings: equation of state — numerical methods — relativity — stars: interiors — stars: neutron

1. INTRODUCTION

The structure of a neutron star is determined by the local balance between the attractive gravitational force and the pressure forces of the neutron-star matter. While the structure of the gravitational field in a neutron star is quite simple, the nuclear interactions that determine the pressure forces are not. These many-body interactions have yet to be fully understood, and consequently the nuclear equation of state at the densities relevant for neutron-star interiors is not well known. In the standard analysis an equation of state must be supplied before the structure equations can be integrated to determine the various observable macroscopic parameters of neutron stars, for example, masses, radii, moments of inertia, surface red-shifts, etc. Since the equation of state is not well known, however, it is not possible to make accurate predictions of these observable neutron-star parameters. This paper presents an alternative analysis of the neutron-star structure equations. A method is developed for determining the equation of state of neutron-star matter directly from a knowledge of the observable masses and radii of these stars. Accurate simultaneous measurements of the mass and radius have yet to be made on any individual neutron star, although this situation could change dramatically when data from LIGO (the Laser Interferometer Gravitational-Wave Observatory) become available (see Abramovici et al. 1992). Whenever and however such data—even from a single neutron star—do become available, the analysis described here will provide interesting information about the nuclear equation of state.

Before discussing this alternative analysis of the neutron-star structure equations, it is appropriate to review briefly the standard. For simplicity, the effects of rotation are neglected here.[1] Thus, the gravitational field of an equilibrium neutron star is taken to be static and spherically symmetric. The equations that determine the structures of these stars were first deduced from Einstein's equation by Tolman (1934) (if not implicitly by Schwarzschild 1916). Those equations were subsequently transformed by Oppenheimer & Volkoff (1939) into the simpler and more useful form that is commonly used today:

$$\frac{dm}{dr} = 4\pi r^2 \rho \,, \tag{1}$$

$$\frac{dp}{dr} = -(\rho + p)\frac{m + 4\pi r^3 p}{r(r - 2m)} \,, \tag{2}$$

where ρ and p are the total energy density and pressure of the matter and m is the "mass" within a given radius r. Since these two equations are consequences of Einstein's equation alone, it is not surprising that they are not sufficient to determine the three functions: ρ, p, and m. Obviously, some microscopic information about the nature of the stellar matter must be supplied before the structure of the star is determined. In the simplest case (which is probably sufficient for neutron stars) this additional information may be given as a barotropic equation of state, that is a nonnegative increasing function

$$\rho = \rho(p) \,. \tag{3}$$

This completes the system of equations (1)–(2)—commonly referred to as the Oppenheimer-Volkoff (OV) equations—whose solutions are the general-relativistic descriptions of non-rotating neutron stars.

The OV equations (1)–(2) are traditionally solved as an initial value problem when an equation of state (3) is given. This is accomplished by specifying the values of the pressure $p = p_c$ and the mass $m = 0$ at the center of the star $r = 0$. Then equations (1)–(2) are integrated outwardly until the surface of the star (where the pressure vanishes) is reached. From the functions $m(r)$ and $p(r)$ obtained in this way it is possible to determine any of the macroscopic properties of the neutron star. For example, the radius of the star R is defined by the expression $p(R) = 0$, and the mass of the star by $M = m(R)$. For a given equation of state, then, the mass $M(p_c)$ and radius $R(p_c)$ may be determined for any value of the central pressure p_c. Thus, the equation of state determines—through the OV equations—the mass-radius relationship for neutron stars.

I find it helpful to visualize the action of the OV equations in determining the mass-radius relationship as a map that takes the curve $[\rho(p), p]$ in the (ρ, p)-plane—the equation of state—into the curve $[M(p_c), R(p_c)]$ in the (M, R)-plane—the mass-

[1] Rotational effects scale as the angular velocity squared, and so are insignificant except in the most rapidly rotating stars. For neutron stars of about 1.4 $M_\odot$ which have rotation periods that exceed 15 ms (i.e., at least 10 times the minimum), the magnitudes of the rotational effects are less than 1% (Friedman, Ipser, & Parker 1986).

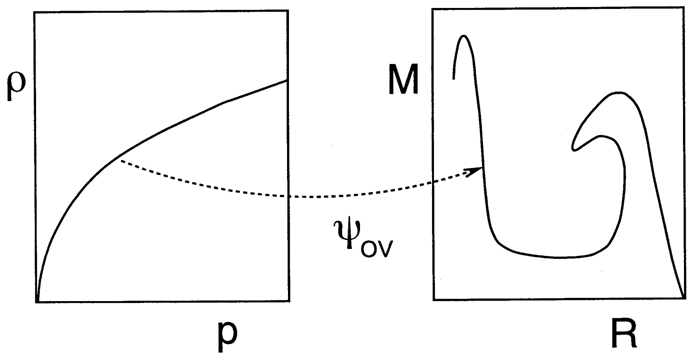

FIG. 1.—A schematic representation of the map—generated by the OV equations—that takes equations of state into mass-radius relationships.

radius relationship. This nonlinear and nonlocal map Ψ_{OV} is illustrated schematically in Figure 1. Arnett & Bowers (1977) were the first to study systematically the properties of this map for a variety of nuclear equations of state. And they were the first to infer—at least qualitatively—information about the equation of state from the observable macroscopic properties of neutron stars. This paper advances their work by showing how a knowledge of the mass-radius relationship can be used to determine the equation of state *quantitatively*. Or in more mathematical terms, this paper shows how to find Ψ_{OV}^{-1}, the inverse of the OV map.

2. INVERTING THE OPPENHEIMER–VOLKOFF MAP

The iterative inversion of the OV map can be accomplished as follows. Assume that $\rho(p)$ is already known for values of $p \le p_i$ for some p_i. And, assume that the mass-radius relationship generated by this partial equation of state agrees with the given mass-radius relationship up to the point (M_i, R_i) corresponding to the stellar model with $p_c = p_i$. To extend the equation of state beyond $\rho_i = \rho(p_i)$, choose a point (M_{i+1}, R_{i+1}) that lies slightly beyond (M_i, R_i) along the given mass-radius relationship. Next, use $m = M_{i+1}$ and $p = 0$ as initial conditions at $r = R_{i+1}$ for the OV equations (1)–(2). These equations may now be integrated inwardly through the outer layers of the stellar model where the equation of state has already been determined, that is, up to the point $r = r_i$ where $p = p_i$. This integration determines, therefore, the mass $m_i = m(r_i)$ and radius r_i of a small otherwise undetermined stellar core. Two things are known, however, about this stellar core: First, the core is nonsingular, and second, the value of $p = p_c$ at the center of this core exceeds p_i by only a small amount. This second fact is a consequence of the continuity of the stellar model with respect to changes in p_c, and the closeness of the point (M_{i+1}, R_{i+1}) to (M_i, R_i) along the mass-radius relationship. These two conditions guarantee that the structure of this stellar core is described by a nonsingular power-series solution to equations (1)–(2). The coefficients in that power series (given explicitly below) are functions of the central density ρ_c and pressure p_c of this core. The quantities ρ_c and p_c may be determined, therefore, by "inverting" those series. Thus, the equation of state may be extended in this way up to the point (ρ_c, p_c). By iteration, then, the entire equation of state may be determined from a given mass-radius relationship.

To implement the numerical inversion of the OV map outlined above, it is convenient to introduce slightly transformed versions of the OV equations. First, introduce the function $h(p)$ defined by the integral

$$h(p) = \int_0^p \frac{d\acute{p}}{\rho(\acute{p}) + \acute{p}} . \tag{4}$$

(While this function is not defined for every equation of state, it is easy to show that it is well defined for those equations of state that produce finite-sized stellar models.) This function is used to reexpress equation (2) as an equation for dh/dr. Second, make m and r the independent variables and h the dependent variable in these equations.[2] In terms of these variables, then, the OV equations (1)–(2) become

$$\frac{dm}{dh} = -\frac{4\pi\rho(h)r^3(r-2m)}{m + 4\pi r^3 p(h)} , \tag{5}$$

$$\frac{dr}{dh} = -\frac{r(r-2m)}{m + 4\pi r^3 p(h)} , \tag{6}$$

where the functions $p(h)$ and $\rho(h)$ are determined by equations (3)–(4). This form of the OV equations has several advantages over the original: (1) The mass m and radius r each enter as dependent variables. Thus, the total mass M and total radius R are simply the boundary values of the functions $m(h)$ and $r(h)$ at the surface of the star $h = 0$. This eliminates the need to solve the equation $p(R) = 0$ to locate the surface. (2) The domain of integration (from $h = h_c$ at the center of the star to $h = 0$ at the surface) is fixed before the equations are solved rather than being fixed as part of the solution. (3) These equations are easier to integrate numerically near the surface of the star since dr/dh is finite there unlike dr/dp.

The transformed OV equations (5)–(6), like the standard equations (1)–(2), are singular at the center of the star. Therefore it is necessary to evaluate the nonsingular solutions near $h = h_c$ analytically. The following are the first two nontrivial terms in the power-series solutions for $r(h)$ and $m(h)$ near $h = h_c$:

$$r(h) = \left[\frac{3(h_c - h)}{2\pi(\rho_c + 3p_c)}\right]^{1/2} \times \left\{1 - \frac{1}{4}\left[\rho_c - 3p_c - \frac{3}{5}\left(\frac{d\rho}{dh}\right)_c\right]\frac{h_c - h}{\rho_c + 3p_c}\right\} , \tag{7}$$

$$m(h) = \frac{4\pi}{3}\rho_c r^3(h)\left[1 - \frac{3}{5}\left(\frac{d\rho}{dh}\right)_c \frac{h_c - h}{\rho_c}\right] . \tag{8}$$

The coefficients in these series are the indicated functions of h_c, $\rho_c = \rho(h_c)$, and $p_c = p(h_c)$. These series play two useful roles in the numerical solution of the OV equations. When the equation of state is given, these series determine $r(h)$ and $m(h)$ in a small neighborhood of the center of the star. A numerical integration can then be used in this case to extend the solution to the remainder of the domain $h_c \ge h \ge 0$. Alternatively, when the equations are being integrated inwardly from the surface of the star toward the center—to invert the OV map—these series determine the structure of the small "undetermined" stellar

[2] Hartle (1978) showed that $r > 2m$ in any nonsingular stellar model. It follows from equation (2) that $dh/dr \le 0$, and so h is a monotonic decreasing function of r. Thus the roles of r and h as independent/dependent variables may be reversed.

core. These series can be "inverted" in this case to determine ρ_c and p_c in terms of the mass m_i and radius r_i of this core:

$$\rho_c = \rho_i + \frac{5}{2}\left(\frac{3m_i}{4\pi r_i^3} - \rho_i\right), \tag{9}$$

$$p_c = p_i + \frac{2\pi}{3}(\rho_i + p_i)(\rho_i + 3p_i)r_i^2\left[1 + \frac{2\pi}{3}(4\rho_i + 3p_i)r_i^2\right] + \frac{\pi}{3}(6\rho_i + 11p_i)\left(\frac{3m_i}{4\pi r_i^3} - \rho_i\right)r_i^2 . \tag{10}$$

The value of h_c may also be determined from a series expansion of equation (4):

$$h_c = h_i + \frac{p_c - p_i}{2(\rho_i + p_i)}\left(3 - \frac{\rho_c + p_c}{\rho_i + p_i}\right). \tag{11}$$

With these mathematical tools in place, the algorithm for inverting the OV map can now be stated explicitly. Assume that a mass-radius relationship is given and that the equation of state is already known for $h \leq h_i$. Let M_i and R_i be the total mass and radius of the star with $h_c = h_i$. Now perform the following sequence of steps iteratively: (1) Choose M_{i+1} and R_{i+1} that lie slightly beyond M_i and R_i along the mass-radius relationship. (2) Use M_{i+1} and R_{i+1} as initial conditions for equations (5)–(6) at $h = 0$, and integrate these equations through the outer layers of the star (where the equation of state is already known) to the point $h = h_i$. This determines the quantities $m_i = m(h_i)$ and $r_i = r(h_i)$ that characterize a small otherwise undetermined stellar core. (3) Use m_i and r_i in equations (9)–(10) to determine ρ_c and p_c. This extends the equation of state up to the point $h = h_c$ given in equation (11).

3. NUMERICAL EXAMPLES

How well does this algorithm work in practice? This question can be explored as follows. First, compute a set of mass-radius data from one of the "realistic" nuclear equations of state. Next, use the algorithm described above to determine a new equation of state from these mass-radius data. And finally, assess the accuracy of the new equation of state by comparing it with the original. Figure 2 displays the two "realistic" nuclear equations of state that are used here for these comparisons. The first, referred to as BJ, is based on the Bethe & Johnson (1974) model 1 equation of state (as tabulated in Pandharipande, Pines, & Smith 1976) for the highest density regime: $\rho \geq 10^{11}$ g cm^{-3}. For lower densities the Baym, Pethick, & Sutherland (1971) equation of state is used. The second equation of state, referred to as RMF (relativistic mean field), is based on the Serot (1979) equation of state for pure neutron matter in the highest density regime: $\rho \geq 3.1 \times 10^{14}$ g cm^{-3}. For lower densities the BJ equation of state described above is used. The BJ equation of state is relatively smooth and devoid of structure around nuclear density where most neutron-star matter exists. In contrast the RMF equation of state has a large discontinuity in its derivative just at nuclear density where it has been artificially patched onto the BJ equation of state. This discontinuity simulates a second-order phase transition, and this second example was selected to be a more severe test of these methods for determining the equation of state. Figure 3 illustrates the mass-radius curves that are determined from these two equations of state by the OV equations.

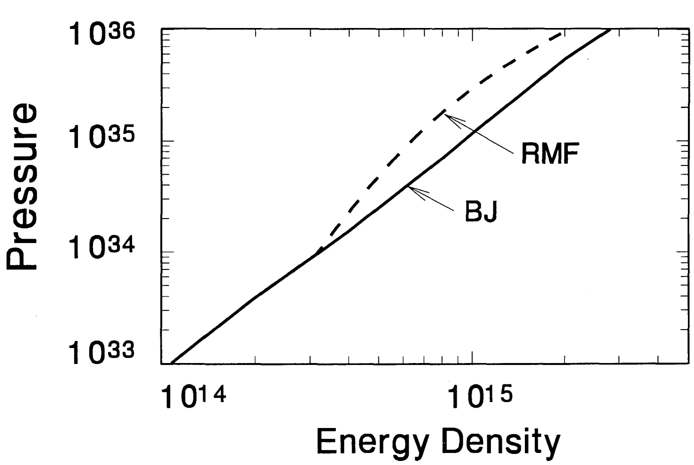

FIG. 2.—The equations of state used to study the inversion of the OV map: BJ is the Bethe & Johnson (1974) equation of state (model 1); and RMF is the relativistic-mean-field equation of state of Serot (1979).

When equations (9)–(10) are used to determine ρ_c and p_c from the mass-radius data illustrated in Figure 3, these quantities are determined quite accurately in the first step. Unfortunately, when this procedure is iterated by updating the equation-of-state table at each step by setting $\rho_{i+1} = \rho_c$, $p_{i+1} = p_c$, and $h_{i+1} = h_c$, the error after n steps is approximately $(-2)^n\epsilon$. The first-step error ϵ may be made arbitrarily small by choosing a small step size, however, the error inevitably grows unacceptably large after many iterations. Thus, the simplest application of the procedure for evaluating the equation of state from neutron star masses and radii fails.

Fortunately, a simple modification of the naive algorithm is stable and accurate. The equation of state computed with equations (9)–(11) falls alternately above and below the correct one. Thus, a line drawn between (ρ_i, p_i) and (ρ_c, p_c) always intersects the correct equation of state. And, this intersection always occurs at a point about one-third of the way between the two since the magnitude of the error doubles (approximately) at each step. Thus, the algorithm is improved by taking this (approximate) intersection point to update the equation-of-state table:

$$\rho_{i+1} = \rho_i + (\rho_c - \rho_i)/3 , \tag{12}$$

$$p_{i+1} = p_i + (p_c - p_i)/3 , \tag{13}$$

$$h_{i+1} = h_i + \frac{p_c - p_i}{18(\rho_i + p_i)}\left(7 - \frac{\rho_c + p_c}{\rho_i + p_i}\right). \tag{14}$$

The equation of state generated iteratively from the data in Figure 3 using equations (12)–(14) has a fractional error, given approximately by $\epsilon(h_{i+1} - h_i)^{5/3}$, that is essentially independent of the number of steps taken. So, this algorithm is stable. The coefficient ϵ that determines the magnitude of this error has the value $\epsilon \approx 8$ for the BJ equation of state, and the somewhat larger value $\epsilon \approx 12$ for RMF. In order to advance the equation of state studied here from $\rho \approx 10^{14}$ g cm^{-3} (where $h \approx 0.03$) to the central density of a 1.4 $M_\odot$ neutron star (where $\rho \approx 1.1 \times 10^{15}$ g cm^{-3} and $h \approx 0.26$ for BJ, or $\rho \approx 5.1 \times 10^{14}$ g cm^{-3} and $h \approx 0.18$ for RMF) with 1% accuracy requires about 10 steps using this algorithm. Thus, the equations of state determined in this way would be indistinguishable from the original curves in Figure 2 if more than about 25 equally spaced points were used to advance the equation of state from 10^{14} g cm^{-3} to the central density of a 1.4 $M_\odot$ neutron star.

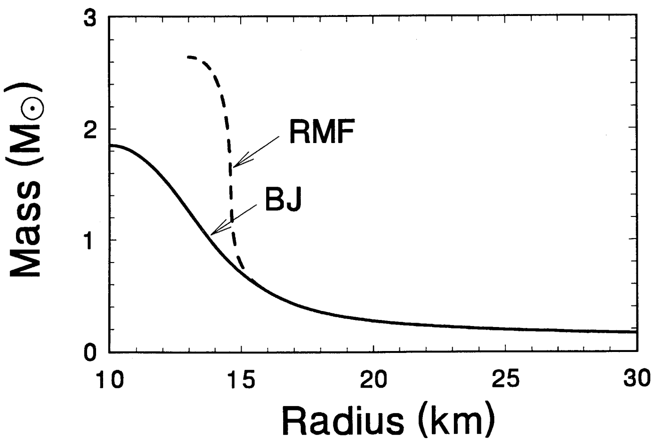

FIG. 3.—The mass-radius relationships determined by the OV equations from the BJ and the RMF equations of state.

The algorithm described above determines the equation of state from a complete mass-radius relationship. This method is not very effective, however, when mass-radius data are available from only a few neutron stars. Since the mass and radius have yet to be measured simultaneously for any individual neutron star, this method is not likely to play a useful role in interpreting observational data in the near future. What will be needed is an algorithm that is capable of deducing some interesting information about the equation of state from the mass and radius of a single neutron star. Since the entire mass-radius relationship is required to determine a complete equation of state, however, this is obviously impossible without making some severe assumptions.

A method is now presented for estimating the nuclear density equation of state from a single neutron-star mass and radius. This method is based on the following two assumptions: First, the equation of state is assumed to be known for densities below some cutoff density ρ_i, taken here to be somewhat below nuclear density. Second, the adiabatic index of the equation of state at higher densities is assumed to be a slowly varying function of the density. This second condition is satisfied by the majority of the "realistic" equations of state that have been published in the literature. However, if there is a phase transition to some other form of matter in the interiors of neutron stars (say to pion or strange-quark condensations), then this second assumption might be violated. When the second condition is satisfied, the equation of state can be approximated for densities between ρ_i and ρ_c by

$$\frac{p}{p_i} = \left(\frac{\rho}{\rho_i}\right)^{\log (p_c/p_i)/\log (\rho_c/\rho_i)} . \qquad (15)$$

Using this expression for the highest-density portion of the equation of state, it is possible to determine the structure of a not-so-small stellar core far more accurately than the power-series expressions in equations (7)–(8). Consequently it is possible to determine the central density and pressure of such a core far more accurately than was possible using equations (9)–(11). This determination may be implemented numerically as follows. First, given a neutron star mass M and radius R, make initial estimates of p_c and ρ_c using equations (9)–(11). Second, integrate the full nonlinear OV equations (5)–(6) outwardly from $h = h_c$ using equation (15) as the highest-density portion of the equation of state. Compare the mass and radius of the stellar model obtained in this way with M and R. If they do not agree to the desired degree of accuracy, then third, adjust p_c and ρ_c with the aid of the numerically evaluated Jacobian matrix $\partial(M, R)/\partial(\rho_c, p_c)$. Finally, iterate this procedure until values of p_c and ρ_c are found that reproduce the given M and R.

This one-point method of determining points on the equation-of-state curve works remarkably well. For masses and radii obtained from the BJ equation of state, and using the cutoff density $\rho_i = 10^{14}$ g cm^{-3}, this method predicts values of p_c and ρ_c that lie within 5% of the original BJ equation of state for every stellar model in the sequence. (The maximum deviation is less than 3% for stars having masses smaller than 1.8 $M_\odot$). This remarkable agreement is due of course to the fact that the BJ equation of state is very well approximated by equation (15) in this density range. For stellar models based on the RMF equation of state the agreement is also fairly good. Figure 4 illustrates the values of p_c and ρ_c determined from a number of different masses and radii using this one-point method with three different values of the cutoff density: $\rho_i = 10^{14}$, 2×10^{14}, and 3.1×10^{14} g cm^{-3}. The maximum deviations of the p_c and ρ_c so obtained are 17%, 14%, and 15% from the original RMF equation of state for the three different cutoffs, respectively. As expected, the presence of the artificial phase transition at $\rho = 3.1 \times 10^{14}$ g cm^{-3} in the RMF equation of state makes the errors far larger than they were for the BJ equation of state. This example shows, nevertheless, that meaningful distinctions between the BJ and the RMF equations of state can be made using this method for densities greater than about twice the phase-transition density. These examples suggest, then, that interesting information can be learned about the nuclear equation of state even from a knowledge of the mass and radius of a single neutron star.

In conclusion I would like to make a few comments about potential generalizations of this work. First, this paper shows how the equation of state may be determined from a knowledge of the masses and radii of stars. I point out that this analysis is applicable to any population of stars composed of fluid having a barotropic equation of state. Thus, it should apply to any population of white-dwarf stars that have uniform chemical composition, for example, pure carbon stars.

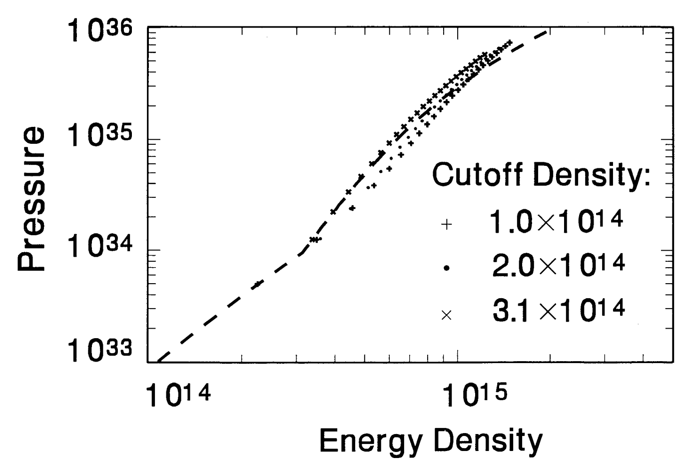

FIG. 4.—The equation of state inferred from the RMF mass-radius data using the "one-step" algorithm. Each equation-of-state point is determined from a single neutron star mass and radius, and a knowledge of the equation of state below the specified cutoff density. For comparison the original RMF equation of state is plotted as the dashed curve.

This analysis will *not* apply, however, to stars having significant internal thermal gradients, for example, main-sequence stars. Second, the equations for the inversion of the OV map are presented here as finite difference equations. While a finite difference form of the equations is needed to implement the inversion numerically, it would be interesting to find an analytic expression for the inversion—for example as an integral-differential equation. Third, the present analysis shows how the equation of state may be deduced from a knowledge of the masses and radii of neutron stars. A similar analysis could be performed to determine the equation of state from (almost) any other pair of observable neutron star parameters, for example, including moments of inertia or surface redshifts. Thus, if other parameters turn out to be easier to observe than the mass and radius, those other parameters could be used to deduce interesting information about the equation of state as well.

I thank Curt Cutler and Steve Detweiler for contributing a number of useful insights on this problem. This research was supported by NSF grant PHY-9019753 and NASA grant NAGW-2951.

REFERENCES

Abramovici, A., et al. 1992, Science, 256, 325
Arnett, W. D., & Bowers, R. L. 1977, ApJS, 33, 415
Baym, G., Pethick, C., & Sutherland, P. 1971, ApJ, 170, 299
Bethe, H. A., & Johnson, M. 1974, Nucl. Phys. A, 230, 1
Friedman, J. L., Ipser, J. R., & Parker, L. 1986, ApJ, 304, 115
Hartle, J. 1978, Phys. Rep., 46, 201
Oppenheimer, J. R., & Volkoff, G. M. 1939, Phys. Rev., 55, 374
Pandharipande, V., Pines, D., & Smith, R. A. 1976, ApJ, 208, 550
Schwarzschild, K. 1916, Sitzung. König. Preuss. Akad. Wissen., 424
Serot, B. D. 1979, Phys. Lett. B, 86, 146
Tolman, R. C. 1934, Relativity Thermodynamics and Cosmology (Oxford: Clarendon Press)