I. INTRODUCTION

Recently the $r$-modes have been found to play an interesting and important role in the evolution of hot young rapidly rotating neutron stars. Andersson [1] and Friedman and Morsink [2] were the first to show that gravitational radiation tends to drive the $r$-modes unstable in all rotating stars. Lindblom, Owen, and Morsink [3] then showed that the coupling of gravitational radiation to the $r$-modes is sufficiently strong to overcome internal fluid dissipation effects and so drive these modes unstable in hot young neutron stars. This result has been verified by Andersson, Kokkotas, and Schutz [4]. This result seemed somewhat surprising at first because the dominant coupling of gravitational radiation to the $r$-modes is through the current multipoles rather than the more familiar and usually dominant mass multipoles. But it is now generally accepted that gravitational radiation does drive unstable any hot young neutron star with angular velocity greater than about 5% of the maximum (the angular velocity where mass shedding occurs). This instability therefore provides a natural explanation for the lack of observed very fast pulsars associated with young supernova remnants.

The $r$-mode instability is also interesting as a possible source of gravitational radiation. In the first few minutes after the formation of a hot young rapidly rotating neutron star in a supernova, gravitational radiation will increase the amplitude of the $r$-mode (with spherical harmonic index $m = 2$) to levels where non-linear hydrodynamic effects become important in determining its subsequent evolution. While the non-linear evolution of these modes is not well understood as yet, Owen et al. [5] have developed a simple non-linear evolution model to describe it approximately. This model predicts that within about one year the neutron star spins down (and cools down) to an angular velocity (and temperature) low enough that the instability is again suppressed by internal fluid dissipation. All of the excess angular momentum of the neutron star is radiated away via gravitational radiation. Owen et al. [5] estimate the detectability of the gravitational waves emitted during this spindown, and find that neutron stars spinning down in this manner may be detectable by the second-generation (“enhanced”) LIGO interferometers out to the Virgo cluster. Bildsten [6] and Andersson, Kokkotas, and Stergioulas [7] have raised the possibility that the $r$-mode instability may also operate in older colder neutron stars spun up by accretion in low-mass x-ray binaries. The gravitational waves emitted by some of these systems (e.g. Sco X-1) may also be detectable by enhanced LIGO [8]. Thus, the $r$-modes of rapidly rotating neutron stars have become a topic of considerable interest in relativistic astrophysics.

The purpose of this paper is to explore further the properties of the $r$-modes of rotating neutron stars. The initial analyses of the $r$-mode instability [1–3] were based on a small angular-velocity expansion for these modes developed originally by Papaloizou and Pringle [9]. This expansion in powers of the angular velocity kept only the lowest-order terms in the expressions for the various quantities associated with the mode: the frequency, velocity perturbation, etc. This lowest-order expansion is sufficient to explore many of the interesting physical properties of these modes, including the gravitational radiation instability. However, some important physical quantities vanish at lowest order and hence a second-order analysis is needed [10]. For example the coupling of the $r$-modes to bulk viscosity vanishes in the lowest-order expansion. Estimates of this important bulk-viscosity coupling to the $r$-modes have been given by Lindblom,
Owen, and Morsink [3], Andersson, Kokkotas, and Schutz [4,11], and Kokkotas and Stergioulas [12]. But (as discussed in more detail in Sec. VI below) none of these is based on the fully self-consistent second-order calculation needed to evaluate this coupling properly. Since bulk viscosity is expected to be the dominant internal fluid dissipation mechanism in hot young neutron stars, it is important to extend the analysis so that this important physical effect can be evaluated accurately.

The dominant internal fluid dissipation mechanism in neutron stars colder than about $10^9$ K is thought to be a superfluid effect called mutual friction [13] caused by the scattering of electrons off the magnetic fields in the cores of vortices. Levin [14] has shown that the importance of the $r$-mode instability in low-mass x-ray binaries depends crucially on the details of the mutual friction damping of these modes. Unfortunately the mutual friction dissipation also vanishes at lowest order in a small angular velocity expansion of the superfluid $r$-modes. Thus in order to evaluate this effect properly, it is also necessary to determine the structure of the $r$-modes of superfluid neutron stars through second order in the angular velocity. This provides another motivation for developing the tools needed to evaluate the second-order rotational effects in the $r$-modes.

In this paper we develop a new formalism for exploring the higher-order rotational effects in the $r$-modes. Our analysis is based on the two-potential formalism [15] in which all physical properties of a mode of a rotating star are expressed in terms of two scalar potentials: a hydrodynamic potential $\delta U$ and the gravitational potential $\delta \Phi$. We define a small angular velocity expansion for the $r$-modes in terms of these potentials, and derive the equations explicitly for the second-order terms. This expansion provides a straightforward and relatively simple way to determine the second-order effects, such as the bulk viscosity coupling, which are of interest to us here. The equations that determine the second-order terms in the $r$-modes form an inhomogeneous hyperbolic boundary value problem that is not amenable to solution by standard numerical techniques. Therefore we have developed new numerical techniques which could well have applications beyond the present problem. In particular these techniques will also be needed to solve the analogous superfluid pulsation equations that determine the effects of mutual friction on these modes.

The time scales derived here for the bulk viscosity damping of the $r$-modes differ considerably from earlier estimates. We find the bulk-viscosity coupling to these modes to be weaker for normal neutron stars than any previous estimates. Consequently the gravitational radiation-driven instability is somewhat more effective at driving unstable the $r$-modes in hot young neutron stars than earlier estimates suggested. Although quantitatively different from earlier estimates, our new values for the bulk-viscosity damping time do not substantially alter the expected spindown scenario in hot young neutron stars. We re-evaluate the critical angular velocity curve (above which the $r$-mode instability sets in) and find no qualitative change from earlier estimates. Our new value for the minimum critical angular velocity is somewhat lower than earlier estimates: about 5% compared to about 8% of the maximum. In very hot young neutron stars there is the possibility that bulk viscosity could re-heat the neutron star (due in part to non-linear effects in the bulk viscosity) and so suppress the instability to some extent [16]. This could result in a significant increase in the time scale required to spin down young neutron stars, and could therefore decrease significantly the detectability of the gravitational radiation emitted. Our new calculation of the bulk viscosity time scale indicates that reheating will not be a major factor in the evolution of young neutron stars. Our calculations also show that the bulk-viscosity coupling in strange stars is somewhat stronger than the initial estimates by Madsen [17]. We find that bulk viscosity completely suppresses the $r$-mode instability in strange stars hotter than $T \approx 5 \times 10^8$ K, in good qualitative agreement with Madsen.

In Sec. II we review the structure of equilibrium stellar models through second order in the angular velocity of the star. In Sec. III we review the two-potential formalism for describing the modes of rotating stars, and derive the small angular velocity expansion of these equations through second order. In Sec. IV we focus our attention on the “classical” $r$-modes, the modes found previously to be subject to the gravitational radiation driven instability. We obtain analytical expressions for the second-order corrections to the frequencies of these modes, and present numerical results for polytropes and for more realistic neutron star models. In Sec. V we develop the numerical techniques needed to find the second-order eigenfunctions for the $r$-modes; we use those techniques to find those eigenfunctions, and we present the results graphically. In Sec. VI we use our new second-order expressions for the $r$-modes to compute the effects of bulk viscosity on the evolution of these modes. In the Appendix we discuss the convergence of the numerical relaxation technique used in Sec. V to solve the unusual hyperbolic boundary value problem for the second-order eigenfunctions.

II. SLOWLY ROTATING STELLAR MODELS

Our analysis of the $r$-modes of rotating stars is based on expanding the equations as power series in the angular velocity $\Omega$ of the star. The first step therefore in obtaining these equations is to find the structures of equilibrium stellar models in a similar power series expansion. This section describes how to solve the equilibrium structure equations for uniformly rotating barotropic stars in such a slow rotation expansion. The solutions will be obtained here up to and including the terms of order $\Omega^2$.

Let $h(p)$ denote the thermodynamic enthalpy of the barotropic fluid:

$$h(p) = \int_0^p \frac{dp'}{\rho(p')}.$$  \hfill (2.1)

where $p$ is the pressure and $\rho$ is the density of the fluid. This definition can always be inverted to determine $p(h)$. The barotropic equation of state, $p = \rho(h)$, then determines $\rho(h) = \rho[p(h)]$. The equations which determine the family
of stationary, axisymmetric uniformly rotating barotropic stellar models are Euler’s equation, which for this case has the simple form

$$0 = \nabla \cdot [h - \frac{1}{2} r^2 (1 - \mu^2) \Omega^2 - \Phi],$$  \hspace{0.5cm} (2.2)

and the gravitational potential equation,

$$\nabla^a \nabla_a \Phi = - 4 \pi G \rho.$$  \hspace{0.5cm} (2.3)

In these expressions \( r \) and \( \mu = \cos \theta \) are the standard spherical coordinates, and \( \Phi \) is the gravitational potential.

We seek solutions to Eqs. (2.2) and (2.3) as power series in the angular velocity \( \Omega \). To that end, we define

$$h(r, \mu) = h_0(r) + h_2(r, \mu) \frac{\Omega^2}{\pi G \rho_0} + \mathcal{O}(\Omega^4),$$  \hspace{0.5cm} (2.4)

$$\rho(r, \mu) = \rho_0(r) + \rho_2(r, \mu) \frac{\Omega^2}{\pi G \rho_0} + \mathcal{O}(\Omega^4),$$  \hspace{0.5cm} (2.5)

$$\Phi(r, \mu) = \Phi_0(r) + \Phi_2(r, \mu) \frac{\Omega^2}{\pi G \rho_0} + \mathcal{O}(\Omega^4),$$  \hspace{0.5cm} (2.6)

where \( \rho_0 \) is the average density of the non-rotating star in the family. Using these expressions then, the first two terms in the solution to Eq. (2.2) are given by

$$C_0 = h_0(r) - \Phi_0(r),$$  \hspace{0.5cm} (2.7)

$$C_2 = h_2(r, \mu) - \frac{1}{2} \pi G \rho_0 r^2 (1 - \mu^2) - \Phi_2(r, \mu),$$  \hspace{0.5cm} (2.8)

where \( C_0 \) and \( C_2 \) are constants. The non-rotating model can be determined in the usual way by solving the gravitational potential equation,

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = - 4 \pi G \rho_0,$$  \hspace{0.5cm} (2.9)

together with Eq. (2.7). The integration constant, \( C_0 \), can be shown to be \( C_0 = - GM_0/R_0 \) by evaluating Eq. (2.7) at the surface of the star. The constants \( M_0 \) and \( R_0 \) are the mass and radius of the non-rotating star.

The second-order contributions to the stellar structure are determined by solving the gravitational potential, Eq. (2.3), together with Eq. (2.8). The second-order density perturbation \( \rho_2 \) is related to \( h_2 \) by

$$\rho_2(r, \mu) = \frac{1}{\rho_0} \frac{d\rho}{dh} h_2(r, \mu).$$  \hspace{0.5cm} (2.10)

Thus using Eq. (2.8), the equation for the second-order gravitational potential can be written in the form

$$\nabla \cdot [h + \nabla \Phi] = 0 = \nabla^a \nabla_a \Phi + 4 \pi G \left( \frac{d\rho}{dh} \right)_0 \Phi_2,$$

$$= - 4 \pi G \left( \frac{d\rho}{dh} \right)_0 \left[ C_2 + \frac{1}{2} \pi G \rho_0 r^2 [1 - P_2(\mu)] \right],$$  \hspace{0.5cm} (2.11)

where \( P_2(\mu) = \frac{1}{2} (3 \mu^2 - 1) \). We note that the right side of Eq. (2.11) splits into a function depending only on \( r \) plus a function of \( r \) multiplied by \( P_2(\mu) \). Since the operator on the left side of Eq. (2.11) acting on \( P_2(\mu) \) gives a function of \( r \) multiplied by \( P_2(\mu) \), it follows that the second-order gravitational potential \( \Phi_2 \) must have a similar splitting:

$$\Phi_2(r, \mu) = \Phi_2(r) + \Phi_2(r) P_2(\mu).$$  \hspace{0.5cm} (2.12)

Thus the partial differential equation (2.11) for \( \Phi_2 \) reduces to a pair of ordinary differential equations for the potentials \( \Phi_2 \) and \( \Phi_{22} \):

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_{20}}{dr} \right) + 4 \pi G \left( \frac{d\rho}{dh} \right)_0 \Phi_{20}$$

$$= - 4 \pi G \left( \frac{d\rho}{dh} \right)_0 \left( C_2 + \frac{1}{2} \pi G \rho_0 r^2 \right),$$  \hspace{0.5cm} (2.13)

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_{22}}{dr} \right) - \frac{6}{r^2} \Phi_{22} + 4 \pi G \left( \frac{d\rho}{dh} \right)_0 \Phi_{22}$$

$$= \frac{1}{2} \pi G^2 \rho_0 r^2 \left( \frac{d\rho}{dh} \right)_0.$$  \hspace{0.5cm} (2.14)

Appropriate boundary conditions are needed to select the unique physically relevant solutions to Eqs. (2.13) and (2.14). In order to ensure that the gravitational potential is non-singular at the center of the star, \( r = 0 \), we must require that \( \Phi_{20} \) and \( \Phi_{22} \) satisfy the following boundary conditions there:

$$0 = \left( \frac{d\Phi_{20}}{dr} \right)_{r=0} = \Phi_{22}(0).$$  \hspace{0.5cm} (2.15)

The potential \( \Phi_{22} \) must also fall to zero as \( r \to \infty \). We can ensure this by requiring that \( \Phi_{22} \) match smoothly at the surface of the star to a potential that in the exterior of the star is proportional to \( P_2(\mu)/r^2 \). It is sufficient therefore to require that \( \Phi_{22} \) satisfy the condition

$$\left( \frac{d\Phi_{22}}{dr} \right)_{r=R_0} = - \frac{3}{2} \Phi_{22}(R_0) \frac{R_0}{R_0}.$$  \hspace{0.5cm} (2.16)

An additional condition is also needed to fix \( \Phi_{20} \). It is customary to consider families of rotating stars which have the same total mass. In this case the monopole part of the exterior gravitational potential is the same for all members of the family. To ensure this, we must require that the potential \( \Phi_{20} \) and its derivative vanish on the surface of the star:
\[ 0 = \Phi_{20}(R_0) = \left( \frac{d\Phi_{20}}{dr} \right)_{r=R_0}. \] (2.17)

It might appear that Eqs. (2.15) and (2.17) now over constrain the potential \( \Phi_{20} \). This would be the case, except that the constant \( C_2 \) that appears on the right side of Eq. (2.13) is still undetermined. The boundary conditions, Eqs. (2.15)–(2.17), are just sufficient, however, to fix uniquely the potentials \( \Phi_{20} \) and \( \Phi_{22} \) together with the integration constant \( C_2 \) as solutions to Eqs. (2.13) and (2.14). We also note that these boundary conditions ensure that

\[ 0 = \int_{-1}^{1} \int_{0}^{R_0} r^2 \rho_2(r, \mu) dr d\mu. \] (2.18)

In summary, then, the thermodynamic functions \( h(r, \mu) \) and \( \rho(r, \mu) \) in slowly rotating barotropic stars are given by Eqs. (2.4) and (2.5), where \( \rho_2(r, \mu) \) and \( h_2(r, \mu) \) are given by

\[
\rho_2(r, \mu) = \left[ \frac{d\rho}{dh} \right]_0 h_2(r, \mu) \\
= \left[ \frac{d\rho}{dh} \right]_0 \left\{ C_2 + \Phi_{20}(r) + \frac{1}{2} \pi G \rho_0 r^2 \right. \\
+ \left[ \Phi_{22}(r) - \frac{1}{2} \pi G \rho_0 r^2 \right] P_2(\mu) \}. \] (2.19)

These expressions for \( h(r, \mu) \) and \( \rho(r, \mu) \) depend only on the structures of the non-rotating star through the functions \( h_0(r) \), \( \rho_0(r) \), and \( (d\rho/dh)_0 \), the potentials \( \Phi_{20} \) and \( \Phi_{22} \) from Eqs. (2.13) and (2.14), and the constant \( C_2 \).

It is also instructive to work out an expression for the surface \( r = R(\mu, \Omega) \) of the rotating star. This surface occurs where the thermodynamic potential \( h[R(\mu, \Omega), \mu] = 0 \). Solving this equation, we find

\[ R(\mu, \Omega) = R_0 + R_2(\mu) \frac{\Omega^2}{\pi G \rho_0} + \mathcal{O}(\Omega^4), \] (2.20)

where \( R_2(\mu) \) is given by

\[
R_2(\mu) = R_{20} + R_{22} P_2(\mu) \\
= \frac{3}{4 \pi G \rho_0 R_0} \left\{ C_2 + \frac{1}{2} \pi G \rho_0 R_0^2 \right. \\
+ \left[ \Phi_{22}(R_0) - \frac{1}{2} \pi G \rho_0 R_0^2 \right] P_2(\mu) \}. \] (2.21)

We have developed a computer code that solves these equations numerically for stars with an arbitrary equation of state. We have tested this code against analytical expressions which can be obtained for a polytropic neutron star equation of state, \( p = K \rho^\gamma \), with \( K \) chosen so that a 1.4\( M_\odot \) model has a radius of \( R_0 = 12.533 \) km. We find that the constants that determine the slowly rotating model for this polytropic case have the values \( C_2 = 0.09802 C_0 \), \( R_{20} = 0.15198 R_0 \), and \( R_{22} = -0.37995 R_0 \). Our numerical results agree with the analytical to floating-point precision.

### III. THE PULSATION EQUATIONS

The modes of any uniformly rotating barotropic stellar model can be described completely in terms of two scalar potentials \( \delta U \) and \( \delta \Phi \) [15]. The potential \( \delta \Phi \) is the Newtonian gravitational potential, while \( \delta U \) determines the hydrodynamic perturbation of the star:

\[ \delta U = \frac{\delta p}{\rho} - \delta \Phi, \] (3.1)

where \( \delta p \) is the Eulerian pressure perturbation, and \( \rho \) is the unperturbed density of the equilibrium stellar model. We assume here that the time dependence of the mode is \( e^{i\omega t} \) and that its azimuthal angular dependence is \( e^{im\varphi} \), where \( \omega \) is the frequency of the mode and \( m \) is an integer. The velocity perturbation \( \delta \mathbf{v} \) is determined in this case by

\[ \delta \mathbf{v} = i Q^{ab} \nabla_b \delta U. \] (3.2)

The tensor \( Q^{ab} \) depends on the frequency of the mode and the angular velocity of the equilibrium star:

\[
Q^{ab} = \frac{1}{(\omega + m \Omega)^2 - 4\Omega^2} \times \left[ (\omega + m \Omega) \delta^{ab} - \frac{4\Omega^2}{\omega + m \Omega} e^{iab} - 2i \nabla^a \nabla^b \right].
\] (3.3)

In Eq. (3.3) the unit vector \( e^a \) points along the rotation axis of the equilibrium star, \( \delta^{ab} \) is the Euclidean metric tensor (the identity matrix in Cartesian coordinates), and \( \nabla^a \) is the velocity of the equilibrium stellar model.

In general, the potentials \( \delta U \) and \( \delta \Phi \) are solutions of the following system of equations [15]:

\[ \nabla_a (\rho Q^{ab} \nabla_b \delta U) = - (\omega + m \Omega) \frac{d\rho}{dh} (\delta U + \delta \Phi), \] (3.4)

\[ \nabla^a \nabla_a \delta \Phi = - 4 \pi G \frac{d\rho}{dh} (\delta U + \delta \Phi), \] (3.5)

subject to the appropriate boundary conditions at the surface of the star for \( \delta U \) and at infinity for \( \delta \Phi \). In order to discuss these boundary conditions in more detail we let \( \Sigma \) denote a function that vanishes on the surface of the star, and which has been normalized so that its gradient, \( n_a = \nabla_a \Sigma \), is the outward directed unit normal vector there, \( n^n = 1 \). The boundary condition on the function \( \delta U \) at the surface of the star, \( \Sigma = 0 \), is to require that the Lagrangian perturbation in the enthalpy \( h \) vanish there, \( \Delta h = 0 \). This condition can be written in terms of the variables used here by noting that

\[ \Delta h = \delta h + \left( \frac{\delta \mathbf{v}^a}{i \kappa \Omega} \right) \nabla_a h, \] (3.6)
where $\kappa$ is related to the frequency of the mode by
\[ \kappa\Omega = \omega + m\Omega. \] (3.7)

Thus using Eqs. (3.1) and (3.2) the boundary condition can be written in terms of $\delta U$ and $\delta \Phi$ as
\[ 0 = [\kappa\Omega(\delta U + \delta \Phi) + Q^{ab}\nabla_a h \nabla_b \delta U]_{\Sigma \Omega}. \] (3.8)

The perturbed gravitational potential $\delta \Phi$ must vanish at infinity, $\lim_{r \to \infty} \delta \Phi = 0$. In addition $\delta \Phi$ and its first derivative must be continuous at the surface of the star. The problem of finding the modes of uniformly rotating barotropic stars is reduced therefore to finding the solutions to Eqs. (3.4) and (3.5) subject to the boundary condition in Eq. (3.8).

The equation for the hydrodynamic potential $\delta U$, Eq. (3.4), has a complicated dependence on the frequency of the mode and the angular velocity of the star through $Q^{ab}$ as given in Eq. (3.3). In the analysis that follows it will be necessary to have those dependences displayed more explicitly. To that end, we re-write Eq. (3.4) and the boundary condition, Eq. (3.8), in the following equivalent forms:
\[ \nabla^a\left[ \rho(\kappa^2 \delta^{ab} - 4z^a z^b) \nabla_b \delta U \right] + \frac{2m \kappa}{\omega} \nabla_a \delta U \]
\[ = - \kappa^2(\kappa^2 - 4)\Omega^2 \frac{d\rho}{dh} (\delta U + \delta \Phi), \] (3.9)
\[ \left[ \kappa^2 \delta^{ab} - 4z^a z^b \right] \nabla_a h \nabla_b \delta U \]
\[ + \frac{2m \kappa}{\omega} \nabla_a \delta U + \kappa^2(\kappa^2 - 4)\Omega^2 (\delta U + \delta \Phi) \]
\[ = 0. \] (3.10)

Here we use the notation $\omega$ for the cylindrical radial coordinate, $\omega = r \sqrt{1 - \mu^2}$, and $\omega^a$ to denote the unit vector in the $\omega$ direction.

Our purpose now will be to derive solutions to Eqs. (3.5) and (3.9) as power series in the angular velocity of the star. To that end we define the expansions of the potentials $\delta U$ and $\delta \Phi$ as
\[ \delta U = R_0^U \frac{\Omega^2}{\pi G \rho_0} + O(\Omega^4), \] (3.11)
\[ \delta \Phi = R_0^\Phi \frac{\Omega^2}{\pi G \rho_0} + O(\Omega^4). \] (3.12)

The normalizations of $\delta U$ and $\delta \Phi$ have been chosen to make the $\delta U_i$ and $\delta \Phi_i$ dimensionless under the assumption that the lowest order terms scale as $\Omega^2$. Here we have limited our consideration to the generalized r-modes [18]; modes which are dominated by rotational effects and whose frequencies vanish linearly therefore in the angular velocity of the star. In this case $\kappa$ [as defined in Eq. (3.7)] is finite in the small angular velocity limit, and so we may expand
\[ \kappa = \kappa_0 + \kappa_2 \frac{\Omega^2}{\pi G \rho_0} + O(\Omega^4). \] (3.13)

Using these expansions, together with those for the structure of the equilibrium star from Eqs. (2.4) and (2.5), it is straightforward to write down order by order the equations for the mode. The lowest order terms in the expansions of Eqs. (3.9) and (3.5) are the following,
\[ \nabla^a [\rho_0(\kappa_0^2 \delta^{ab} - 4z^a z^b) \nabla_b \delta U_0] + \frac{2m \kappa_0}{\omega} \nabla^a \delta U_0 = 0, \] (3.14)
\[ \nabla^a \nabla_a \delta \Phi_0 = -4 \pi G \frac{d\rho}{dh} \delta U_0 + \frac{2m \kappa_0}{\omega} \nabla^a \delta U_0 \] (3.15)

Similarly, the lowest order term in the expansion of the boundary condition is
\[ \left[ (\kappa_0^2 \delta^{ab} - 4z^a z^b) \nabla_a h_0 \nabla_b \delta U_0 + \frac{2m \kappa_0}{\omega} \nabla^a \delta U_0 \right]_{r = R_0} = 0. \] (3.16)

Continuing on to second order, the equations for the potentials are
\[ \nabla^a [\rho_0(\kappa_2^2 \delta^{ab} - 4z^a z^b) \nabla_b \delta U_2] + 2m \kappa_2 \frac{\Omega^2}{\pi G \rho_0} \nabla^a \delta U_2 \]
\[ + \nabla^a [\rho_2(\kappa_0^2 \delta^{ab} - 4z^a z^b) \nabla_b \delta U_0 + 2\kappa_0 \kappa_2 \rho_0 \nabla^a \delta U_0] \]
\[ + \frac{2m}{\omega} \nabla^a (\kappa_1 \nabla_a \rho_0 + \kappa_0 \nabla_a \rho_2) \delta U_0 \]
\[ = - \kappa_0^2(\kappa_2^2 - 4) \pi G \rho_0 \frac{d\rho}{dh} (\delta U_0 + \delta \Phi_0), \] (3.17)
\[ \nabla^a \nabla_a \delta \Phi_2 = -4 \pi G \frac{d\rho}{dh} (\delta U_2 + \delta \Phi_2) \]
\[ - 4 \pi G \frac{d^2 \rho}{dh^2} h_2 (\delta U_0 + \delta \Phi_0). \] (3.18)

The second-order boundary condition is somewhat more complicated; it must include two types of terms. The first type comes from the second-order terms in the expansion of Eq. (3.8) in powers of the angular velocity. The second type comes from the fact that the boundary condition is to be imposed on the actual surface of the rotating star, not the surface $r = R_0$. This second type of term is the correction to the lowest-order boundary condition, Eq. (3.16), needed to impose it on the actual boundary of the star (to second order in the angular velocity). Hence the terms of the second type are proportional to $R_2$, the second-order change in the radius of the star from Eq. (2.21). The resulting boundary condition is
\[
\begin{align*}
&\left( \kappa_0^2 \delta^{ab} - 4 z^a z^b \right) \nabla_a h_0 \nabla_b \delta U_2 + \frac{2m \kappa_0}{\omega} \omega^a \nabla_a h_0 \delta U_2 + \left( \kappa_0^2 \delta^{ab} - 4 z^a z^b \right) \nabla_a h_2 \nabla_b \delta U_0 + \frac{2m \kappa_0}{\omega} \omega^a \nabla_a h_2 \delta U_0 \\
&+ 2 \kappa_0 \kappa_2 \omega^a \nabla_a \delta U_0 + \frac{2m \kappa_2}{\omega} \omega^a \nabla_a h_0 \delta U_0 + \kappa_0^2 (\kappa_0^2 - 4) \pi \bar{\rho}_0 (\delta U_0 + \delta \Phi_0) \\
&+ R_2 r^3 \nabla_c \left[ \left( \kappa_0^2 \delta^{ab} - 4 z^a z^b \right) \nabla_a h_0 \nabla_b \delta U_0 + \frac{2m \kappa_0}{\omega} \delta U_0 \omega^a \nabla_a h_0 \right] \bigg|_{r = R_0} = 0. \tag{3.19}
\end{align*}
\]

In summary, then, Eqs. (3.17) and (3.18) together with the boundary condition, Eq. (3.19), determine the second-order terms in the structure of any generalized \( r \)-mode.

**IV. CLASSICAL \( r \)-MODES**

There exists a large class of modes in rotating barotropic stellar models whose properties are determined primarily by the rotation of the star \[18,19\]. We refer to these as generalized \( r \)-modes. In this section we restrict our attention however to those modes which contribute primarily to the gravitational radiation driven instability. These "classical" \( r \)-modes (which were studied first by Papaloizou and Pringle \[9\]) are generated by hydrodynamic potentials of the form (see e.g. Lindblom and Ipser \[18\])

\[
\delta U_0 = \alpha \left( \frac{r}{R_0} \right)^{m+1} P_{m+1}(\mu) e^{i m \varphi}. \tag{4.1}
\]

It is straightforward to verify that this \( \delta U_0 \) is a solution to Eq. (3.14) if the eigenvalue \( \kappa_0 \) has the value

\[
\kappa_0 = \frac{2}{m + 1}. \tag{4.2}
\]

This \( \delta U_0 \) and \( \kappa_0 \) also satisfy the boundary condition, Eq. (3.16), without further restriction at the boundary (and at every point within the star as well). The gravitational potential \( \delta \Phi_0 \) must have the same angular dependence as \( \delta U_0 \). Thus, \( \delta \Phi_0 \) must (through a slight abuse of notation) have the form

\[
\delta \Phi_0 = \alpha \delta \Phi_0(r) P_{m+1}(\mu) e^{im \varphi}. \tag{4.3}
\]

The gravitational potential, Eq. (3.15), reduces to an ordinary differential equation then for \( \delta \Phi_0(r) \):

\[
\begin{align*}
\frac{d^2 \delta \Phi_0}{dr^2} + \frac{2}{r} \frac{d \delta \Phi_0}{dr} + \left[ 4 \pi G \frac{dp}{dh} \right]_0 \frac{(m + 1)(m + 2)}{r^2} \delta \Phi_0 &= 0
\end{align*}
\]

Once \( \delta U_0 \) and \( \delta \Phi_0 \) are known, it is straightforward to evaluate the perturbations in other thermodynamic quantities to this order. For example \( \delta \rho_0 = \rho_0 \delta h_0 = \rho_0 (\delta U_0 + \delta \Phi_0) \). And it is straightforward to evaluate then the velocity perturbation to this order using Eq. (3.2).

We next consider the second-order contributions to the \( r \)-modes. First, let us analyze the second-order equation for the potential \( \delta U \), Eq. (3.17). This equation contains two types of terms: those proportional to \( \delta U_2 \) and those that are not. We will consider those terms not proportional to \( \delta U_2 \) as source terms, and we evaluate them now. It is convenient to break these source terms into three groups. The first group is proportional to \( \rho_2 \). These terms can be simplified by recalling that \( \delta U_0 \) satisfies Eq. (3.14) for any spherically symmetric density distribution. Then, using the fact from Eq. (2.19) that \( \rho_2(r, \mu) = \rho_{20}(r) + \rho_{22}(r) \rho_2(\mu) \), we find

\[
\nabla_a \left[ \rho_2 (\kappa_0^2 \delta^{ab} - 4 z^a z^b) \nabla_b \delta U_0 \right] + \frac{2m \kappa_0}{\omega} \omega^a \nabla_a \delta U_0 + \rho_{22} \frac{2m \kappa_0}{\omega} + \frac{12m(m + 2)}{(m + 1)^2} \frac{\rho_{22}}{r^2} \delta U_0. \tag{4.5}
\]

The second group of terms is proportional to \( \kappa_2 \). These terms have the following simplified form:

\[
\nabla^a (2 \kappa_0 \kappa_2 \rho_0 \nabla_a \delta U_0) + \frac{2m \kappa_2}{\omega} \omega^a \nabla_a \rho_0 \delta U_0 \]

\[
= 2(m + 2) \kappa_2 \frac{1}{r} \frac{d \rho_0}{dr} \delta U_0. \tag{4.6}
\]

Thus, combining these terms with those on the right side of Eq. (3.17), we obtain the following expression for the equation that determines \( \delta U_2 \) for the classical \( r \)-modes:

\[
\nabla_a \left[ \rho_2 \left( \frac{4 \delta^{ab} - 4 z^a z^b}{(m + 1)^2} \right) \nabla_b \delta U_2 \right] + \frac{4m \omega^a \nabla_a \rho_0}{(m + 1) \omega} \delta U_2 \\
= \frac{12m(m + 2)}{(m + 1)^2} \rho_{22} \frac{1}{r^2} \delta U_0 - 2(m + 2) \kappa_2 \frac{1}{r} \frac{d \rho_0}{dr} \delta U_0 \\
+ 16 \pi G \bar{\rho}_0 \frac{m(m + 2)}{(m + 1)^2} \frac{d \rho_0}{dh} \delta U_0 + \delta \Phi_0. \tag{4.7}
\]

A similar reduction can also be made on the second-order boundary condition, Eq. (2.19). We collect similar terms together to obtain the following simplifications:
\[
\left(\frac{\kappa_0^2}{r^2} - 4z^2e^2\right)\nabla_\alpha h_\beta \delta U_\alpha + \frac{2m}{\varpi} \frac{\kappa_0}{r} \nabla_\alpha h_\beta \delta U_\alpha = 0,
\]
\[
= - \frac{12m(m+2)}{(m+1)^2} \frac{h_{22}}{r^2} \delta U_0, \quad (4.8)
\]
\[
R_2 r^2 \nabla_c \left[ \left( \frac{\kappa_0^2}{r^2} - 4z^2e^2 \right) \nabla_\alpha h_\beta \delta U_\alpha + \frac{2m}{\varpi} \frac{\kappa_0}{r} \delta U_0 \nabla_\alpha h_\beta \right] = 0. \quad (4.9)
\]

The latter follows from the fact that the expression in Eq. (3.16) is zero everywhere if \( \delta U_0 \) is given by Eq. (4.1). Combining these simplified expressions together gives the following form for the boundary condition that constrains \( \delta U_2 \):

\[
\left\{ \left[ \frac{4}{(m+1)^2} - 4z^2e^2 \right] \nabla_\alpha h_\beta \delta U_\alpha + \frac{4m}{\varpi} \nabla_\beta h_\alpha \delta U_0 + \frac{2m}{\varpi} \frac{\kappa_0}{r} \delta U_0 \frac{1}{r} \frac{dh_0}{dr} \delta U_0 = 0 \right\}_{r=R_0} \quad (4.10)
\]

We note that the operator on the left side of Eq. (4.7) which acts on \( \delta U_2 \) is identical to the operator that acts on \( \delta U_0 \) from the lowest-order equation (3.14). We also note that the right side of Eq. (4.7) is a function of \( r \) multiplied by the angular function \( P_{m+1}(\mu) e^{im\phi} \). These facts allow us to derive a simple formula for the second-order eigenvalue \( \kappa_2 \) in terms of known quantities. Multiply the left side of Eq. (4.7) by \( \delta U_0^m \) and integrate over the interior of the star. This integral vanishes because this operator is symmetric and \( \delta U_0 \) also satisfies Eq. (3.14). This implies that the integral of \( \delta U_0^m \) multiplied by the right side of Eq. (4.7) also vanishes. This integral gives the following expression for the eigenvalue \( \kappa_2 \) once the angular integrals are performed:

\[
\kappa_2 \int_0^{R_0} \left[ \frac{r}{R_0} \right]^{2m+2} \frac{d\rho_0}{dr} \frac{1}{r^2} dr = 6m \left[ \frac{r}{R_0} \right]^{2m+2} \left[ \frac{r}{R_0} \right]^{m+1} \frac{8\pi G\rho_0 m}{(m+1)^4} + \int_0^{R_0} \frac{r}{R_0}^{m+1} \left[ \frac{r}{R_0} \right]^{m+1} \frac{d\rho_0}{dr} \frac{1}{r^2} dr.
\]

We have evaluated Eq. (4.11) numerically to determine \( \kappa_2 \) for a variety of equations of state. Table I presents the values of \( \kappa_2 \) for the classical \( r \)-modes with \( 2 \leq m \leq 6 \) of stars with polytropic equations of state. We also present in Fig. 1 a graph of the frequency \( \omega/\Omega = \kappa - 2 \) of the \( m = 2 \) classical \( r \)-modes computed for 1.4M\(_\odot\) stellar models based on seven realistic equations of state [20]. The dashed line in Fig. 1 corresponds to the lowest-order approximation of the \( r \)-mode frequency \( \omega/\Omega = \kappa - 2 \), which is the same for any equation of state. The solid curves are based on the second-order formula \( \omega/\Omega = \kappa - 2 + \kappa_2 \Omega^2/\pi G\rho_0 \). It is interesting to see in Fig. 1 that the higher-order terms make only small (up to about 12\%) at the highest angular velocities) corrections to the frequencies of these modes for stars with realistic equations of state. We also note that the general tendency of the frequency of these modes to be smaller than that predicted by the lowest-order expression is consistent with the results found by Lindblom and Ipser [18] for the Maclaurin spheroids. An analytical expression for \( \kappa_2 \) can be obtained from Eq. (4.11) for the uniform density case by performing the indicated integrals analytically. The resulting expression is equivalent to Eq. (6.10) of Lindblom and Ipser [18].

### Table I. The second-order eigenvalues \( \kappa_2 \) of the classical \( r \)-modes for stars with polytropic equations of state \( p = K\rho^{1+1/n} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m = 2 )</th>
<th>( m = 3 )</th>
<th>( m = 4 )</th>
<th>( m = 5 )</th>
<th>( m = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.57407</td>
<td>0.59766</td>
<td>0.54720</td>
<td>0.49074</td>
<td>0.44044</td>
</tr>
<tr>
<td>0.5</td>
<td>0.41718</td>
<td>0.43861</td>
<td>0.40415</td>
<td>0.36406</td>
<td>0.32782</td>
</tr>
<tr>
<td>1.0</td>
<td>0.29883</td>
<td>0.32054</td>
<td>0.29946</td>
<td>0.27250</td>
<td>0.24729</td>
</tr>
<tr>
<td>1.5</td>
<td>0.21183</td>
<td>0.23426</td>
<td>0.22369</td>
<td>0.20693</td>
<td>0.19019</td>
</tr>
<tr>
<td>2.0</td>
<td>0.14777</td>
<td>0.17084</td>
<td>0.16846</td>
<td>0.15961</td>
<td>0.14942</td>
</tr>
<tr>
<td>2.5</td>
<td>0.10091</td>
<td>0.12426</td>
<td>0.12808</td>
<td>0.12532</td>
<td>0.12016</td>
</tr>
<tr>
<td>3.0</td>
<td>0.06716</td>
<td>0.09024</td>
<td>0.09859</td>
<td>0.10039</td>
<td>0.09905</td>
</tr>
<tr>
<td>3.5</td>
<td>0.04334</td>
<td>0.06556</td>
<td>0.07699</td>
<td>0.08210</td>
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<td>0.02692</td>
<td>0.04773</td>
<td>0.06102</td>
<td>0.06839</td>
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</tr>
<tr>
<td>4.5</td>
<td>0.01589</td>
<td>0.03487</td>
<td>0.04896</td>
<td>0.05768</td>
<td>0.06252</td>
</tr>
</tbody>
</table>

### Fig. 1. Angular velocity dependence of the frequencies of the classical \( m = 2 \) \( r \)-modes for 1.4M\(_\odot\) stellar models based on seven realistic neutron star equations of state. The dashed curve is based on the lowest-order expression for \( \omega/\Omega = \kappa - 2 \), while the solid curves are based on the second-order expression \( \omega/\Omega = \kappa - 2 + \kappa_2 \Omega^2/\pi G\rho_0 \).
eigenfunctions $\delta U_2$ and $\delta \Phi_2$ of the classical $r$-modes. Once $\delta U_2$ is known, the solution of Eq. (3.18) to determine $\delta \Phi_2$ is straightforward. Thus our discussion will concentrate on the more difficult problem of solving Eq. (4.7) for $\delta U_2$. It will be convenient to introduce the notation

$$D(\delta U_2) = \nabla_a \left[ \rho_0 \left( \frac{4 \delta^{ab} - 4 \delta^a \delta^b}{(m+1)^2} \right) \nabla_b \delta U_2 \right] + \frac{4 m \sigma \nabla_a \rho_0}{(m+1) \sigma} \delta U_2, \quad (5.1)$$

for the differential operator that appears on the left side of Eq. (4.7). Thus Eq. (4.7) can be written in the form

$$D(\delta U_2) = F, \quad (5.2)$$

where

$$F = \frac{12 m (m+2)}{(m+1)^2} \frac{1}{r} \delta U_0 - 2 (m+2) \kappa_2 r \frac{d \rho_0}{dr} \delta U_0 + 16 \pi G \frac{m (m+2)}{(m+1)^4} \frac{d \rho}{dh} \left( \delta U_0 + \delta \Phi_0 \right). \quad (5.3)$$

The problem of solving Eq. (5.2) numerically is a somewhat non-standard problem that is made difficult by two facts. First, the operator $D$ has a non-trivial kernel: $D(\delta U_0) = 0$. Many of the straightforward numerical techniques fail in this case. Second, the operator $D$ is hyperbolic. There appears to be little previous work on solving hyperbolic boundary value problems of this type.

We solve Eq. (5.2) here using a variation of the standard relaxation method commonly used to solve elliptic partial differential equations [21]. To that end we introduce a fictitious time parameter $\lambda$ and convert Eq. (5.2) into an evolution equation

$$\partial_\lambda \delta U_2 = D(\delta U_2) - F. \quad (5.4)$$

The idea is to impose as initial data for Eq. (5.4) a guess for $\delta U_2$, and then to evolve these data (as a function of $\lambda$) until a stationary ($\partial_\lambda \delta U_2 = 0$) state is reached. If successful, the late time solution ($\lim_{\lambda \to \infty} \delta U_2$) to Eq. (5.4) will also be a solution to Eq. (5.2).

We implement the relaxation method to solve Eq. (5.2) by using a discrete representation of the functions and differential operators. Let $u_{i}^{\lambda}$ denote the discrete representation of the function $\delta U_2$ evaluated at the fictitious time $\lambda_i$. The index $i$ (and later $j$ and $k$ as well) takes on values from 1 to $N$ where $N$ is the dimension of the particular discretization used. Similarly the discrete representation of the differential operator $D$ of Eq. (5.2) is denoted $D_{i}^{\lambda}$, and the representation of the right side of Eq. (5.2) is denoted $F_{i}$. Thus the discrete representation of Eq. (5.2) is simply

$$D_{i}^{\lambda} u_{i}^{\lambda} = F_{i}. \quad (5.5)$$

Summation is implied for pairs of repeated indices [e.g. $i$ on the left side of Eq. (5.5)]. The algebraic equation (5.5) cannot be solved by the most straightforward direct numerical techniques because the operator $D$ has a nontrivial kernel (mentioned above). Consequently the matrix $D_{i}^{\lambda}$ has no inverse.

Thus we are lead to introduce the evolution equation (5.4). We use the "implicit" form of the discrete representation of Eq. (5.4):

$$O_{i}^{\lambda} u_{i+1}^{\lambda} = \left( D_{i}^{\lambda} - \frac{1}{\Delta \lambda} F_{i} \right) u_{i+1}^{\lambda} = F_{i} - \frac{1}{\Delta \lambda} u_{i}^{\lambda}. \quad (5.6)$$

In Eq. (5.6), $O_{i}^{\lambda}$ is the $N$ dimensional identity matrix, and $\Delta \lambda$ is the relaxation time step. Given $u_{i}^{\lambda}$ we solve Eq. (5.6) for $u_{i+1}^{\lambda}$ by direct solution of the linear algebraic equation. We use the band-diagonal linear equation solver LINSIS from EISPACK to compute $(O_{i}^{\lambda})^{-1} F_{i} - u_{i}^{\lambda} / \Delta \lambda$. We find that this can be computed stably and accurately for almost any value of the relaxation timestep $\Delta \lambda$.

Unfortunately solving Eq. (5.6) iteratively does not yield the desired solution to Eq. (5.5) in the limit of large $n$. Instead the solution grows exponentially, becoming in the limit of large $n$ closer and closer to a non-trivial solution to the homogeneous equation, $\delta U_0$. Fortunately, this malady is easily corrected. Let $\bar{u}_{i}$ denote the discrete representation of $\delta U_0$; thus $D_{i}^{\lambda} \bar{u}_{i} = 0$ since $\delta U_0$ is in the kernel of $D$. Also we let $\bar{u}_{i}$ denote the discrete representation of the co-vector associated with $\bar{u}_{i}$. In particular choose $\bar{u}_{i}$ so that $\bar{u}_{i} \bar{u}_{j}$ is the discrete representation of the integral of $|\delta U_0|^2$. Then the matrix

$$P_{i}^{\lambda} = I_{i}^{\lambda} - \frac{\bar{u}_{i} \bar{u}_{i}}{\bar{u}_{i}} \quad (5.7)$$

is the discrete representation of the operator that projects functions into the subspace orthogonal to $\delta U_0$. We use this projection in conjunction with Eq. (5.6) to define a modified relaxation scheme to determine $u_{i+1}^{\lambda}$ iteratively:

$$u_{i+1}^{\lambda} = P_{i}^{\lambda} (O_{i}^{\lambda})^{-1} F_{i} / \Delta \lambda. \quad (5.8)$$

By applying the projection $P_{i}^{\lambda}$ after each relaxation step we ensure that the exponentially growing kernel is removed from the solution. We find that the iteration scheme defined in Eq. (5.8) does converge quickly and stably to a solution of Eq. (5.5). In the Appendix we discuss the reason this numerical relaxation method works even in the case of the unusual hyperbolic boundary-value problem considered here. We show that convergence is guaranteed for sufficiently large values of the relaxation time step $\Delta \lambda$, and that it also converges for either sign of $\Delta \lambda$.

In order to implement this inversion scheme we need explicit discrete representations of these operators. We find it convenient to work in spherical coordinates $r$ and $\mu = \cos \theta$. In terms of these coordinates then, the differential operator $D$ has the form
A similar spherical representation is also needed for the boundary condition, Eq. (5.9), of one of the odd-order Legendre polynomials $D_l$. We find that after about five steps with the iteration scheme described in Eq. (5.8) converges rapidly. We begin the iteration by setting $u_{0}^{0}=0$ and find that after about five steps with $\Delta \lambda = -10^{0} R_{0}/\rho_{c}$ (where $\rho_{c}$ is the central density) the changes in $u_{0}^{0}$ from one iteration to the next become negligible.

Since the eigenfunction $\delta U_{2}$ is somewhat complicated we present several different representations of it graphically. Figure 2 depicts the functions $\delta U_{2}(r, \mu_{k})$ with $\mu_{k}$ located at the grid points used in our integration: the roots of $P_{l}(\mu) = 0$ in this case. We present in Fig. 3 another representation of this $\delta U_{2}$ in which we graph the functions $\delta U_{2}(r_{k}, \mu)$ for $r_{k} = (k/5) R_{0}$. This graph gives a clearer picture of the angular structure of $\delta U_{2}$. Finally we give in Fig. 4 another representation of the function $\delta U_{2}$ in which we decompose the angular structure of $\delta U_{2}$ into spherical harmonics by defining the functions $f_{k}(r)$:

$$
\delta U_{2}(r, \mu) = \sum_{k=1}^{\infty} f_{k}(r) P_{m+2k-1}^{m}(\mu).
$$

We find numerically that the $f_{k}(r)$ are negligibly small except for the smallest few values of $k$. In Fig. 4 we graph the first three $f_{k}(r)$.

To measure the degree to which $\delta U_{2}$ satisfies the original differential equation, we define

$$
\epsilon = \frac{\int |D(\delta U_{2}) - F|^2 r^2 dr d\mu}{\int |F|^2 r^2 dr d\mu}.
$$

We find that the value of $\epsilon$ achieved by a given solution is approximately $\epsilon \approx (4.3/N_{r})^{2}$ where $N_{r}$ is the number of radial grid points used in the discretization [23]. It is instructive to compare $\epsilon$ to the quantity

$$
\epsilon_{0} = \frac{\int |D(\delta U_{0})|^2 r^2 dr d\mu}{\int |F|^2 r^2 dr d\mu},
$$

which measures the degree to which $\delta U_{0}$ is in the kernel of $D$. Since analytically $D(\delta U_{0}) = 0$, the deviation of $\epsilon_{0}$ from zero is a measure of the accuracy of our discrete representa-
Each of the terms on the right side of Eq. (6.1) is proportional to $\Omega^3$ at lowest order. And thus the second-order quantities $\delta U_2$, $\kappa_2$ etc. that we have evaluated in the preceding sections are needed to evaluate $\delta \sigma$.

Bulk viscosity causes the energy associated with a perturbation to be dissipated according to the formula

$$\left(\frac{d\bar{E}}{dt}\right)_B = -\int \zeta \delta \sigma \delta \sigma^* d^3 x, \quad (6.2)$$

where $\zeta$ is the bulk viscosity coefficient, and $\bar{E}$ is the energy of the perturbation as measured in the co-rotating frame of the fluid. The energy $\bar{E}$ can be expressed as an integral of the fluid perturbations:

$$\bar{E} = \frac{1}{2} \left( \int \rho \delta v^a \delta v^*_a + \delta U \delta \rho^* \right) d^3 x. \quad (6.3)$$

Bulk viscosity causes the energy in a mode to decay (or grow) exponentially with time. We can evaluate the imaginary part of the frequency of a mode that results from bulk-viscosity effects by combining Eqs. (6.2) and (6.3). The result, which defines the bulk-viscosity damping time, $\tau_B$, is given by

$$\frac{1}{\tau_B} = -\frac{1}{2\bar{E}} \left( \frac{d\bar{E}}{dt} \right)_B. \quad (6.4)$$

In order to evaluate $1/\tau_B$ we need to have explicit expressions for the various terms that appear in the integrands of Eqs. (6.2) and (6.3). The energy $\bar{E}$, for example, can be expressed as the integral

$$\bar{E} = \frac{\alpha^2 \pi}{2m} (m+1)^3 (2m+1)! R_0^4 \Omega^2 \int_0^{R_0} \rho_0(r) \left( \frac{r}{R_0} \right)^{2m+2} dr + \mathcal{O}(\Omega^4), \quad (6.5)$$

by performing the angular integrals indicated in Eq. (6.3) [24].

In the dissipation integral, Eq. (6.2), an explicit expression for the bulk viscosity $\zeta$ is needed. In standard neutron-star matter the dominant form of bulk viscosity is due to the emission of neutrinos via the modified URCA process [25]. An approximate expression for this form of the bulk-viscosity coefficient is [26]

$$\zeta = 6.0 \times 10^{-5} \frac{\rho^2 T^6}{\kappa^4 \Omega^2}, \quad (6.6)$$

where all quantities are expressed in cgs units. For the case of the classical $r$-modes the expansion $\delta \sigma$ that appears in Eq. (6.2) can be expressed explicitly in terms of the potentials $\delta U_0$, $\delta \Phi_0$, and $\delta U_2$.
\[
\delta \sigma = i \frac{\rho}{\rho_0} \left( \frac{d\rho}{dh} \right)_0 \frac{R_0^2 \Omega^3}{2 \pi G \rho_0} \frac{m+1}{2m+1} \times \left\{ \frac{m(m+1)}{r} \frac{dU_0}{dr} \delta U_2 + \left[ 1 - (m+1)^2 \mu^2 \right] \frac{dU_0}{dr} \frac{\partial \delta U_2}{\partial r} \right\} \\
- \left( m+1 \right)^2 \mu (1-\mu^2) \frac{dU_0}{dr} \frac{\partial \delta U_2}{\partial \mu} - 3m(m+2)h_{22} \frac{\delta U_0}{r^2} \\
+ \kappa_2 \frac{2}{2r} \frac{dU_0}{dr} \frac{\delta U_0}{r} \\
- 4\pi G \rho_0 \left( \frac{m(m+2)}{m+1} \right)^2 \frac{dU_0}{dr} \frac{\delta \Phi_0}{r^2} \right\} + \mathcal{O}(\Omega^5) .
\]

(6.7)

We note that \( \delta \sigma \) is just the last term in the expression given in Eq. (6.7) for \( \delta \sigma \). It is the only term in Eq. (6.7) that depends only on the lowest-order perturbation quantities: the others depend on higher-order corrections through \( \delta U_2 \), \( \kappa_2 \) or \( h_{22} \). We define the approximate bulk-viscosity time scale \( \tau_B \) in analogy with Eq. (6.4) by replacing \( \delta \sigma \) with \( \delta \sigma \) in Eq. (6.2).

The bulk-viscosity contribution to the imaginary part of the frequency, \( 1/\tau_B \), is proportional to \( \Omega^2 \). This follows from Eqs. (6.2) and (6.4) because \( E \) scales as \( \Omega^2 \) from Eq. (6.5), \( \xi \) as \( \Omega^{-2} \) from Eq. (6.6), and \( \delta \sigma \) as \( \Omega^3 \) from Eq. (6.7). We note that a previous calculation of this bulk-viscosity time scale by Andersson, Kokkotas, and Schutz [4] reported that \( 1/\tau_B \) was proportional to \( \Omega^{1.77} \). The bulk-viscosity damping time, \( 1/\tau_B \), also scales with temperature as \( T^6 \).

Thus, it is convenient to define \( \tilde{\tau}_B \): the bulk viscosity time scale evaluated at \( \Omega^2 = \pi G \rho_0 \) and \( T = 10^9 \) K,

\[
\frac{1}{\tau_B} = \frac{1}{\tilde{\tau}_B} \left[ \frac{T}{10^9 \text{ K}} \right]^6 \left( \frac{\Omega^2}{\pi G \rho_0} \right) + \mathcal{O}(\Omega^4) .
\]

(6.9)

We have evaluated \( \tilde{\tau}_B \) numerically for the \( m=2 \) r-mode (the one most unstable to gravitational radiation) of a 1.4M_\odot stellar model with the polytropic equation of state discussed in Sec. II, and find \( \tilde{\tau}_B = 2.01 \times 10^{11} \) s. This value is longer by the factor 3.7(\sqrt{\pi G \rho_0/\Omega})^{0.23} than that found by Andersson, Kokkotas, and Schutz [4]. This discrepancy may be due to the fact that their calculation was based on the second-order formalism of Saio which was reported to contain errors by Smeyers and Martens [11].

For comparison, we have also re-evaluated the approximate timescale \( \tau_B \) described above and find \( \tilde{\tau}_B = 7.04 \times 10^9 \) s. This approximation was based on the assumption that the average value of the Lagrangian change in the density \( \Delta \rho \) [which appears on the right side of Eq. (6.1)] would be similar in magnitude to the average value of the Eulerian change in the density \( \delta \rho \) [which appears on the right side of Eq. (6.8)]. Our results here show that \( \delta \rho \) is on average a factor of 5.3 larger than \( \Delta \rho \) for the \( m=2 \) r-mode. We also note that the value of the approximate time scale \( \tilde{\tau}_B \) found here is about 10 times the value reported by Lindblom, Owen, and Morsink [3]. This discrepancy is due to an error in Eq. (4) of that paper: the right side of that equation should be multiplied by an additional overall factor of \( (2l+1)/[(l(l+1) \sqrt{2l+3}) \right]. Consequently the numerical bulk viscosity damping time estimate \( \tau_B \) given there must be multiplied by \( l^2(l+1)^2(2l+3)/(2l+1)^2 \), or about 10.08 for the \( l=2 \) case, in agreement with our present calculation.

The most interesting application of these dissipative timescales is to evaluate the stability of rotating neutron stars to the gravitational radiation driven instability in the r-modes [3]. The imaginary part of the frequency of the r-modes, \( \Im \omega \), includes contributions from gravitational radiation \( \overline{\omega}_{GR} \) in addition to shear \( \eta \) and bulk \( \tau_B \) effects. The general expression for \( \Im \omega(\Omega,T) \), a function of the temperature \( T \) and angular velocity \( \Omega \) of the star, is given by

\[
\frac{1}{\tau(\Omega,T)} = \frac{1}{\tau_{GR}} \left( \frac{\Omega^2}{\pi G \rho_0} \right)^3 + \frac{1}{\tau_s} \left( \frac{T}{10^9 \text{ K}} \right)^2 \left( \frac{\Omega^2}{\pi G \rho_0} \right) + \frac{1}{\tau_B} \left( \frac{T}{10^9 \text{ K}} \right)^6 \left( \frac{\Omega^2}{\pi G \rho_0} \right) .
\]

(6.10)

The bulk viscosity time scale \( \tilde{\tau}_B = 2.01 \times 10^{11} \) s has been evaluated in this paper, while the gravitational radiation time scale, \( \tau_{GR} = 3.26 \) s, and the shear viscosity timescale \( \tau_s = 2.52 \times 10^8 \) s, were obtained by Lindblom, Owen and Morsink [3] for the polytropic stellar model discussed in Sec. II. It is interesting to determine from this expression the critical angular velocity \( \Omega_c \):
Fig. 5 occurs at \( \Omega_c = 0.0301 \sqrt{\pi G \rho_0} \) which is 4.51% of the approximate maximum angular velocity \( \frac{7}{2} \sqrt{\pi G \rho_0} \).

Stars composed of strange quark matter are subject to a \( r \)-mode instability in hot strange stars. Using our polytropic stellar model described in Sec. II, we have computed the bulk-viscosity damping time of the \( r \)-modes. We also thank J. Creighton, S. Detweiler, J. Ipser, N. Stergioulas, K. Thorne, and R. Wagoner for helpful conversations concerning this work. This research was supported by NSF grants AST-9417371 and PHY-9796079, and by NASA grant NAG5-4093.

APPENDIX: NUMERICAL RELAXATION

The operator \( D \) defined in Eq. (5.1) is symmetric, in the sense that

\[
\int g^* D(f) d^3x = \left[ \int f^* D(g) d^3x \right]^*, \tag{A1}
\]

for arbitrary functions \( f \) and \( g \) in any stellar model where \( \rho_0 = 0 \) on the surface. Thus the discrete representation of the operator \( D_{ij} \) will be a Hermitian matrix, and consequently \( D_{ij} \) will have a complete set of eigenvectors. Let \( e_i^a \) denote the eigenvector corresponding to eigenvalue \( d_a : D_{ij} e_i^a = d_a e_j^a \). Since these eigenvectors form a complete set, we can express any vector as a linear combination of them. Thus we take \( F^i = \sum_a D_{a}^{i} e_j^a \), \( u_n = \sum_a u_n^a e_j^a \), etc. The numerical relaxation scheme indicated in Eq. (5.8) can be re-expressed therefore in the eigenvector basis as

\[
u_{n+1}^a = \frac{\Delta \lambda F^a - u_n^a}{\Delta \lambda d_a - 1}. \tag{A2}\]

The role of the projection operator \( P^i_{ij} \) is merely to remove from Eq. (A2) the component corresponding to the zero eigenvalue. The recurrence relation, Eq. (A2), can be solved analytically:

\[
u_{n+1}^a = \lambda F^a \sum_{k=0}^{n} (-x_a)^k, \tag{A3}\]

where

\[
x_a = \frac{1}{\Delta \lambda d_a - 1}. \tag{A4}\]

This series converges as long as \( |x_a| < 1 \). Since the projection operator has eliminated the one equation where \( d_a = 0 \), it is easy to choose \( \Delta \lambda \) so that \( |x_a| < 1 \) for all \( a \), e.g., by taking \( \Delta \lambda \) sufficiently large. Thus, the sequence \( u_{n+1}^a \) converges to
SECOND-ORDER ROTATIONAL EFFECTS ON THE $r$-.

\[
\lim_{n \to \infty} \alpha_n^a = \frac{x_a \Delta \lambda F^a}{1 + x_a} = \frac{F^a}{d^a}. \tag{A5}
\]

Thus the implicit relaxation scheme converges to the desired solution to Eq. (5.2).

In contrast to the implicit relaxation method defined in Eq. (5.8), the analogous explicit relaxation method does not converge at all for this problem. An analysis similar to that carried out above reveals that the criterion for convergence of the explicit scheme is that $|\Delta \lambda d^a + 1| < 1$. Clearly this can only hold for operators $D$ where the eigenvalues all have the same sign (as is the case when $D$ is elliptic) and only when $\Delta \lambda$ has the correct sign. Our limited experience with hyperbolic operators $D$ is that their eigenvalues have both signs. Consequently it is not surprising that our attempts at explicit numerical relaxation fail in this case.

[10] Throughout this paper we use “second order” to describe terms in the expansion of a quantity in powers of the angular velocity which are two powers in the angular velocity higher than the lowest-order terms. For example, if the lowest-order velocity perturbation were normalized so that it was linear in the angular velocity, then the second-order terms would be proportional to the angular velocity cubed.
[11] Andersson, Kokkotas, and Schutz [4] analyze the structure of the $r$-modes using the second-order formalism developed by H. Saio, Astrophys. J. 256, 717 (1982). However, this formalism was reported to contain errors by P. Smeyers and L. Mar- tens, Astron. Astrophys. 125, 193 (1983) which result in incorrect expressions for the second-order angular perturbations. This formalism would not be expected then to describe the bulk-viscosity coupling to these modes correctly.
[16] Y. Levin (private communication).
[20] The seven equations of state used to obtain the curves in Fig. 1 are BJJ, DiazII, FP, HKP, PandN, WFF3, and WGG. Definitions of these abbreviations, and references to the original literature on these equations of state, are given in M. Salgado, S. Bonazzola, E. Gourgoulhon, and P. Haensel, Astron. Astrophys. 291, 155 (1994).
[22] We have also written a second totally independent code that uses a uniformly spaced angular grid and standard three-point angular differencing. We find the results of the two independent codes to be completely equivalent. However, the code based on the uniformly spaced angular grid requires many more angular grid points than the code described more fully in the text to achieve the same accuracy.
[23] The value of $\epsilon$ is essentially independent of the number of angular grid points $N_{\alpha}$ because the angular dependence of the function $\delta U_{\alpha}$ is determined by only a few terms when expressed as the series in Eq. (5.11). For such functions the angular differencing method discussed in Ipser and Lindblom [15] is essentially exact.
[24] We note that the normalization of these modes used here [as defined in Eq. (4.1)] differs from that used in Lindblom, Owen, and Morsink [3]. The normalization parameter $\alpha$ used here is related to the one used there (call it $\alpha'$) by

$$
(\alpha')^2 = \frac{\alpha^2 \pi}{m} (m+1)^2 (2m+1)!
$$