Gravitational Radiation Instability in Hot Young Neutron Stars

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We show that gravitational radiation drives an instability in hot young rapidly rotating neutron stars. This instability occurs primarily in the \( l = 2 \) \( r \)-mode and will carry away most of the angular momentum of a rapidly rotating star by gravitational radiation. On the time scale needed to cool a young neutron star to about \( T = 10^9 \) K (about one year) this instability can reduce the rotation rate of a rapidly rotating star to about 0.076\( \Omega_K \), where \( \Omega_K \) is the Keplerian angular velocity where mass shedding occurs. In older colder neutron stars this instability is suppressed by viscous effects, allowing older stars to be spun up by accretion to larger angular velocities.

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Recently Andersson [1] discovered (and Friedman and Morsink [2] confirmed more generally) that gravitational radiation couples these modes primarily through the current multipoles, rather than the usual mass multipoles. We also evaluate the effects of internal fluid dissipation which tends to suppress this instability. We find that gravitational radiation is stronger than viscosity dissipation which tends to suppress this instability. We show that gravitational radiation drives an instability in hot young rapidly rotating neutron stars. This instability occurs primarily in the \( l = 2 \) \( r \)-mode and will carry away most of the angular momentum of a rapidly rotating star by gravitational radiation. On the time scale needed to cool a young neutron star to about \( T = 10^9 \) K (about one year) this instability can reduce the rotation rate of a rapidly rotating star to about 0.076\( \Omega_K \), where \( \Omega_K \) is the Keplerian angular velocity where mass shedding occurs. In older colder neutron stars this instability is suppressed by viscous effects, allowing older stars to be spun up by accretion to larger angular velocities.

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The \( r \)-modes of rotating barotropic Newtonian stars are solutions of the perturbed fluid equations having (Eulerian) velocity perturbations

\[
\delta \bar{v} = \alpha R \Omega \left( \frac{r}{R} \right)^l \bar{Y}^B_{lm} e^{i\omega t},
\]

where \( R \) and \( \Omega \) are the radius and angular velocity of the unperturbed star, \( \alpha \) is an arbitrary constant, and \( \bar{Y}^B_{lm} \) is the magnetic-type vector spherical harmonic defined by

\[
\bar{Y}^B_{lm} = [l(l + 1)]^{-1/2} r \bar{\nabla} \times (r \bar{\nabla} Y_{lm}).
\]

Equation (4) is the complete expression for \( \delta \rho \) to order \( \Omega^2 \). The next order terms are proportional to \( \Omega^4 \).

Our interest here is to study the evolution of these modes due to the dissipative influences of viscosity and gravitational radiation. For this purpose it is useful to consider the effects of radiation on the evolution of the energy of the mode (as measured in the corotating frame)

\[
E = \frac{1}{2} \int \rho \delta \bar{v} \cdot \delta \bar{v}^* + \left( \frac{\delta \rho}{\rho} - \delta \Phi \right) \delta \rho^* d^3 x.
\]

This energy evolves on the secular time scale of the dissipative processes. The general expression for the time...
derivative of $\dot{E}$ for a mode with time dependence $e^{i\omega t}$ and azimuthal angular dependence $e^{im\phi}$ is
\[
\frac{d\dot{E}}{dt} = -\int (2\eta \delta\sigma^{ab}\delta\sigma^{*}_{ab} + \xi \delta\sigma^{*}\delta\sigma) d^3x
- \omega(\omega + m\Omega) \sum_{l=2} N_l \omega^{2l} (|\delta D_{lm}|^2 + |\delta J_{lm}|^2).
\] (7)

The thermodynamic functions $\eta$ and $\xi$ that appear in Eq. (7) are the shear and bulk viscosities of the fluid. (The viscous forces are driven by the shear $\delta\sigma_{ab}$ and expansion $\delta\sigma$ of the perturbation, defined by the usual expressions $\delta\sigma_{ab} = \frac{1}{2}(\nabla_a \delta\nu_b + \nabla_b \delta\nu_a - \frac{2}{3} \delta\sigma_{ab} \nabla_c \delta\nu^c)$, (8)
\[\delta\sigma = \nabla_a \delta\nu^a.\] (9)

Gravitational radiation couples to the evolution of the mode through the mass $\delta D_{lm}$ and current $\delta J_{lm}$ multipole moments of the perturbed fluid,
\[\delta D_{lm} = \int \delta\rho r^l \tilde{Y}_{lm}^* d^3x,\] (10)
\[\delta J_{lm} = \frac{2}{c} \sqrt{\frac{l}{l+1}} \int r^l (\delta\rho \tilde{\delta\nu} + \delta\rho \tilde{\nu} \cdot \tilde{Y}_{lm}^* d^3x,\] (11)
with coupling constant
\[N_l = \frac{4\pi G (l+1)(l+2)}{c^{2l+1} [(l+1)!]^2} \Omega^2 \equiv \frac{4\pi G (l+1)(l+2)}{c^{2l+1} [(l+1)!]^2} \Omega^2 < 0.\] (12)

The terms in the expression for $d\dot{E}/dt$ due to viscosity and the gravitational radiation generated by the mass multipoles are well known [6]. The terms involving the current multipole moments have been deduced from the general expressions given by Thorne [7].

We can now use Eq. (7) to evaluate the stability of the r-modes. Viscosity always tends to decrease the energy $\dot{E}$, while gravitational radiation may either increase or decrease it. The sum that appears in Eq. (7) is positive definite; thus the effect of gravitational radiation is determined by the sign of $\omega(\omega + m\Omega)$. For r-modes this quantity is negative definite:
\[\omega(\omega + l\Omega) = -\frac{2(l-1)(l+2)}{(l+1)^2} \Omega^2 < 0.\] (13)
Therefore gravitational radiation tends to increase the energy of these modes. For small angular velocities the energy $\dot{E}$ is positive definite: The positive term $|\delta\tilde{\nu}|^2$ in Eq. (6) (proportional to $\Omega$) dominates the indefinite term $(\delta\rho/\rho - \delta\Phi) |\delta\rho|^2$ (proportional to $\Omega^4$). Thus, gravitational radiation tends to make every r-mode unstable in slowly rotating stars. This confirms the discovery of Andersson [1] and the more general arguments of Friedman and Morsink [2]. To determine whether these modes are actually stable or unstable in rotating neutron stars, therefore, we must evaluate the magnitudes of all the dissipative terms in Eq. (7) and determine which dominates.

Here we estimate the relative importance of these dissipative effects in the small angular velocity limit using the lowest order expressions for the r-mode $\delta\tilde{\nu}$ and $\delta\rho$ given in Eqs. (1) and (4). The lowest order expression for the energy of the mode $\dot{E}$ is
\[\dot{E} = \frac{1}{2} \frac{\dot{E}}{\Omega} \int_0^R \rho r^{2l+2} dr.\] (14)

The lowest order contribution to the gravitational radiation terms in the energy dissipation comes entirely from the current multipole moment $\delta J_{ll}$. This term can be evaluated to lowest order in $\Omega$ using Eqs. (1) and (11):
\[\delta J_{ll} = \frac{2\alpha \Omega}{c R^{l+1}} \int \frac{l}{l+1} \int_0^R r^4 \rho r^{2l+2} dr.\] (15)

The other contributions from gravitational radiation to the dissipation rate are all higher order in $\Omega$. The mass multipole moment contributions are higher order because (a) the density perturbation $\delta\rho$ from Eq. (4) is proportional to $\Omega^2$ while the velocity perturbation $\delta\tilde{\nu}$ is proportional to $\Omega$; and (b) the density perturbation $\delta\rho$ generates gravitational radiation at order $2l + 4$ in $\omega$ while $\delta\tilde{\nu}$ generates radiation at order $2l + 2$.

The contribution of gravitational radiation to the imaginary part of the frequency of the mode $1/\tau_{\text{GR}}$ can be computed as follows:
\[1/\tau_{\text{GR}} = -\frac{1}{2\dot{E}} \frac{d\dot{E}}{dt}.\] (16)

Using Eqs. (14)–(16) we obtain an explicit expression for the gravitational radiation time scale associated with the r-modes:
\[\frac{1}{\tau_{\text{GR}}} = -\frac{32\pi G \Omega^{2l+2}}{c^{2l+3}} \times \frac{(l-1)!}{(2l+1)!} \left[\frac{l+2}{l+1}\right]^{2l+2} \int_0^R \rho r^{2l+2} dr.\] (17)

The time derivative of the energy due to viscous dissipation is driven by the shear $\delta\sigma_{ab}$ and the expansion $\delta\sigma$ of the velocity perturbation. The shear can be evaluated using Eqs. (1) and (8) and its integral over the constant r two-spheres performed in a straightforward calculation. Using the formulas for the viscous dissipation rate Eq. (7) and the energy Eq. (14), we obtain the contribution of shear viscosity to the imaginary part of the frequency of the mode,
\[\frac{1}{\tau_v} = (l-1)(2l+1) \int_0^R \eta r^{2l} dr \int_0^R \rho r^{2l+2} dr.\] (18)

The expansion $\delta\sigma$, which drives the bulk viscosity dissipation in the fluid, can be reexpressed in terms of the density perturbation. The perturbed mass conservation law gives the relationship $\delta\sigma = -i(\omega + m\Omega)\Delta\rho/\rho$, where $\Delta\rho$ is the Lagrangian perturbation in the density. The perturbation analysis used here is not of sufficiently
We have evaluated these time scales for neutron stars as well as the imaginary part of the frequency due to bulk viscosity. The dissipation time scales for this polytropic model discussed above are 1.4 $\times 10^5$ s for $\Omega = 0.083$ and 3.26 s for $\Omega = 0.043$. The maximum angular velocity for any star occurs when the material at the surface effectively orbits the star. This “Keplerian” angular velocity $\Omega_K$ is very nearly $\frac{1}{2} \sqrt{\pi G \rho}$ for any equation of state. Thus the minimum critical angular velocity due to instability of the r-modes is about 0.065$\Omega_K$ for any equation of state [10].

To determine how rapidly a young neutron star is allowed to spin after cooling, we must compare the rate it cools with the rate it loses angular momentum sufficiently small angular velocities, viscosity dominates and the mode is stable. For sufficiently large $\Omega$, however, gravitational radiation will dominate and drive the mode unstable. It is convenient to define a critical angular velocity $\Omega_c$, where the sign of the imaginary part of the frequency changes from positive to negative: $1/\tau(\Omega_c) = 0$. If the angular velocity of the star exceeds $\Omega_c$ then gravitational radiation reaction dominates viscosity and the mode is unstable.

For a given temperature and mode $l$ the equation for the critical angular velocity, $0 = 1/\tau(\Omega_c)$, is a polynomial of order $l + 1$ in $\Omega_c^2$, and thus each mode has its own critical angular velocity. However, only the smallest of these (always the $l = 2$ r-mode here) represents the critical angular velocity of the star. Figure 1 depicts the critical angular velocity for a range of temperatures relevant for neutron stars. The solid curve in Fig. 1 represents the critical angular velocity for the polytropic model discussed above. Figure 2 depicts the critical angular velocities for 1.4$M_\odot$ neutron star models computed from a variety of realistic equations of state [8]. Figure 2 illustrates that the minimum critical angular velocity (in units of $\sqrt{\pi G \rho}$) is extremely insensitive to the equation of state. The minima of these curves occur at $T = 2 \times 10^9$ K, with $\Omega_c = 0.043 \sqrt{\pi G \rho}$. The maximum angular velocity for any star occurs when the material at the surface effectively orbits the star. This “Keplerian” angular velocity $\Omega_K$ is very nearly $\frac{1}{2} \sqrt{\pi G \rho}$ for any equation of state. Thus the minimum critical angular velocity due to instability of the r-modes is about 0.065$\Omega_K$ for any equation of state [10].
by gravitational radiation. We approximate the cooling with a simple model based on the emission of neutrinos through the modified URCA process [11]. We compute the time evolution of the angular velocity of the star by setting $dJ/dt = J/\tau$, where $J$ is the angular momentum of the star and $\tau$ is the time scale given in Eq. (22). The result is a simple first order differential equation for $\Omega(t)$ which we solve for initial angular velocity $\Omega = \Omega_K$ and initial temperature $10^{14}$ K. The solution is shown as the dash-dotted line in Fig. 1. The gravitational radiation time scale is so short that the star radiates away its angular momentum almost as quickly as it cools. The angular velocity of the star decreases from $\Omega_K$ to 0.076$\Omega_K$ in a period of about one year [12]. Thus, we conclude that young neutron stars will be spun down by the emission of gravitational radiation within their first year to a rotation period of about $13P_{\text{min}}$, where $P_{\text{min}} = 2\pi/\Omega_K$. The Crab pulsar with present rotation period 33 ms and initial period 19 ms (based on the measured braking index) rotates more slowly than this limit if $P_{\text{min}} < 1.5$ ms.

Our analysis is based on the assumption that a young hot neutron star may be modeled as an ordinary fluid. Once the star cools below the superfluid transition temperature (about $10^9$ K) the analysis presented here must be modified [13]. We expect the $r$-mode instability to be completely suppressed (with $\Omega_c = \Omega_K$) when the star becomes a superfluid [14]. This makes it possible for old recycled pulsars to be spun up to large angular velocities by accretion if they are not reheated above $10^9$ K in the process. If nonperfect fluid effects enter above $10^9$ K, however, the spin-down process may be terminated at a higher angular velocity than the 0.076$\Omega_K$ figure computed here. The detection of a young fast pulsar [15] would provide evidence for such effects at temperatures higher than $10^9$ K. Magnetic fields could also damp these modes; however preliminary estimates based on standard magnetosphere-mode coupling models [16] suggest that such damping is too weak to suppress the relatively low frequency $r$-mode instability.

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[10] Note that the minimum $\Omega_c$ is rather small, and thus the small $\Omega$ expansions used here are expected to be quite good. Also, the approximation used to evaluate $\tau_B$ in Eq. (19) is not expected to have a large effect on this minimum value; e.g., if $\tau_B$ were off by a factor of 10, then the high $T$ portion of the curve in Fig. 1 would be moved along the $T$ axis by a factor of about $1.5 \approx (10)^{1/6}$. Since the shear viscosity influence on the curve (the dashed line in Fig. 1) is so flat in this temperature range, the minimum value of $\Omega_c$ would not be significantly changed.
[11] S. L. Shapiro and S. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars (Wiley, New York, 1983). Viscosity also converts rotational energy into heat. However, in the high temperature range this energy is converted primarily by bulk viscosity into neutrinos which are quickly radiated away. We do not expect this rotational reheating to significantly impede the cooling process.
[12] Our assumption that the spin-down time scale is $\tau$ of Eq. (22) is based on the fact that the $r$-mode will grow (within a few minutes) to a point where the mode contains a substantial fraction of the total angular momentum of the star before being saturated by nonlinear effects. Equation (22) is a low $\Omega$ expansion so the early time values of $\tau$ are also somewhat uncertain. Fortunately, the final angular velocity state of the star is quite insensitive to the details of the cooling and spin-down.
[13] The formation of a solid crust below $\rho = 2 \times 10^{14}$ g/cm$^3$ also affects these modes; however, this density appears to be too low to affect the $r$-mode instability.