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CRITICAL ANGULAR VELOCITIES OF ROTATING NEUTRON STARS

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ABSTRACT

The critical angular velocities associated with the gravitational radiation secular instability of rotating neutron stars are computed. Corrections are given for some errors in previously published work, and new calculations are presented of the effects of post-Newtonian gravitation and hydrodynamics on these critical angular velocities.

Subject headings: hydrodynamics — instabilities — stars: neutron — stars: rotation

1. INTRODUCTION

The maximum angular velocities of rotating neutron stars (and hence the minimum pulsation periods of pulsars) are probably determined by a secular instability driven by gravitational radiation reaction (Friedman 1983; Wagoner 1984). This instability has been studied (Ipser & Lindblom 1989; 1990, hereafter Paper I; 1991, hereafter Paper II) by evaluating the appropriate modes of rotating neutron stars, and testing to see whether the imaginary parts of their frequencies are positive (stable) or negative (unstable). The imaginary part of the frequency, Im $(\omega) \equiv 1/\tau$, of any particular mode is influenced by gravitational radiation reaction and by internal fluid dissipation processes: bulk and shear viscosity. For very weak dissipation, it is possible (and convenient) to represent the imaginary part of the frequency as a sum of separate contributions from each dissipative process:

$$\frac{1}{\tau} = \frac{1}{\tau_{\rm GR}} + \frac{1}{\tau_\eta} + \frac{1}{\tau_\zeta}, \tag{1}$$

where $1/\tau_{\rm GR}$ represents the contribution due to gravitational radiation, while $1/\tau_\eta$ and $1/\tau_\zeta$ represent the contributions due to shear and bulk viscosity respectively. Explicit expressions for each of these dissipative contributions are given, for example, in Paper II in terms of the eigenfunction of the particular mode and the structure of the equilibrium rotating star.

For sufficiently small angular velocities, the imaginary parts of the frequencies of all modes are positive (in most stars), and therefore (almost all) slowly rotating stars are stable (Lindblom & Hiscock 1983). In rapidly rotating stars, however, $1/\tau$ may become negative for some mode(s), signaling the onset of an instability. The critical angular velocity associated with a particular mode is defined, then, to be the smallest angular velocity where that mode is unstable, i.e., the smallest positive root of the imaginary part of the frequency of that mode

$$\frac{1}{\tau(\Omega_c)} = 0 . \tag{2}$$

Since it is not known a priori which mode will be responsible for limiting the angular velocity of a star, however, a somewhat more elaborate definition of the critical angular velocity of a star is actually needed. The idea is that the critical angular velocity of a rotating star be defined as the maximum angular velocity at which a *stable* star of given mass may rotate. The critical angular velocity of a rotating star may be defined, therefore, to be the minimum (actually the greatest lower bound) solution to the set of equations (2) for all pulsation modes of the star.

The problem of solving the set of equations (2) explicitly to find the critical angular velocity of a particular stellar model is made simpler by considering the modes having angular dependence $e^{im\varphi}$ separately for each value of the integer m. For these modes it is possible to reexpress equation (2) in a form that is better suited to numerical solution (Lindblom 1986; Paper II),

$$\Omega_c = \frac{\omega_m(0)}{m} \times \left\{\alpha_m(\Omega_c) + \gamma_m(\Omega_c)\left[\frac{\tau_{\rm GR}(0)}{\tau_\eta(0)} + \tilde{\epsilon}_m(\Omega_c)\,\frac{\tau_{\rm GR}(0)}{\tau_\zeta(0)}\right]^{1/(2l+1)}\right\} . \tag{3}$$

The functions $\alpha_m(\Omega)$, $\gamma_m(\Omega)$, and $\tilde{\epsilon}_m(\Omega)$ describe the angular velocity dependence of the frequency (and its imaginary part) of the mode in question. These functions are defined by

$$\alpha_m(\Omega) = \frac{\omega_m(\Omega) + m\Omega}{\omega_m(0)}, \tag{4}$$

$$\gamma_m(\Omega) = \frac{\omega_m(\Omega)}{\omega_m(0)}\left[\frac{\tau_\eta(0)}{\tau_{\rm GR}(0)}\,\frac{\tau_{\rm GR}(\Omega)}{\tau_\eta(\Omega)}\right]^{1/(2l+1)}, \tag{5}$$

and

$$\tilde{\epsilon}_m(\Omega) = \frac{\tau_\zeta(0)}{\tau_\eta(0)}\,\frac{\tau_\eta(\Omega)}{\tau_\zeta(\Omega)}, \tag{6}$$

where $\omega_m(\Omega)$ represents the real part of the frequency of the mode. The functions $\alpha_m(\Omega)$, $\gamma_m(\Omega)$, and $\tilde{\epsilon}_m(\Omega)$ are defined to be dimensionless and have numerical values that are of order unity. Graphs of these functions are given in Paper II.

It is straightforward to determine the critical angular velocity associated with a particular mode by solving equation (3) once the frequency (and its imaginary part) of the nonrotating star [$\omega_m(0)$, $\tau_{\rm GR}(0)$, and $\tau_\zeta(0)$] and the functions $\alpha_m(\Omega)$, $\gamma_m(\Omega)$, and $\tilde{\epsilon}_m(\Omega)$ describing the angular velocity dependence of these frequencies are known. The techniques for determining these quantities are described in detail in Paper II. The purpose of this paper is to present corrections to some errors contained in the results presented in Paper II and to estimate the effects of post-Newtonian gravitation and hydrodynamics on the critical angular velocities based on the analysis of Cutler & Lindblom (1992).

2. CORRECTIONS

The techniques developed in Paper II for evaluating the effects of dissipation on the modes of rotating stars are correct (as far as I know). And the computer program written to implement those techniques computes correctly (as far as I know) the frequencies and damping times of the modes of rotating stars. However, the portion of the code that printed out the dissipative damping times of nonrotating stars mislabeled some of the output. In the printed output the actual dissipative damping times were confused with a "back of the envelope" estimate of these quantities (see e.g., Cutler & Lindblom 1987). Unfortunately, the printed output was interpreted literally in compiling the data contained in Table 2 of Paper II. Thus, all of the dissipative damping times contained in that table are incorrect (except for the $n = 0$ polytropes which were computed analytically). The correct values (recomputed using the techniques described in Paper II) are given here in Table 1.

All frequencies and damping times in Table 1 are given in units of $\Omega_0 \equiv (3GM/4R^3)^{1/2}$, where M is the mass of the star; R is its radius; and G is Newton's constant. We recall that for polytropes the frequency of the mode ω_m/Ω_0 is independent of M and R. In contrast the gravitational damping time $\tau_{\rm GR}\,\Omega_0$ scales as $(R/M)^{(2l+1)/2}$. All of the stellar models used in computing Table 1 have $M = 1.5\ M_\odot$ and $R = 17.171$, 14.245, 12.533, and 9.822 km for the $n = 0$, 3/4, 1, and 5/4 polytropes, respectively. The viscosities of neutron star matter are temperature dependent (see Paper II). The damping times reported here in Table 1 assume a temperature of $T = 10^9$ K. The viscous damping time due to neutron-neutron scattering $\tau_{\eta_n}\Omega_0$ scales as $R^{17/4}T^2/M^{3/4}$; the damping time due to electron-electron scattering $\tau_{\eta_e}\Omega_0$ scales as $R^{7/2}T^2/M^{1/2}$; and the bulk viscous damping time $\tau_\zeta\Omega_0$ scales as $M^{1/2}R^{1/2}/T^6$. These scalings can be deduced from the analytical expressions for the various quantities given in Paper II, and these scalings have been verified in the numerical output of the computer program.

The error in the dissipative damping times of nonrotating stars did not affect any of the results pertaining to the functions $\alpha_m(\Omega)$, $\gamma_m(\Omega)$, or $\epsilon_m(\Omega)$ that were presented in Paper II. Thus, Figures 1–16 of Paper II are correct (as far as I know). I also point out that the formula used in this work (and previously in Cutler, Lindblom, & Splinter 1990, and in Paper II) for the bulk viscosity is $\zeta = 6.0 \times 10^{25}\ (\rho_{15}/\omega)^2 T_9^6$. Sawyer's (1989) published version of this formula contains a typographical error (Sawyer 1994). Fortunately the formula used in our work was free of that error.

The critical angular velocities of rotating neutron stars presented in Paper II were determined by finding the smallest root of equation (3) above for the modes that are expected to contribute most to the gravitational radiation secular instability: the $l = m$ modes with $2 \le m \le 6$. Equation (3) depends on the damping times of nonrotating stars $\tau_{\rm GR}(0)$, $\tau_\eta(0)$, and $\tau_\zeta(0)$ and consequently the critical angular velocities reported in Paper II are incorrect. Figures 1–3 give corrected representations of the critical angular velocities presented originally as Figures 17–19 of Paper II. These figures give the critical angular velocities as functions of the temperature of the neutron star matter. The critical angular velocities are given in units of $\Omega_{\rm max}$ the maximum angular velocity for which an equilibrium stellar model exists. These maximum angular velocities have the values $\Omega_{\rm max} = 0.648\Omega_0$, $0.639\Omega_0$, and $0.626\Omega_0$ for the $n = 3/4$, 1, and 5/4 polytropes, respectively. The revised curves are qualitatively similar to those given in Paper II. However, the revised critical angular velocities are somewhat larger (by about 2% of $\Omega_{\rm max}$) and the temperature where the critical angular velocity is smallest has been lowered from about 5×10^9 to about 2×10^9 K. The main reason for these changes is that the bulk viscous damping times reported in

TABLE 1

DAMPING TIMES AND PULSATION FREQUENCIES FOR 1.5 $M_\odot$ NONROTATING POLYTROPES

$l = m$	n[a]	$\omega(0)$[b]/Ω_0	$\tau_{\rm GR}\,\Omega_0$	$\tau_{\eta_n}\Omega_0$	$\tau_{\eta_e}\Omega_0$	$\tau_\zeta\Omega_0$
2......	0	1.033	2.84×10^3	1.07×10^{13}	3.80×10^{12}	...
	3/4	1.291	9.79×10^2	4.13×10^{12}	1.05×10^{12}	4.55×10^{15}
	1	1.416	5.60×10^2	1.73×10^{12}	5.08×10^{11}	1.49×10^{15}
	5/4	1.543	2.43×10^2	4.39×10^{11}	1.63×10^{11}	5.90×10^{14}
3......	0	1.512	1.65×10^5	3.81×10^{12}	1.36×10^{12}	...
	3/4	1.822	4.66×10^4	2.19×10^{12}	5.25×10^{11}	3.38×10^{15}
	1	1.960	2.40×10^4	1.05×10^{12}	2.85×10^{11}	1.19×10^{15}
	5.4	2.095	8.49×10^3	3.06×10^{11}	1.03×10^{11}	5.12×10^{14}
4......	0	1.886	1.03×10^7	1.98×10^{12}	7.04×10^{11}	...
	3/4	2.212	2.44×10^6	1.52×10^{12}	3.48×10^{11}	3.34×10^{15}
	1	2.351	1.14×10^6	8.05×10^{11}	2.06×10^{11}	1.21×10^{15}
	5/4	2.480	3.28×10^5	2.61×10^{11}	8.16×10^{10}	5.61×10^{14}
5......	0	2.202	7.35×10^8	1.21×10^{12}	4.32×10^{11}	...
	3/4	2.531	1.47×10^8	1.18×10^{12}	2.59×10^{11}	3.24×10^{15}
	1	2.667	6.17×10^7	6.73×10^{11}	1.63×10^{11}	1.32×10^{15}
	5/4	2.789	1.45×10^7	2.38×10^{11}	6.96×10^{10}	6.53×10^{14}
6......	0	2.481	5.94×10^{10}	8.23×10^{11}	2.92×10^{11}	...
	3/4	2.806	1.00×10^{10}	9.63×10^{11}	2.05×10^{11}	3.58×10^{15}
	1	2.939	3.82×10^9	5.88×10^{11}	1.37×10^{11}	1.48×10^{15}
	5/4	3.053	7.27×10^8	2.23×10^{11}	6.17×10^{10}	7.72×10^{14}

[a] The index n is the parameter in the polytropic equation of state: $p = \kappa\rho^{1+1/n}$.

[b] The frequencies and damping times are given in units of $\Omega_0 = (\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star.

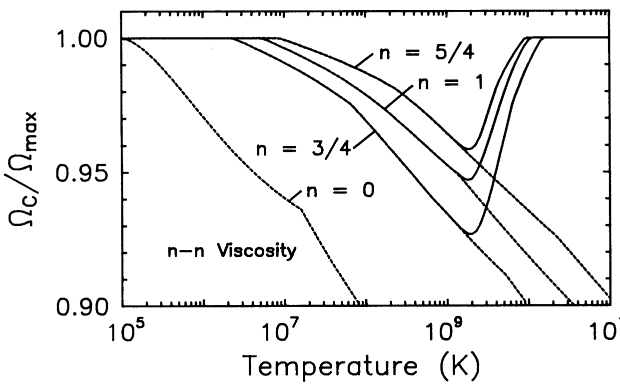

FIG. 1.—Critical angular velocities Ω_c as functions of the temperature for 1.5 $M_\odot$ polytropes of indices $n = 0$, 3/4, 1, and 5/4. The angular velocities are given in units of Ω_{max}, the maximum angular velocity for which an equilibrium stellar model exists of the same mass. The dashed curves ignore the effects of bulk viscosity.

Paper II were considerably too large. Thus, bulk viscosity is more efficient in suppressing the gravitational radiation instability than reported previously, and this leads to larger critical angular velocities.

3. POST-NEWTONIAN EFFECTS

The mathematical techniques needed to incorporate the effects of general relativity theory (in the post-Newtonian approximation) into the description of the pulsations and stability of rotating stellar models have been developed by Cutler (1991) and by Cutler & Lindblom (1992). And those mathematical techniques have been used to compute the angular velocity dependence of the frequencies, $\omega_m(\Omega)$, of the modes that contribute strongly to the gravitational radiation secular instability. In particular $\omega_m(0)$ and the functions $\alpha_m(\Omega)$ were determined for $n = 1$ polytropes in the post-Newtonian approximation (Cutler & Lindblom 1992) for the $l = m$ modes with $2 \leq m \leq 6$. These improved frequencies can be used along with the Newtonian values for the dissipative damping times [$\tau_{GR}(0)$, $\tau_\eta(0)$, and $\tau_\zeta(0)$] and the functions $\gamma_m(\Omega)$ and $\tilde{\epsilon}_m(\Omega)$ to determine "improved" estimates of the critical angular velocities of rotating neutron stars from equation (3). The results of such a calculation are shown in Figure 4.

The post-Newtonian effects included here cause the gravitational radiation secular instability to set in at significantly lower angular velocities than suggested by the purely Newtonian calculation! This additional instability is caused by the frequencies of the modes $\omega_m(\Omega)$ passing through zero at lower angular velocities, Ω/Ω_0, in the post-Newtonian calculation. This change in sign of the frequency is what drives the gravitational radiation secular instability. The results presented in Figure 4 are not a complete post-Newtonian description of the pulsations, however. Post-Newtonian effects should also be included in the calculation of the dissipation timescales [$\tau_{GR}(0)$, $\tau_\eta(0)$, and $\tau_\zeta(0)$] and the functions $\gamma_m(\Omega)$ and $\tilde{\epsilon}_m(\Omega)$. Although it is impossible to predict a priori whether these additional post-Newtonian effects will tend to strengthen or to suppress the secular instability, these preliminary results do provide a strong incentive for completing the post-Newtonian analysis.

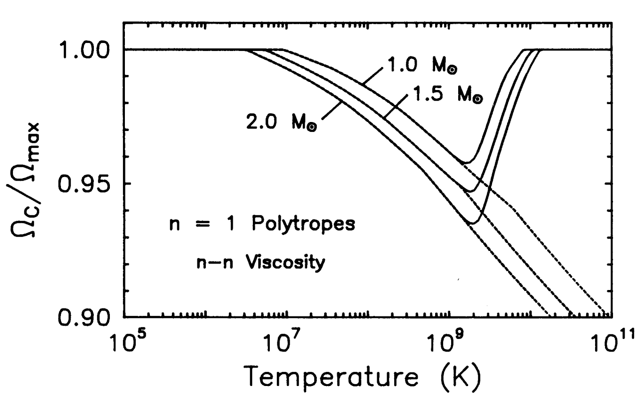

FIG. 2.—Critical angular velocities Ω_c as functions of the temperature for 1.0, 1.5, and 2.0 $M_\odot$ polytropes of index $n = 1$. The angular velocities are given in units of Ω_{max}, the maximum angular velocity for which an equilibrium stellar model exists of the same mass. The dashed curves ignore the effects of bulk viscosity.

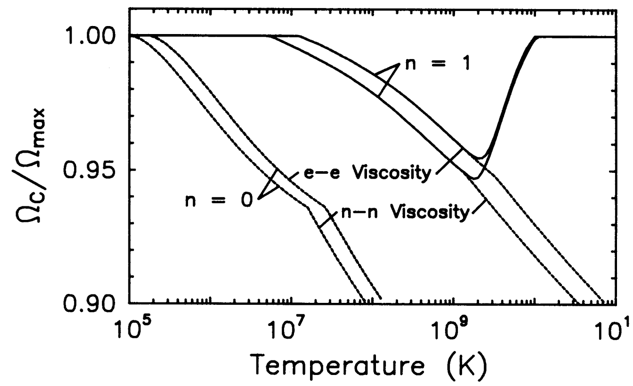

FIG. 3.—Critical angular velocities Ω_c as functions of the temperature based on two different expressions for the shear viscosity: neutron-neutron scattering viscosity η_n and electron-electron scattering viscosity η_e. The angular velocities are given in units of Ω_{max}, the maximum angular velocity for which an equilibrium stellar model exists of the same mass. The dashed curves ignore the effects of bulk viscosity.

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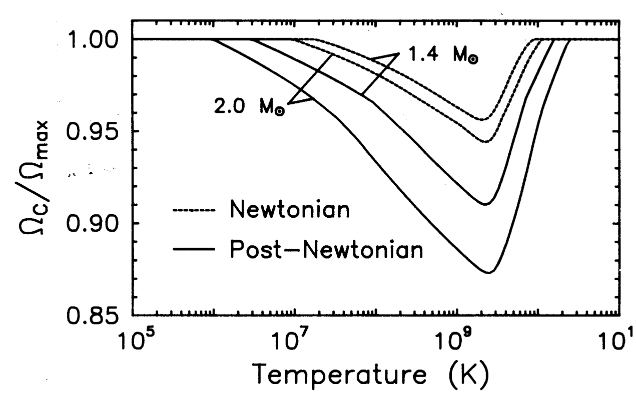

FIG. 4.—Critical angular velocities Ω_c as functions of the temperature for 1.4 and 2.0 $M_\odot$ polytropes of index $n = 1$ using electron-electron scattering shear viscosity. The dashed curves use Newtonian gravitation and hydrodynamics to evaluate the modes while the solid curves use post-Newtonian gravitation and hydrodynamics. The angular velocities are given in units of $\Omega_{max} = 0.639\Omega_0$, the maximum angular velocity for which an equilibrium stellar model exists in the Newtonian theory.

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