Effect of a neutron-star crust on the *r*-mode instability

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(Received 18 June 2000; published 26 September 2000)

The presence of a viscous boundary layer under the solid crust of a neutron star dramatically increases the viscous damping rate of the fluid *r*-modes. We improve previous estimates of this damping rate by including the effect of the Coriolis force on the boundary-layer eigenfunction and by using more realistic neutron-star models. If the crust is assumed to be perfectly rigid, the gravitational radiation driven instability in the *r*-modes is completely suppressed in neutron stars colder than about 1.5×10^8 K. Energy generation in the boundary layer will heat the star, and will even melt the crust if the amplitude of the *r*-mode is large enough. We solve the heat equation explicitly (including the effects of thermal conduction and neutrino emission) and find that the *r*-mode amplitude needed to melt the crust is $\alpha_c \approx 5\times10^{-3}$ for maximally rotating neutron stars. If the *r*-mode saturates at an amplitude larger than α_c, the heat generated is sufficient to maintain the outer layers of the star in a mixed fluid-solid state analogous to the pack ice on the fringes of the Arctic Ocean. We argue that in young, rapidly rotating neutron stars this effect considerably delays the formation of the crust. By considering the dissipation in the ice flow, we show that the final spin frequency of stars with *r*-mode amplitude of order unity is close to the value estimated for fluid stars without a crust.

PACS number(s): 04.30.Db, 04.40.Dg, 97.60.Jd

I. INTRODUCTION

The *r*-modes (fluid oscillations governed primarily by the Coriolis force) have been the focus of considerable attention over the past few years (see Friedman and Lockitch [1] for a review). The gravitational-radiation-driven instability of these modes has been proposed as an explanation for the observed relatively low spin frequencies of young neutron stars and of accreting neutron stars in low-mass x-ray binaries (LMXBs) as well. The *r*-mode instability may also provide a source of gravitational waves detectable by the ''enhanced'' Laser Interferometric Gravitational Wave Observatory (LIGO) and VIRGO interferometer, the configurations whose operation is expected to begin in about the year 2006. In real neutron stars this instability can only occur when the gravitational-radiation driving time scale of the *r*-mode is shorter than the time scales of the various internal dissipation processes that may occur in the star. In this paper we re-examine and re-calculate some of the dissipation timescales associated with the crust of the neutron star.

Recently Bildsten and Ushomirsky [2] made the first estimate of the effect of a solid crust on the *r*-mode instability. They found that the shear dissipation in the viscous boundary layer between the solid crust and the fluid core decreases the viscous damping time scale by more than a factor of 10^5 in old, accreting neutron stars and more than 10^7 in hot, young neutron stars. The viscous damping time scale in these stars is thus comparable to the gravitational radiation driving time scale, and so Bildsten and Ushomirsky concluded that the *r*-mode instability is unlikely to play a role in old, accreting neutron stars. In hot, young neutron stars they also predicted that this boundary-layer damping mechanism severely limits the ability of the *r*-mode instability to reduce the angular momentum of the star, and hence to produce detectable amounts of gravitational radiation.

However, the debate over the relevance of the *r*-mode instability to the observed spin periods of neutron stars is not settled. Andersson *et al.* [3] corrected a minor numerical factor in the work of Bildsten and Ushomirsky [2] and used different neutron-star parameters to obtain a significantly different result for the critical frequency of the onset of the *r*-mode instability. Andersson *et al.* [3] estimated the critical frequency (the frequency of rotation of the star at which the driving and damping time scales are equal) to be about 40% lower than the estimate of Bildsten and Ushomirsky [2]. Based on this new estimate, and contrary to the conclusions of Bildsten and Ushomirsky [2], Andersson *et al.* [3] inferred that the *r*-mode instability is likely to be the mechanism that limits the LMXB spin periods and those of other millisecond pulsars as well.

The calculations of Bildsten and Ushomirsky [2] and of Andersson *et al.* [3] depend on an extremely simple model of the boundary layer which neglects the Coriolis force, the dominant restoring force for the *r*-modes. Rieutord [4], building on the work of Greenspan [5], improved this model by finding the self-consistent solution to the linearized Navier-Stokes equations (including the Coriolis force terms) throughout the boundary layer. Rieutord's model of the boundary layer includes the correct angular structure and results in dissipative time scale estimates that are significantly shorter than those of Andersson *et al.* [3]. Coincidentally, Rieutord's estimates of the critical spin frequencies agree rather closely with the original estimates of Bildsten and

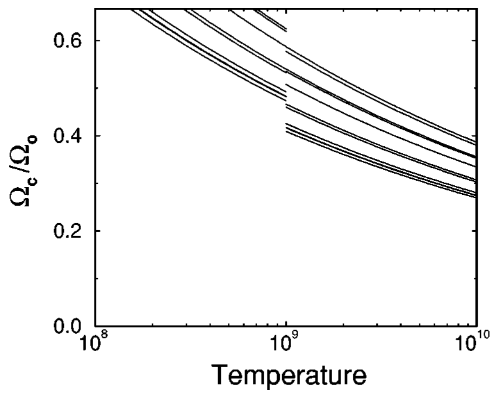

FIG. 1. Temperature dependence of the critical angular velocities of $1.4M_\odot$ neutron stars constructed with a number of realistic equations of state.

Ushomirsky [2]. These changes in the values of the boundary layer dissipation time scale, and the corresponding changes in the conclusions regarding the physical relevance of the *r*-mode instability for neutron stars with crusts, have prompted us to revisit this subject once again.

Our aim is to provide a complete, careful re-derivation of recent results [2–4], including the effects of non-uniform density stellar models—an important factor neglected by Andersson *et al.* [3] and Rieutord [4]. We use a variety of stellar models to explore the sensitivity of the results to the poorly known neutron-star equation of state. In summary (see Fig. 1), we find that the critical frequency is even greater (by 25%–50% depending on the equation of state) than that estimated by Bildsten and Ushomirsky [2] and Rieutord [4], and double to triple that estimated by Andersson *et al.* [3]. Our new results make it appear unlikely that the *r*-mode instability is responsible for limiting the spin periods of the LMXBs.

However, the interpretation of our results is somewhat complicated by the recent work of Levin and Ushomirsky [6]. They showed that mechanical crust-core coupling can reduce the relative velocity between the crust and the core, thereby reducing the shear and the viscous dissipaton in the boundary layer. Let $\Delta v/v$ denote the difference between the velocities in the inner edge of the crust and outer edge of the core divided by the velocity of the core. In the static, rigid crust case $\Delta v/v=1$, but Levin and Ushomirsky [6] find that this quantity can lie anywhere in the range $0.05 \lesssim \Delta v/v \lesssim 1$. They find that the precise value of $\Delta v/v$ varies in a complicated manner with the spin frequency of the star, and is quite sensitive to the thickness of the crust and other model parameters. Throughout this paper we assume that the crust is static, i.e. that $\Delta v/v=1$, but indicate at appropriate points in the text how to rescale various quantities (such as dissipation time scales) for $\Delta v/v<1$. Our calculations therefore provide only an upper limit on the critical frequency for the onset of the *r*-mode instability in neutron stars with crusts. However, our limits are easily adjusted once a more realistic value of $\Delta v/v$ is known.

In young neutron stars, the *r*-mode instability is still a viable mechanism for spindown even if $\Delta v/v=1$. In the presence of a crust, the majority of the viscous dissipation is confined to a very thin boundary layer. If the *r*-mode amplitude is larger than some critical value, the resulting heating in this layer is so intense that it can compete with neutrino cooling and heat the crust-core interface to the melting temperature. This possibility was first suggested by Owen [7], who crudely estimated the critical amplitude to be $\alpha_c \approx 10^{-3}$, and the idea of crust re-melting was suggested again by Andersson *et al.* [3]. Here we perform a comprehensive, self-consistent analysis of this heating effect, including conductive transport of heat into the core and the crust, and eventual radiation of the excess thermal energy by neutrinos. We find that, for maximally rotating neutron stars, the critical dimensionless *r*-mode amplitude needed to heat the crust-core interface to the melting temperature at the equator is $\alpha_c \approx 5\times10^{-3}$. [This value depends on the spin frequency and is somewhat larger away from the equator; see Eq. (42) and Fig. 4.] If the *r*-mode amplitude grows to a value exceeding α_c in a hot, young neutron star, the crust will not form as usual and the neutron star will spin down to a much lower frequency than would have been possible were a crust present.

What happens if the *r*-mode amplitude does exceed the critical value, α_c? If the *r*-mode completely melts the crust, the boundary layer and the heating are removed and the outer layers of the neutron star quickly cool back down to melting temperature. If the crust completely cools and solidifies, the boundary-layer heating due to the *r*-mode quickly heats the crust back up to melting temperature. It is clear that, in the presence of a strong enough *r*-mode, neither a solid crust nor a pure fluid is possible. We imagine the situation to be similar to the pack ice on the Arctic Ocean. While the *r*-mode amplitude exceeds the critical value α_c in this picture, the outer layers will be composed of chunks of crustal ''ice'' floating in the fluid at the melting temperature of about 10^{10} K. The dissipation mechanism in this pack ice will be a combination of macroscopic viscosity due to collisions between chunks of ice and microscopic viscosity due to boundary layers bordering the chunks. Calculation of the viscosity in this situation would be very complicated but for the fact that the system must be maintained close to the melting temperature. The size of the ice chunks (and other variables controlling the viscosity) will adjust themselves so that the heat dissipated in the ice flow balances the neutrino cooling. This allows us to estimate the *r*-mode damping time scale quantitatively in the pack ice, without knowing the details of this complicated process. We find that, for *r*-mode saturation amplitudes of order unity, the final spindown frequency is little changed from that of the purely fluid model of the instability considered over the past few years.

The rest of this paper is organized as follows. In Sec. II we re-derive the velocity profile of the *r*-modes in the boundary layer, and in Sec. III we re-derive the energy dissipation in the boundary layer using techniques and notation that will be more familiar to the relativistic astrophysics community. In Sec. IV we apply these results to the question of the stability of the *r*-modes in hot, young neutron stars and in older, colder, accreting ones. In Sec. V we derive the thermal structure of the boundary layer and find the *r*-mode amplitude α_c necessary to heat the bottom of the crust to melting. In Sec.

VI we argue that the presence of an r-mode with an amplitude greater than α_c will in fact prevent crust formation, and instead lead to the pack-ice flow described above. We compute the effective dissipation in this flow, and consider the implications of the delayed crust formation for the develo-pent of the r-mode instability. In Sec. VII we summarize and discuss some of the implications of our results. In the Appendix we summarize the relevant thermodynamic properties of the neutron star matter near the crust-core interface.

II. STRUCTURE OF THE BOUNDARY LAYER

We begin by re-deriving the structure of the r-modes in the boundary layer near a rigid solid crust. Our analysis improves the initial studies of Bildsten and Ushomrisky [2] and of Andersson *et al.* [3] by properly evaluating the angular structure of the boundary layer. We follow closely the more recent work of Rieutord [4], but employ a notation that is more familiar to the relativistic astrophysics community and improve his estimates by allowing non-uniform density stellar models.

Let us decompose the fluid perturbation representing an r-mode into two parts: the eigenfunctions describing the mode in the zero viscosity limit, δv^a and $\delta U \equiv \delta p/\rho$, and the corrections that must be added to these when viscosity is present, $\delta\tilde{v}^a$ and $\delta\tilde{U}$. The velocity correction $\delta\tilde{v}^a$ must be chosen so that the relative velocity between the fluid core and the solid crust vanishes: for the case of a rigid crust this is equivalent to $0=\delta v^a+\delta\tilde{v}^a$. Thus the viscous corrections to the velocity field are not small, at least near the crust. The correction to the hydrodynamic potential $\delta\tilde{U}$, however, will turn out to be small everywhere.

The equations for the viscous corrections to the velocity field are obtained by expanding the Navier-Stokes equation to first order in the amplitude of the perturbation. We assume that the equilibrium star is rigidly rotating with angular velocity Ω, and that the temporal and angular dependence of the mode is $e^{i\omega t+im\varphi}$. As usual in boundary-layer theory, we assume that the fluid functions change much more rapidly in the direction perpindicular to the boundary. Thus we assume that the angular derivatives are much smaller than radial derivatives, and neglect them. Under these assumptions, the equations that determine the viscous corrections to the fluid flow are

$$i(\omega+m\Omega)\delta\tilde{v}^r-2\Omega r\sin^2\theta\,\delta\tilde{v}^\varphi=-\partial_r\delta\tilde{U}+\tfrac{2}{3}\,\frac{\eta}{\rho}\partial_r^2\delta\tilde{v}^r, \tag{1}$$

$$i(\omega+m\Omega)\delta\tilde{v}^\varphi+\frac{2\Omega}{r}\delta\tilde{v}^r+2\Omega\cot\theta\,\delta\tilde{v}^\theta=\frac{\eta}{\rho}\partial_r^2\left(\delta\tilde{v}^\varphi-\tfrac{1}{3}\delta\tilde{v}^r\right), \tag{2}$$

$$i(\omega+m\Omega)\delta\tilde{v}^\theta-2\Omega\cos\theta\sin\theta\,\delta\tilde{v}^\varphi=\frac{\eta}{\rho}\partial_r^2\left(\delta\tilde{v}^\theta-\tfrac{1}{3}\delta\tilde{v}^r\right), \tag{3}$$

where η and ρ are the viscosity and density of the fluid, respectively. These equations assume only that the viscous corrections vary much more rapidly with r than with the angular coordinates. Thus we generalize the analysis of Rieutord [4]: we do not assume *a priori* that the fluid flow is incompressible or that the equilibrium is of uniform density.

The solutions to these equations depend exponentially on r with a characteristic length scale d:

$$d^2=\left.\frac{\eta}{2\Omega\rho}\right|_{r=R_c}. \tag{4}$$

The radial velocity correction $\delta\tilde{v}^r$ must vanish both at the boundary and deep within the fluid. Given the exponential nature of the solutions, it follows that $\delta\tilde{v}^r=0$ everywhere within the fluid. With this simplification, Eqs. (2) and (3) determine the velocity corrections $\delta\tilde{v}^\theta$ and $\delta\tilde{v}^\varphi$, while Eq. (1) determines $\delta\tilde{U}$ in terms of $\delta\tilde{v}^\varphi$. Equations (2) and (3) may be re-written in the following form:

$$\left[d^2\partial_r^2-i\left(\frac{\sigma}{2\Omega}+\cos\theta\right)\right](\delta\tilde{v}^\theta+i\sin\theta\,\delta\tilde{v}^\varphi)=0, \tag{5}$$

$$\left[d^2\partial_r^2-i\left(\frac{\sigma}{2\Omega}-\cos\theta\right)\right](\delta\tilde{v}^\theta-i\sin\theta\,\delta\tilde{v}^\varphi)=0, \tag{6}$$

where $\sigma=\omega+m\Omega$ is the mode frequency in the rotating frame.

It is straightforward now to write down the general solutions to these equations, and then to impose the boundary condition, $0=\delta v^a+\delta\tilde{v}^a$, at the inner edge of the crust $r=R_c$. These solutions are given by

$$\delta\tilde{v}^\theta=-\delta v^\theta(R_c)\Lambda_+(r,\theta)-i\sin\theta\,\delta v^\varphi(R_c)\Lambda_-(r,\theta), \tag{7}$$

$$\sin\theta\,\delta\tilde{v}^\varphi=-\sin\theta\,\delta v^\varphi(R_c)\Lambda_+(r,\theta)+i\,\delta v^\theta(R_c)\Lambda_-(r,\theta), \tag{8}$$

where $\delta v^a(R_c)$ is the standard non-viscous r-mode velocity perturbation, evaluated at the radius of the inner edge of the crust $r=R_c$. The functions $\Lambda_\pm(r,\theta)$ are defined by

$$\Lambda_\pm(r,\theta)=\tfrac{1}{2}e^{-\zeta\sqrt{i(\cos\theta+\sigma/2\Omega)}}\pm\tfrac{1}{2}e^{-\zeta\sqrt{i(\cos\theta-\sigma/2\Omega)}}, \tag{9}$$

where ζ is the dimensionless radial parameter,

$$\zeta=\frac{R_c-r}{d}, \tag{10}$$

and the characteristic thickness of the boundary layer, d, is defined in Eq. (4). For r-modes the frequency of the mode (as measured in the co-rotating frame of the fluid) that appears in the definition of $\Lambda_\pm$ has the value $\sigma/2\Omega=1/(m+1)$. For rapidly rotating neutron stars $d\approx(10^8\text{ K}/T)$ cm, where T is the temperature in the boundary layer. Therefore, $d/R_c\ll 1$.

III. DISSIPATION IN THE BOUNDARY LAYER

The shear of the r-mode velocity field is dominated by the rapid radial change in $\delta\widetilde{v}^a$ through the boundary layer. Thus, up to terms of order d/R_c, the square of the shear tensor in the boundary layer is given by

$$\delta\sigma^*_{ab}\delta\sigma^{ab} = \tfrac{1}{2}R_c^2(|\partial_r\delta\widetilde{v}^\theta|^2 + \sin^2\theta\,|\partial_r\delta\widetilde{v}^\varphi|^2). \qquad (11)$$

The angular structure of $\delta\widetilde{v}^a$ in Eqs. (7) and (8) is determined in part by the structure of the dissipation-free velocity field δv^a. For convenience we may write the non-dissipative r-mode velocity field as

$$\delta v^\theta = -iAr^{m-1}\sin^{m-1}\theta\, e^{i(\omega t+m\varphi)}, \qquad (12)$$

$$\delta v^\varphi = Ar^{m-1}\sin^{m-2}\theta\cos\theta\, e^{i(\omega t+m\varphi)}, \qquad (13)$$

where A is a normalization constant. It is straightforward then to evaluate the square of the shear tensor:

$$\delta\sigma^*_{ab}\delta\sigma^{ab} = \frac{|A|^2 r^{2m}}{8\,d^2}F(r,\theta), \qquad (14)$$

where

$$F(r,\theta) = \sin^{2m-2}\theta[(1-\cos\theta)^2 p_+^2 e^{-\zeta p_+} + (1+\cos\theta)^2 p_-^2 e^{-\zeta p_-}] \qquad (15)$$

and

$$p_\pm = \sqrt{2|\cos\theta \pm 1/(m+1)|}. \qquad (16)$$

Now integrate the energy dissipation rate due to shear viscosity over the fluid interior to the crust, ignoring terms of order d/R_c:

$$\frac{d\widetilde{E}}{dt} = -\int 2\,\eta\,\delta\sigma^*_{ab}\delta\sigma^{ab}d^3x = 2\sqrt{2}\pi|A|^2\Omega\, dR_c^{2m+2}\rho_c\mathcal{I}_m, \qquad (17)$$

where $\mathcal{I}_m$ is defined by

$$\mathcal{I}_m = \int_0^\pi \sin^{2m-1}\theta(1+\cos\theta)^2\sqrt{|\cos\theta - 1/(m+1)|}\,d\theta. \qquad (18)$$

For the case of primary interest to us, $m=2$, this integral has the value $\mathcal{I}_2 = 0.80411$ [4].

Now we can define the viscous timescale for dissipation in the boundary layer:

$$\frac{1}{\tau_v} = -\frac{1}{2\widetilde{E}}\frac{d\widetilde{E}}{dt}. \qquad (19)$$

Using the expression for $d\widetilde{E}/dt$ derived above and the usual expression for the energy $\widetilde{E}$ we find

$$\tau_v = \frac{1}{2\Omega}\frac{2^{m+3/2}(m+1)!}{m(2m+1)!!\mathcal{I}_m} \times \sqrt{\frac{2\Omega R_c^2\rho_c}{\eta_c}}\int_0^{R_c}\frac{\rho}{\rho_c}\left(\frac{r}{R_c}\right)^{2m+2}\frac{dr}{R_c}. \qquad (20)$$

Here the quantities R_c, ρ_c and η_c are the radius, density, and the viscosity of the fluid at the outer edge of the core. We note that, while the viscous dissipation rate in the boundary layer, Eq. (17), depends only on quantities local to the boundary layer (d, ρ_c, R_c, Ω), the time scale τ_v depends also on the global structure of the mode. This is due to the fact that, while most of the energy dissipation takes place in the boundary layer, most of the energy in the mode is not localized there. For realistic neutron stars this expression, Eq. (20), for the viscous time scale is about a factor of 2 larger than the one obtained by Rieutord [4], who assumed a uniform-density stellar model.

In deriving our expression for the viscous boundary layer dissipation time scale, Eq. (20), we assumed that the crust is rigid and hence static in the rotating frame. The motion of the crust due to the mechanical coupling to the core [6] effectively increases τ_v by a factor of $(\Delta v/v)^{-2}$.

IV. STABILITY OF THE *r*-MODES

In this section we evaluate viscous timescales for neutron stars (based on slowly rotating Newtonian models) constructed from a set of ‘‘realistic’’ equations of state, and use these time scales to evaluate the stability of these stars. In neutron stars colder than about 10^9 K the shear viscosity is expected to be dominated by electron-electron scattering. The viscosity associated with this process is given by [8,9]

$$\eta^{ee} = 6.0\times 10^6\rho^2 T^{-2}, \qquad (21)$$

where all quantities are given in cgs units, and T is measured in K. For temperatures above about 10^9 K, neutron-neutron scattering provides the dominant dissipation mechanism. In this range the viscosity is given by [8,9]

$$\eta^{nn} = 347\rho^{9/4}T^{-2}. \qquad (22)$$

We find it useful to factor the angular velocity and temperature dependence from the viscous time scale defined in Eq. (20). Thus we define a fiducial viscous time scale $\widetilde{\tau}_v$ such that

$$\tau_v = \widetilde{\tau}_v\left(\frac{\Omega_0}{\Omega}\right)^{1/2}\left(\frac{T}{10^8\ \mathrm{K}}\right), \qquad (23)$$

where $\Omega_0 = \sqrt{\pi G\bar{\rho}}$. We have evaluated these fiducial viscous time scales (for each type of viscous dissipation) for 1.4 $M_\odot$ neutron star models based on a variety of realistic equations of state [10] as well as the standard $n=1$ polytrope with a radius of 12.53 km. These results are summarized in Table I, along with other relevant properties of these stellar models. In particular we also include the total radius R, the radius of the fluid core R_c, the energy contained in an ex-

TABLE I. Properties of $1.4M_\odot$ neutron stars with rigid crusts for densities below $\rho_c=1.5\times10^{14}$ g/cm^3. Times are given in seconds, lengths in kilometers, temperatures in units of 10^8 K, and energies in units of 10^{51} ergs.

EOS[a]	R	R_c	$\tilde{e}_c$	$\tilde{e}$	$\tilde{\tau}_{GR}$	$\tilde{\tau}_v^{ee}$	T_c^{ee}	$\tilde{\tau}_v^{nn}$	T_c^{nn}
n1poly	12.53	11.01	1.94	2.53	4.25	23.3	1.69	52.0	0.76
BJI	15.04	12.16	1.30	1.72	13.11	24.6	4.96	54.8	2.23
DiazII	15.80	12.60	1.20	1.67	17.37	24.3	6.64	54.2	2.98
FP	13.18	11.34	1.93	2.18	5.24	25.9	1.88	57.8	0.84
Glend1	16.47	12.74	0.97	1.53	25.32	23.4	10.1	52.2	4.51
Glend2	16.84	12.95	0.96	1.53	28.12	23.7	11.0	52.8	4.95
HKP	16.43	13.27	1.32	1.76	18.41	24.6	6.97	54.8	3.13
PandN	12.77	10.85	1.84	2.08	4.84	26.9	1.67	59.9	0.75
SHW	14.92	12.89	1.97	2.23	8.40	25.0	3.11	55.8	1.40
WFF3	12.98	11.21	2.10	2.33	4.52	27.7	1.52	61.8	0.68
WGW	14.64	12.38	1.85	2.17	8.32	26.8	2.88	59.8	1.29

[a]The various equations of state used here are described in Bonazzola, Frieben, and Gourgoulhon [10].

cited r-mode, $\tilde{e}=\tilde{E}(\Omega_0/\alpha\Omega)^2$ ergs, for the case of a fully fluid star, the energy contained in an excited r-mode, $\tilde{e}_c$, in the case with rigid crust, the fiducial gravitational radiation time scale $\tilde{\tau}_{GR}$, and a critical temperature T_c (defined below). We note that the τ_v values listed in Table I presume that the crust is rigid and does not move in the corotating frame. To take into account the motion of the crust, multiply $\tilde{\tau}_v^{ee}$ and $\tilde{\tau}_v^{nn}$ by $(\Delta v/v)^{-2}$ [11].

The gravitational radiation time scale is evaluated here using the formalism developed by Lindblom, Owen, and Morsink [12]. This time scale is given by

$$\frac{1}{\tau_{GR}}=\frac{32\pi G\Omega^{2m+2}}{c^{2m+3}}\frac{(m-1)^{2m}}{[(2m+1)!!]^2}\times\left(\frac{m+2}{m+1}\right)^{2m+2}\int_0^{R_c}\rho r^{2m+2}dr. \tag{24}$$

Since we presume that the crust is static in the rotating frame, the integral in Eq. (24) extends only over the interior of the fluid core, rather than throughout the star. Consequently, these gravitational radiation time scales are somewhat longer (typically about 30% longer) than those computed earlier for hotter, completely fluid stars [11]. The fiducial gravitational radiation time scale $\tilde{\tau}_{GR}$ given in Table I is defined by

$$\tau_{GR}=\tilde{\tau}_{GR}\left(\frac{\Omega_0}{\Omega}\right)^{2m+2}. \tag{25}$$

Gravitational radiation tends to drive the r-modes unstable, while viscosity suppresses the instability. We define the critical angular velocity Ω_c, above which the r-mode is unstable, by the condition $\tau_v=\tau_{GR}$:

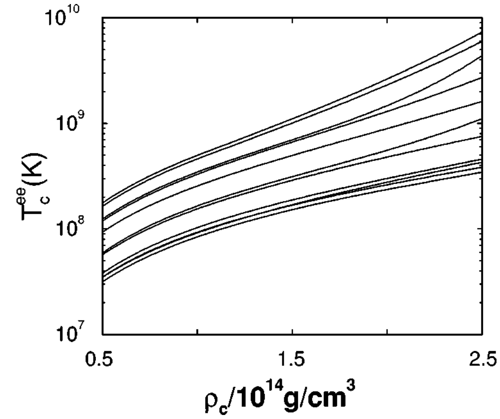

FIG. 2. Critical temperature T_c as a function of the crust formation density ρ_c for a number of realistic equations of state.

$$\frac{\Omega_c}{\Omega_0}=\left(\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_v}\right)^{2/11}\left(\frac{10^8\ \mathrm{K}}{T}\right)^{2/11}. \tag{26}$$

Figure 1 illustrates the temperature dependence of the critical angular velocity for $1.4M_\odot$ neutron stars constructed from a variety of realistic equations of state. The discontinuities in these curves at $T=10^9$ K occur because the superfluid transition changes the viscosity from electron-electron scattering at low temperatures to neutron-neutron scattering. The outer edge of the core in these models is taken to be $\rho_c=1.5\times10^{14}$ g cm^{-3} [13]. Equation (26) and the curves in Fig. 1 neglect the motion of the crust in the rotating frame; i.e., they assume that $\Delta v/v=1$. In general, Ω_c is a factor of $(\Delta v/v)^{4/11}$ smaller than that given in Eq. (26). Thus, Ω_c displayed in Fig. 1 provides an upper bound on the critical frequency for realistic neutron stars. In these expressions for Ω_c we neglect the effects of bulk viscosity, which are unimportant for $T\lesssim10^{10}$ K.

The angular velocity of a neutron star can never exceed some maximum value $\Omega_{\max}\approx\frac{2}{3}\Omega_0$ [10]. Thus, there is a critical temperature below which the gravitational radiation instability is completely suppressed by viscosity. This critical temperature is given by

$$\frac{T_c}{10^8\mathrm{K}}=\left(\frac{\Omega_0}{\Omega_{\max}}\right)^{11/2}\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_v}\approx\left(\frac{3}{2}\right)^{11/2}\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_v}. \tag{27}$$

In terms of T_c then the critical angular velocity can be expressed in a particularly simple form:

$$\frac{\Omega_c}{\Omega_0}=\frac{\Omega_{\max}}{\Omega_0}\left(\frac{T_c}{T}\right)^{2/11}\approx\frac{2}{3}\left(\frac{T_c}{T}\right)^{2/11}. \tag{28}$$

The values of this critical temperature are given in Table I for the case $\rho_c=1.5\times10^{14}$ g cm^{-3}. Since the exact density ρ_c where crust formation begins is only poorly known, we explore in Fig. 2 the dependence of T_c on ρ_c. The motion of the crust would change T_c of Eq. (27) by the factor $(\Delta v/v)^2$, and so the curves in Fig. 2 provide an upper bound on T_c for realistic neutron star models.

V. THERMAL STRUCTURE OF THE BOUNDARY LAYER

Viscous dissipation in the boundary layer between the core and the crust deposits thermal energy into a very thin layer of material. Previous authors assumed that this heat is effectively conducted away from the boundary layer, so that the star remains essentially isothermal. Clearly, this situation is idealized, and viscous dissipation will raise the temperature at the crust-core interface. The viscous heating competes with thermal conduction away from the boundary layer, as well as neutrino emission from the crust and the core. In this section we evaluate the *r*-mode amplitude needed to raise the temperature at the boundary layer to a given value. Our particular interest is to determine the minimum *r*-mode amplitude needed to raise the temperature to $T_m \approx 10^{10}(\rho/\rho_c)^{1/3}$ K, the melting temperature of the crust (see the Appendix for a discussion of the microphysics employed in this paper). If this occurs, the *r*-modes may continue to be unstable to much lower angular velocities (see Sec. VI).

In the discussion that follows, it will be necessary to know the explicit temperature dependences of the thermal conductivity and neutrino emissivity. As described in the Appendix, these temperature dependences are

$$\kappa = \frac{\tilde{\kappa}}{T}, \tag{29}$$

$$\epsilon = \tilde{\epsilon} T_{10}^n, \tag{30}$$

where $T_{10} = T/10^{10}$ K, and the exponent in the emissivity equation is $n=8$ in the fluid core and $n=6$ in the crust. The prefactors $\tilde{\kappa}$ and $\tilde{\epsilon}$ are independent of temperature and are given by $\tilde{\kappa}_< = 1.5\times 10^{31}$, $\tilde{\epsilon}_< = 8.6\times 10^{28}$ in the core and $\tilde{\kappa}_> = 2.8\times 10^{30}$, $\tilde{\epsilon}_> = 1.5\times 10^{25}$ in the crust. The quantities κ and ϵ are in cgs units. Note that for notational convenience in subsequent computations, the temperature in Eq. (29) is measured in K, while in Eq. (30) it is measured in 10^{10} K.

Let us first neglect the thermal conductivity altogether, and ask what *r*-mode amplitude is necessary to heat the boundary layer to T_m if the heat is radiated exclusively by neutrino emission from the boundary layer itself. The rate $d\tilde{E}/dt$ at which the shear deposits energy into the vicinity of the boundary layer is given by Eq. (17), where the amplitude A is related to the dimensionless *r*-mode amplitude, as defined by Lindblom, Owen and Morsink [12], via

$$\alpha = \sqrt{\frac{16\pi}{5}} \frac{R}{\Omega} A. \tag{31}$$

We equate this rate to the neutrino emission rate in the boundary layer, $4\pi R_c^2 d\tilde{\epsilon}_< T_{10}^8$, to obtain $\alpha_c(\text{local})$, the *r*-mode amplitude necessary to keep the crust-core interface at the melting temperature if thermal conduction is not important:

$$\alpha_c(\text{local}) = \frac{R}{R_c}\left(\frac{16\pi\sqrt{2}\,\tilde{\epsilon}_< T_{m,10}^8}{5\mathcal{I}_2 \rho_c R_c^2 \Omega^3}\right)^{1/2} = 1.3\times 10^{-4}\left(\frac{\Omega_0}{\Omega}\right)^{3/2} T_{m,10}^4. \tag{32}$$

In this equation the numerical prefactor has been evaluated using the standard $n=1$ polytropic stellar model. This estimate sets the lower bound on the critical melting amplitude, since heat conduction increases the volume that radiates heat by neutrino emission, and hence more viscous dissipation is required to maintain the interface at a certain temperature.

We now show how to evaluate the effect of finite conductivity. The general equation for the thermal evolution of the material in a neutron star is

$$C_v \partial_t T = -\epsilon + \vec{\nabla}\cdot(\kappa \vec{\nabla} T) + 2\eta\,\delta\sigma_{ab}^*\delta\sigma^{ab}, \tag{33}$$

where T is the temperature, C_v is the specific heat at constant density, ϵ is the neutrino emissivity, and κ is the thermal conductivity. This is a time-dependent equation, and in general the overall cooling of the star due to Urca neutrino emission must be followed along with the heating in the boundary layer. Let l denote the thickness of the region adjacent to the boundary layer whose temperature is raised above the ambient by the viscous dissipation. On a time scale $t_{\text{diff}} \approx C_v l^2/\kappa$ the heat generated in the boundary layer will diffuse throughout this larger region. In the neutron-star matter near the boundary layer this time scale is $t_{\text{diff}} \approx 8\times 10^{-5}(l/d)^2$ s, which depends on the temperature only through the boundary layer thickness, $d \propto T^{-1}$. The neutron star as a whole cools on the Urca cooling time scale, $t_{\text{cool}} \approx 30\, T_{10}^{-6}$ s [14]. If the diffusion time scale is shorter than the cooling time scale, then the temperature distribution in the region near the boundary layer will estabilish a quasi-equilibrium state in which the excess heat is conducted into a large enough volume for it to be radiated away by neutrinos. The temperature at the inner edge of this region slowly decreases, tracking the overall cooling of the star. As we shall see below, the width of this quasi-equilibrium layer is a few hundred times the thickness of the boundary layer itself. Thus $t_{\text{diff}} < t_{\text{cool}}$ and so a quasi-equilibrium state exists in which $C_v \partial_t T$ can be neglected compared to the heat conduction term $\partial_r(\kappa\partial_r T)$.

In the quasi-equilibrium region, but outside the boundary layer, the temperature distribution is described approximately by

$$\partial_r(\kappa \partial_r T) = \epsilon. \tag{34}$$

Using the simple dependence of κ and ϵ on temperature, given by Eqs. (29) and (30), it is straightforward to obtain a first integral of Eq. (34),

$$\left(\frac{\partial T_{10}}{\partial r}\right)^2 = \frac{2\tilde{\epsilon}}{n\tilde{\kappa}} T_{10}^2 (T_{10}^n - T_{0,10}^n), \tag{35}$$

where T_0 is the ambient temperature in the core where the heat flux $\kappa\partial_r T$ tends to zero.

The viscous energy generation in the boundary layer is determined by the shear given by Eq. (14). Here we re-express this energy generation rate as

$$2\,\eta\delta\sigma^*_{ab}\delta\sigma^{ab}=\alpha^2\frac{5\rho_c R_c{}^4\Omega_0^3}{32\pi R^2}\left(\frac{\Omega}{\Omega_0}\right)^3 F(r,\theta), \tag{36}$$

where the function $F(r,\theta)$ is defined in Eq. (15). This function falls off exponentially away from the boundary layer with the length scale d. Thus F can be reasonably approximated as a delta function,

$$F(r,\theta)\approx\sqrt{\frac{4\eta}{3\Omega\rho_c}}\frac{\delta(r-R_c)}{f^2(\theta)}, \tag{37}$$

where the angular function $f(\theta)$ is defined by

$$f^{-2}(\theta)=\sqrt{\tfrac{3}{8}}\sin^2\theta[(1-\cos\theta)^2p_+ +(1+\cos\theta)^2p_-]. \tag{38}$$

Now we return to the full equation for the thermal distribution in the vicinity of the boundary layer:

$$\partial_r(\kappa\partial_r T)=\epsilon-2\,\eta\delta\sigma^*_{ab}\delta\sigma^{ab}. \tag{39}$$

This equation can be integrated analytically, using Eq. (35), when we approximate the heating in the viscous boundary layer by Eq. (37). The r-mode amplitude α_c needed to raise the temperature to the value T_m is

$$\alpha_c^2\frac{5R_c^4\sqrt{\eta\rho_c\Omega^5}}{16\sqrt{3}\pi R^2 f^2(\theta)}=\sqrt{\tfrac{1}{4}\tilde{\epsilon}_<\tilde{\kappa}_<(T^8_{m,10}-T^8_{0,10})}+\sqrt{\tfrac{1}{3}\tilde{\epsilon}_>\tilde{\kappa}_>(T^6_{m,10}-T^6_{0,10})}. \tag{40}$$

For the values of the microphysical parameters described above, the critical amplitude satisfies

$$\frac{\alpha_c^2}{f^2(\theta)T^5_{m,10}}\left(\frac{\Omega}{\Omega_0}\right)^{5/2}=8.0\times10^{-6}\sqrt{1-\frac{T_0^8}{T_m^8}}+\frac{5.3\times10^{-8}}{T_{m,10}}\sqrt{1-\frac{T_0^6}{T_m^6}}. \tag{41}$$

We illustrate in Fig. 3 the dependence of this critical amplitude on the ambient temperature T_0 for the case $T_m=10^{10}$ K and $\theta=\pi/2$. The solid curve is the analytical expression given in Eq. (41). For comparison, we also include a selection of points computed numerically by solving the full differential equation (39) without making the assumption that the heat source is a delta function. Clearly, the analytical approximation is extremely good. We also see that, because of the steep dependence of the neutrino cooling rate on the temperature, the value of α_c is rather insensitive to T_0, except when $T_0\approx T_m$.

The prefactor of the second term on the right side of Eq. (41) is much smaller than the prefactor of the first term. Thus we see that the core plays a much more important role in determining α_c than the crust. This is easy to understand: At $T=T_m$, the neutrino emissivity in the core is approximately 5000 times larger than in the crust. Since we demand that the crust-core interface be kept at a fixed temperature, much more heat is emitted on the core side of the interface than on the crust side. Therefore, the heat flux into the core must be much higher than the flux into the crust. Since the heat flux into the crust can be neglected (i.e. $\tilde{\epsilon}_<\tilde{\kappa}_<\gg\tilde{\epsilon}_>\tilde{\kappa}_>$), we may obtain a much simpler expression for the critical amplitude:

$$\alpha_c\approx2.8\times10^{-3}T^{5/2}_{m,10}f(\theta)\left(\frac{\Omega_0}{\Omega}\right)^{5/4} \tag{42}$$

for the case $T_0\ll T_m$. For a maximally rotating neutron star, with $\Omega_{\max}\approx\frac{2}{3}\Omega_0$, the prefactor in the above equation is 4.7×10^{-3}.

The function $f(\theta)$, illustrated in Fig. 4, determines the angular dependence of the critical melting amplitude. Melting the crust near the poles requires a higher r-mode amplitude than melting at the equator. However, over a wide range of angles the critical amplitude needed to heat the interface to a given temperature is the same as the equatorial ($\theta=\pi/2$) value to within a factor of 2.

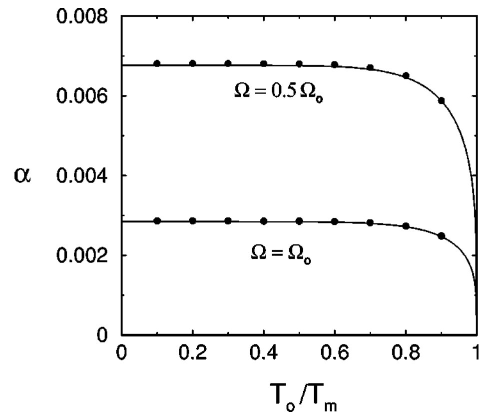

FIG. 3. Dependence of the critical amplitude α on T_0 for stars with $\Omega=0.5\Omega_0$. The smooth curve is the analytical expression given in Eq. (40), while the points are numerical solutions to the heat equation without the delta-function approximation for the boundary layer heating term.

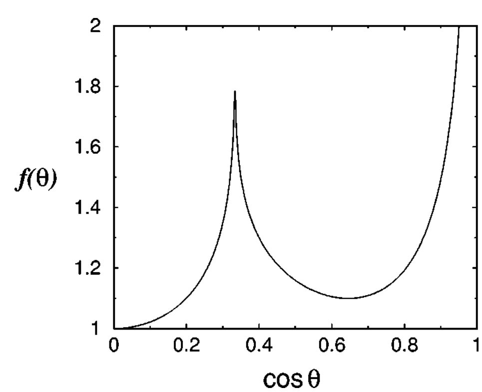

FIG. 4. Function f that determines the angular dependence of the critical amplitude α.

The critical melting amplitude α_c is roughly 20 times $\alpha_c(\text{local})$, our estimate that neglects the effect of thermal conduction, Eq. (32). From this we may deduce that the thickness of the layer into which thermal energy is conducted before being radiated away is of order $l/d \approx \alpha_c^2/\alpha_c^2(\text{local}) \approx 400$. From our numerical solutions, we confirm that the thermal flux in the core falls from its value at the crust-core interface to half that value at approximately $500d$. Thus the thickness of the layer that radiates the excess thermal energy is small enough (just a few centimeters) to justify the quasi-equilibrium analysis which we used.

Finally, we note that the heating rate used in this calculation neglects the motion of the crust in the rotating frame [6]. The motion of the crust increases the critical melting amplitude by the factor of $(\Delta v/v)^{-1}$.

VI. DISSIPATION IN THE ICE FLOW

As a young neutron star cools, its temperature quickly falls below the freezing temperature and a crust begins to form. If no unstable *r*-mode is present, a solid crust forms as usual. However, if the star is rapidly rotating, the $m=2$ *r*-mode will be driven unstable, and may grow to an appreciable amplitude even before the crust begins to form. Therefore, the crust would have to form in the presence of the shearing motion of the *r*-mode. Even in the presence of the crust, the *r*-mode is unstable at high enough frequencies, as illustrated in Fig. 1. Once the amplitude of the mode grows beyond the critical value given in Eq. (42), the dissipation in the boundary layer would be sufficient to re-melt the crust if it formed. If the crust fails to form or is completely melted away, the boundary layer heating would dissappear and the dissipation would not be sufficient to prevent freezing. Clearly, the state of the outer layers of the star can be neither pure solid nor pure fluid. We imagine that instead a mixed state will form: a neutron-star ice flow, much like the pack ice on the surface of the Arctic Ocean.

Microscopic viscosity will be replaced in this ice flow by dissipation that is dominated by collisions between macroscopic chunks of crust, the boundary layers around the ice chunks, or other mechanisms. However, regardless of the detailed dissipation mechanism, this flow is self-regulating: as long as the amplitude of the *r*-mode remains above the critical value of Eq. (42), the dissipation must, on average, be enough to keep the crust at around the melting temperature. Any less dissipation, and the crust would completely freeze, thus giving rise to a boundary layer which would re-melt the crust. Any more dissipation and the crust would simply melt, and then promptly re-form as the fluid viscosity alone is insufficient to keep the outer layers hot. The assumption of self-regulating flow allows us to compute the dissipation rate *regardless* of the details of the dissipation mechanism.

The dominant energy sink at high temperatures is bremsstrahlung neutrino emission, with emissivity given by Eq. (A6). Thus the viscous dissipation in the ice flow must just balance the losses due to neutrino emission. Presuming that the outer layers of the neutron star are kept at the local melting temperature given by Eq. (A2), we can determine the total energy dissipated in the ice flow, and hence an effective time scale τ_{ice} on which this ice flow dissipation will damp the *r*-mode:

$$\frac{1}{\tau_{\text{ice}}} = \frac{1}{2\tilde{E}} \int \tilde{\epsilon}\, T_{m,10}^6 \, d^3x. \tag{43}$$

Displaying explicitly the scalings of τ_{ice} with the mode amplitude α and the angular velocity Ω, we find

$$\tau_{\text{ice}} = \alpha^2 \left(\frac{\Omega}{\Omega_0}\right)^2 \tilde{\tau}_{\text{ice}}. \tag{44}$$

The fiducial timescales $\tilde{\tau}_{\text{ice}}$ range from 1.4×10^8 s to 1.3×10^9 s for the ‘‘realistic’’ equations of state discussed above. Unlike the other dissipative time scales in this problem, this one depends on the mode amplitude α.

The *r*-mode will continue to be unstable, and thus will continue to spin down the neutron star by emitting gravitational radiation, as long as the gravitational radiation time scale is shorter than the ice flow viscosity time scale, and its amplitude remains above the value needed to sustain the ice flow. The critical angular velocity where the equality is achieved and the mode becomes stable is

$$\frac{\Omega_c}{\Omega_0} = \left(\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_{\text{ice}}}\right)^{1/8} \alpha^{-1/4}. \tag{45}$$

For $\alpha = 1$, this critical angular velocity has the value $0.093\Omega_0$ for the standard $n=1$ polytropic stellar model and values that range from $0.086\Omega_0$ to $0.14\Omega_0$ for the realistic models considered here. (Note that the gravitational radiation time scale τ_{GR} used here is the one appropriate for a mode that extends to the surface of the star, not the one given in Table I for a mode that extends only to the edge of the core.) Thus we see that if the amplitude of the *r*-mode saturates at a value that is close to unity, the instability will continue—even in the presence of a crust—until the angular velocity of the star is a small fraction of Ω_0.

Is the ice-flow picture described above consistent? Clearly the dissipation due to molecular viscosity is insufficient to keep the outer layers of the neutron star hot, but can collisions between macroscopic ice chunks accomplish this? Let us estimate the characteristic size of the ice chunks needed to produce the needed macroscopic viscosity. We assume that the ice chunks are of average size D and the mean free path between them is λ. If $\lambda \gg D$, then the collisions between the chunks are rare, and do not significantly raise the viscosity compared to the purely fluid value (see Sec. 22 of Landau and Lifshitz [15]). We therefore consider the regime where the ice chunks are densely packed, with $D \gtrsim \lambda$, and behave like a fluid. We follow the discussion of Haff [16] and Borderies *et al.* [17].

The ice chunks mainly follow the *r*-mode velocity flow, but due to the collisions between themselves acquire a random component of velocity, $\tilde{v}$. The viscosity in such a flow is given by $\eta_{\text{ice}} \approx \rho D^2 \tilde{v}/\lambda$ [16,17], and is a factor of D/λ bigger than the usual molecular viscosity. This is because each particle transports momentum across a distance D while only having traveled the (possibly much smaller) distance λ between each collision.

The collisions between the chunks are inelastic with a fraction γ of the collision energy per chunk, $\rho D^3\tilde{v}^2$, being converted to heat. Considerations of icy particle collisions in the temperature regime appropriate for Saturn's rings put $\gamma \approx 0.7$–0.9 (see Borderies *et al.* [18] for a review). Since our flow is near the melting temperature, we expect very inelastic collisions as well, i.e., $\gamma \approx 1$. The collision rate is just $\tilde{v}/\lambda$, so the energy dissipation per unit volume in the flow is $\gamma\rho\tilde{v}^3/\lambda$. For the r-modes the shear dissipation rate is $2\eta_{\rm ice}\delta\sigma^*_{ab}\delta\sigma^{ab} \approx \eta_{\rm ice}\alpha^2\Omega^2$, so $\tilde{v} \approx \gamma^{-1/2}\alpha\Omega D$. Thus, when the collisions are inelastic, the random component of the ice chunk velocity is comparable to the velocity difference between the neighboring chunks in the overall ice flow. The viscosity of this granular flow is therefore given by

$$\eta_{\rm ice} \approx \frac{\rho\alpha\Omega D^3}{\gamma^{1/2}\lambda} \approx 10^{18}\ \frac{\alpha D^3}{\gamma^{1/2}\lambda}\left(\frac{\rho}{\rho_c}\right)\left(\frac{\Omega}{\Omega_0}\right), \tag{46}$$

where λ and D are measured in centimeters and $\eta_{\rm ice}$ is in cgs units. Not surprisingly, this viscosity is much larger than the microscopic viscosities given by Eqs. (21) and (22).

The energy dissipation rate in the ice flow is obtained by integrating $\eta_{\rm ice}\delta\sigma^*_{ab}\delta\sigma^{ab}$ over the outer layers of the star:

$$\frac{dE}{dt} \approx \frac{(\alpha\Omega D)^3 M_{\rm cr}}{\gamma^{1/2}\lambda}, \tag{47}$$

where $M_{\rm cr}$ is the mass of the material at densities less than ρ_c. Using the r-mode energy for an $n=1$ polytrope (see Table I), the damping time due to this form of viscous dissipation is

$$\tau_{\rm ice} \approx 10^8\ \text{s}\ \frac{\gamma^{1/2}\lambda}{\alpha D^3}\left(\frac{\Omega_0}{\Omega}\right)\left(\frac{0.05M}{M_{\rm cr}}\right). \tag{48}$$

The heat deposited by the ice flow into the star is radiated by neutrinos on the time scale evaluated in Eq. (44). Equating these two time scales gives an estimate of the average ice chunk size necessary to keep the crust at the local melting temperature:

$$D \approx 1\ \text{cm}\ \left[\left(\frac{10^8\ \text{s}}{\tilde{\tau}_{\rm ice}}\right)\left(\frac{\gamma^{1/2}\lambda}{D}\right)\left(\frac{\Omega_0}{\alpha\Omega}\right)^3\left(\frac{0.05M}{M_{\rm cr}}\right)\right]^{1/2}. \tag{49}$$

Our estimate of the ice chunk size depends only weakly on the unknown inelasticity of the collisions and the ratio λ/D. We do not expect λ to be much smaller than D, because if it were, the ice chunks would probably lock together into bigger pieces, leading to increased friction and dissipation. Moreover, as argued above, we expect $\gamma \approx 1$ as well. Thus, for typical values of $\tilde{\tau}_{\rm ice}$ evaluated above, the value of the ice chunk size needed to keep the outer layers of the star at the melting temperature is $D \approx 1$ cm.

Several conclusions can be drawn from this estimate. First, the smaller the mode amplitude, the larger the ice chunks have to be in order to provide enough friction (recall that the viscosity is proportional to D^2) to keep the fluid at the melting temperature in the face of neutrino losses. The chunk size exceeds the radius of the star when $\alpha \lesssim 10^{-4}(\Omega_0/\Omega)(\lambda/D)^{1/3}$. Hence, if the mode amplitude is smaller than this value, the ice-flow mechanism is unable to keep the outer part of the star in a melted state, and it must freeze. Since this amplitude is smaller than the critical melting amplitude computed in the previous section, this does not further restrict the range of mode amplitudes where the ice flow is expected to occur. Second, even for $\alpha \approx 1$ the chunk size is small but not microscopic. So our picture of the ice flow is consistent throughout the regime of interest.

The argument that the ice flow maintains the star's outer layers near melting depends on the ability of the r-mode to re-heat a solid crust back to the verge of melting, should a crust form through some fluctuation. Up to this point we assumed that this heating is instantaneous once α exceeds α_c. This is not an issue if $\Omega > \Omega_c$ [the critical angular velocity for the r-mode instability, Eq. (26)], since the amplitude of the r-mode is growing due to radiation reaction in this case. However, if $\Omega < \Omega_c$, the r-mode is decaying due to the dissipation in the boundary layer even while it is reheating the crust. Can the r-mode re-heat the crust and reform the ice flow before being damped? A simple comparison of the energy needed to melt the crust, E_m, to the mode energy, $\tilde{E}$ (see Table I), is inadequate, since it neglects the continuing energy input from gravitational radiation reaction.

The r-mode melts the crust on a time scale

$$\tau_m = \frac{E_m}{\beta(d\tilde{E}/dt)_v} = \tau_v\frac{E_m}{2\beta\tilde{E}}, \tag{50}$$

where the factor β is the fraction of heat flowing into the crust, which is given approximately by $\beta = (4\tilde{\epsilon}_>\tilde{\kappa}_>/3\tilde{\epsilon}_<\tilde{\kappa}_<)^{1/2} \approx 0.007$ [see Eq. (35)]. While it is melting the crust, the r-mode is also damped at the rate $1/\tau_d = 1/\tau_v - 1/\tau_{GR}$. The damping rate is exactly zero on the stability curve $\Omega = \Omega_c$. However, since the r-mode has a finite amplitude, it will continue to spin the star down below the stability line while melting the crust.

Let $\Delta\Omega$ be the change in spin frequency, due to the r-mode evolution, during the time τ_m. For small α, the spin-down rate is given by Eq. (3.14) of Owen *et al.* [19]:

$$\frac{1}{\Omega}\frac{d\Omega}{dt} \approx \frac{0.2\alpha^2}{\tau_v}. \tag{51}$$

Thus, in the time it takes to melt the crust, the star spins down by $\Delta\Omega/\Omega_c \approx 0.1\alpha^2 E_m/\beta\tilde{E}$ (which is independent of α since $\tilde{E} \propto \alpha^2$). By using a Taylor expansion, it is easy to show that the r-mode damping time for $\Omega = \Omega_c - \Delta\Omega$ is approximately

$$\tau_d \approx \frac{2}{11}\frac{\Omega_c}{\Delta\Omega}\tau_v = \frac{2\beta\tilde{E}}{\alpha^2 E_m}\tau_v . \qquad (52)$$

If $\tau_m < \tau_d$, we conclude that the *r*-mode can re-melt the crust and so re-create the ice flow before being damped out. This condition places the following limit on the mode amplitude α:

$$\alpha \gtrsim \frac{E_m}{2\beta\tilde{e}}\left(\frac{\Omega_0}{\Omega}\right)^2 \approx 3\times10^{-3}\ \left(\frac{\Omega_0}{\Omega}\right)^2. \qquad (53)$$

In the above equation, we assumed that the energy required to re-melt the crust is of order kT per nucleus (since the crust is near the melting temperature), i.e. $E_m \approx 10^{47} T_{m,10}$ erg. This amplitude is very close, both in the absolute magnitude and in the scaling with Ω and T_m, to the critical melting amplitude α_c. Hence, as long as the *r*-mode amplitude is large enough to raise the crust-core interface to the melting temperature, the energy contained in the mode is enough to melt the crust even if the *r*-mode is no longer linearly unstable. Since the criterion (53) does not depend on $\Delta v/v$ while α_c [Eq. (42)] increases with $\Delta v/v$, the conclusion does not change even when the motion of the crust is taken into account.

VII. DISCUSSION

We have computed stability curves (critical frequency as a function of temperature) for the onset of the *r*-mode instability in a neutron star with a (laminar) viscous boundary layer under a solid crust. We improve previous calculations by including the effect of the Coriolis force (the dominant restoring force) on the boundary layer and by using realistic neutron-star models—two important ingredients which have not been combined in previous work. Our stability results are summarized in Eqs. (27) and (28), Table I, and Figs. 1 and 2. If the neutron star crust is rigid and does not move in the rotating frame, then our results imply that the *r*-modes are not unstable in any of the accreting neutron stars in LMXBs or millisecond pulsars. However, if the relative velocity amplitude between the core and the crust, $\Delta v/v$, is significantly smaller than 1, as recent calculations for constant-density stars suggest [6], then our results constitute only the upper limits on the critical frequencies and temperatures for the onset of the *r*-mode instability. A self-consistent calculation of the *r*-mode eigenfunctions in the presence of a realistic crust is necessary to settle (at least for unmagnetized neutron stars) the question of linear stability of the *r*-modes.

We find that localized heating in the boundary layer between the solid crust and the fluid core can successfully compete both with the heat conduction away from the boundary layer and with neutrino emission. In Sec. V, we computed the critical *r*-mode amplitude, Eq. (42), needed to raise the temperature of the boundary layer to the crust melting temperature while the interior of the star far away from the boundary layer remains at a much lower temperature. The amplitude required for this is rather small, of order 10^{-2}, and is comparable to that necessary to break up the crust via purely mechanical coupling.

Based on the smallness of the melting amplitude, we argue in Sec. VI that the spindown scenarios for neutron stars with crusts advanced by previous authors may need to be significantly modified. As long as the *r*-mode amplitude remains above the melting value, a completely solid crust is not possible. Instead, the outer layers of a neutron star will resemble an ice pack which flows along with the *r*-mode motion. Viscosity in the ice pack adjusts itself to maintain the outer layers of the neutron star near the crust's melting temperature, regardless of the details of the dissipation in the ice flow. The ice flow can persist even if the *r*-mode is no longer linearly unstable according to the instability criterion for neutron stars with crusts, since an *r*-mode with an amplitude greater than some minimum value [given in Eq. (53) for the case $\Delta\Omega || \Omega_c$] can melt the crust before being damped by boundary-layer friction. Therefore, if the crust is melted either during spindown of young neutron stars or in the final stage of a thermal runaway in LMXBs [20], the final spin frequency is not set by the boundary-layer damping time, but instead by the balance between $\tau_{\rm ice}$ and $\tau_{\rm GR}$. This leads to much smaller final spin frequencies than computed by previous authors. A neutron star with an *r*-mode amplitude large enough to create the ice pack can spin down to a frequency nearly as low as a purely fluid star [see Eq. (45)], provided the saturation amplitude of a purely fluid *r*-mode of is order unity.

Very recently, Wu *et al.* [21] evaluated the effect of a turbulent boundary layer between the crust and the core on the stability of the *r*-modes. They find that the turbulent dissipation time scale depends on the *r*-mode amplitude and therefore determines a saturation amplitude, which they estimate (in our notation) as $\alpha_{\rm sat} \approx 1.5 \times 10^{-2} (\Omega/\Omega_0)^5 (\Delta v/v)^{-3}$, or about 2.0×10^{-3} for a maximally rotating star with a rigid crust. Inserting Eq. (10) of Wu *et al.* [21] into our Eq. (39), we find that a turbulent boundary layer changes the critical melting amplitude at the equator to

$$\alpha_c \approx 5.6\times10^{-4}\left(\frac{\Omega_0}{\Omega}\right)\left(\frac{\Delta v}{v}\right)^{-1} \qquad (54)$$

or $\alpha_c \approx 7.9\times10^{-4}$ for a maximally rotating polytropic neutron star with a rigid crust. Comparing with Eq. (42), we conclude that the crust can be heated to the melting temperature at even smaller amplitudes than in the presence of a laminar boundary layer. Thus, if the *r*-mode amplitude in a newborn neutron star exceeds the critical value given above when the temperature drops below the crust's melting temperature, then crust formation will be delayed (as we argued in Sec. VI). In this case the star will spin down to frequencies close to those predicted for crustless fluid stars, provided the saturation amplitude of the *r*-mode is of order unity.

ACKNOWLEDGMENTS

We thank C. Cutler, P. Goldreich, Y. Levin, and B. Schutz for helpful discussions concerning this work. This work has been supported by NSF Grant Nos. PHY-9796079 and AST-9618537, and by NASA Grant Nos. NAG5-4093 and NGC5-7034.

APPENDIX: THERMODYNAMIC PROPERTIES OF NEUTRON-STAR MATTER

The strength of electrostatic interactions between the nuclei in neutron-star matter is typically expressed in terms of the Coulomb coupling parameter,

$$\Gamma = \frac{Z^2 e^2}{akT}, \tag{A1}$$

where Z is the nuclear charge, and $a=(3A_{\rm cell}m_b/4\pi\rho)^{1/3}$ is the radius of the Wigner-Seitz cell with $A_{\rm cell}$ nucleons in it. According to the recent molecular-dynamics simulations of Farouki and Hamaguchi [22], such a classical one-component plasma crystallizes when $\Gamma \gtrsim 173$. However, the details of the nucleon-nucleon interaction that determine Z and $A_{\rm cell}$ near the bottom of the crust, $\rho=\rho_c$, are still not completely understood. Depending on the particular interaction model, values of Z ranging from approximately 30 to 50 and $A_{\rm cell}$ in the range of 500 to 1000 are obtained. These values result in melting temperatures within a factor of 2 of 10^{10} K. (See Pethick and Ravenhall [23] for a review of the nuclear physics, Douchin and Haensel [24] for the latest calculation, and Haensel [25] for a discussion of the melting temperatures appropriate for young neutron stars and LMXBs.) Moreover, for some nuclear force models, nuclei at $\rho \gtrsim 10^{14}$ g cm^{-3} may resemble rods, plates, and tubes, rather than spheres (Lorentz *et al.* [26]). The melting temperatures of such exotic phases of matter have not been calculated, but it is reasonable to assume [27] that they will be comparable to those of ''ordinary'' matter with spherical nuclei. In the light of these uncertainties, we adopt

$$T_m = 10^{10}\left(\frac{\rho}{\rho_c}\right)^{1/3} \text{ K}, \tag{A2}$$

as the fiducial melting temperature of crustal matter.

The thermal conductivity in the neutron-star core is dominated by electron-electron collisions. A convenient fit to the conductivity is given by Flowers and Itoh [28]:

$$\kappa = \frac{1.5\times 10^{21}}{T_{10}} \text{ erg cm}^{-1}\text{ s}^{-1}\text{ K}^{-1}, \tag{A3}$$

at $\rho=\rho_c$, where $T_{10}=T/10^{10}$ K. The neutrino emissivity in the core is given by the usual modified Urca formula [29,30]:

$$\epsilon = 8.6\times 10^{28}\, T_{10}^8 \text{ erg cm}^{-3}\text{ s}^{-1}, \tag{A4}$$

at $\rho=\rho_c$. These lead to the expressions for the core values, $\tilde{\kappa}_< = 1.5\times 10^{31}$ and $\tilde{\epsilon}_< = 8.6\times 10^{28}$, used in Sec. V.

The microphysics in the crust is, in general, quite a bit more complicated. However, we are primarily interested in the conditions at or near the melting temperature. For $T \gtrsim 2\times 10^{10}$ K, the crustal conductivity is also dominated by electron-electron collisions, and even at $T\approx 10^{10}$ K, to within a factor of 2,

$$\kappa \approx \kappa_{ee} \approx \frac{2.8\times 10^{20}}{T_{10}} \text{ erg cm}^{-1}\text{ s}^{-1}\text{ K}^{-1}, \tag{A5}$$

where we approximated the fits of Flowers and Itoh [28] for the temperature regime of interest. For temperatures lower than this, the conductivity is dominated by electron-phonon scattering, which is approximately constant, $\kappa_{e\text{-}ph} \approx 10^{20}$ for temperatures in excess of the Debye temperature, $T_d \approx 5\times 10^9(\rho/\rho_c)^{1/2}$ K. For $T \ll T_d$, the electron-phonon scattering has the same temperature dependence (but a different prefactor) as electron-electron scattering, $\kappa_{e\text{-}ph} \approx 1\times 10^{19}/T_{10}$. We also find that the recent calculations of neutrino-pair bremsstrahlung in the crust (Haensel *et al.* [31] and Kaminker *et al.* [32]) can be reasonably approximated by

$$\epsilon \approx 1.5\times 10^{25}\left(\frac{\rho}{\rho_c}\right) T_{10}^6 \text{ erg cm}^{-3}\text{ s}^{-1}. \tag{A6}$$

These expressions lead to the crust values $\tilde{\kappa}_> = 2.8\times 10^{30}$ and $\tilde{\epsilon}_> = 1.5\times 10^{25}$ that we use in Sec. V. Since the crust does not play a significant role in determining the temperature at the boundary layer (as shown in Sec. V), including a detailed treatment of the microphysics there is not essential.

[1] J. L. Friedman and K. H. Lockitch, Prog. Theor. Phys. Suppl. **136**, 121 (1999).

[2] L. Bildsten and G. Ushomirsky, Astrophys. J. Lett. **529**, L33 (2000).

[3] N. Andersson, D. I. Jones, K. D. Kokkotas, and N. Stergioulas, Astrophys. J. Lett. **534**, L75 (2000).

[4] M. Rieutord, astro-ph/0003171.

[5] H. P. Greenspan, *The Theory of Rotating Fluids* (Cambridge University Press, Cambridge, England, 1969).

[6] Y. Levin and G. Ushomirsky, astro-ph/0006028.

[7] B. J. Owen, to appear in the Proceedings of the 3rd Edoardo Amaldi Conference on Gravitational Waves, gr-qc/9912065.

[8] E. Flowers and N. Itoh, Astrophys. J. **230**, 847 (1979).

[9] C. Cutler and L. Lindblom, Astrophys. J. **314**, 234 (1987).

[10] S. Bonazzola, J. Frieben, and E. Gourgoulhon, Astrophys. J. **460**, 379 (1996).

[11] For a star with a non-rigid crust the mode energy $\tilde{e}_c$ is slightly greater and the gravitational radiation time scale τ_{GR} is slightly shorter than the values displayed in Table I. However, the differences depend on $\Delta v/v$ in a manner more complicated than simple scaling, and therefore we do not explore them in this paper.

[12] L. Lindblom, B. J. Owen, and S. M. Morsink, Phys. Rev. Lett. **80**, 4843 (1998).

[13] C. J. Pethick, D. G. Ravenhall, and C. P. Lorenz, Nucl. Phys. **A584**, 675 (1995).

[14] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (Wiley, New York, 1983).
[15] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1959).
[16] P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
[17] N. Borderies, P. Goldreich, and S. Tremaine, Icarus **63**, 406 (1985).
[18] N. Borderies, P. Goldreich, and S. Tremaine, in *Planetary Rings*, edited by R. Greenberg and A. Brahic (University of Arizona Press, Tucson, AZ, 1984), p. 713.
[19] B. J. Owen, L. Lindblom, C. Cutler, B. F. Schutz, A. Vecchio, and N. Andersson, Phys. Rev. D **58**, 084020 (1998).
[20] Y. Levin, Astrophys. J. **517**, 328 (1999).
[21] Y. Wu, C. D. Matzner, and P. Arras, astro-ph/0006123.
[22] R. T. Farouki and S. Hamaguchi, Phys. Rev. E **47**, 4330 (1993).
[23] C. J. Pethick and D. G. Ravenhall, Annu. Rev. Nucl. Part. Sci. **45**, 429 (1995).
[24] F. Douchin and P. Haensel, Phys. Lett. B **485**, 107 (2000).
[25] P. Haensel, in *Relativistic Gravitation and Gravitational Radiation*, edited by J.-P. LaSota and J.-A. Marck (Cambridge University Press, Cambridge, England, 1997).
[26] C. P. Lorenz, D. G. Ravenhall, and C. J. Pethick, Phys. Rev. Lett. **70**, 379 (1993).
[27] C. J. Pethick and A. Y. Potekhin, Phys. Lett. B **427**, 7 (1998).
[28] E. Flowers and N. Itoh, Astrophys. J. **250**, 750 (1981).
[29] B. L. Friman and O. V. Maxwell, Astrophys. J. **232**, 541 (1979).
[30] D. G. Yakovlev and K. P. Levenfish, Astron. Astrophys. **297**, 717 (1995).
[31] P. Haensel, A. D. Kaminker, and D. G. Yakovlev, Astron. Astrophys. **314**, 328 (1996).
[32] A. D. Kaminker, C. J. Pethick, A. Y. Potekhin, V. Thorsson, and D. G. Yakovlev, Astron. Astrophys. **343**, 1009 (1999).