Instabilities of Rotating Neutron Stars

Lee Lindblom

Theoretical Astrophysics 130-33
California Institute of Technology, Pasadena, CA 91125, USA
lindblom@tapir.caltech.edu

Abstract
This paper reviews the analysis of instabilities of rapidly rotating stellar models. Particular emphasis is given here to those instabilities driven by dissipative processes (e.g. viscosity and gravitational radiation emission) that are expected to play a significant role in influencing the observable properties of rotating neutron stars.

1 Introduction

Observations are beginning to provide a wealth of information about rapidly rotating stars in which relativistic effects play an important role. Measurements of the periods of pulsars show that neutron stars can rotate with periods as short as 1.56ms (Backer et al. 1989). Measurements of the orbital elements of binary systems containing pulsars now give accurate determinations of the masses of about ten neutron stars (Thorsett et al. 1993). To understand the meaning of these (and other related) observations, the appropriate theoretical tools must be developed for analyzing the structures and stability of rapidly rotating relativistic stellar models. The techniques for constructing and analyzing equilibrium models from a given equation of state are now well understood (Friedman et al. 1986, Cook et al. 1994). Thus it is relatively easy to compare the observable macroscopic properties (e.g. masses, angular velocities, etc.) of these models with the observations. The inverse problem of determining the poorly known equation of state from the observable properties of relativistic stars is only beginning to be understood however (Lindblom 1992).
In addition to a thorough understanding of the structures of equilibrium stellar models, the stability of these models must also be understood in order to interpret the observations. Stability theory is required, for example, to determine the ranges of masses and angular velocities present in stable (and thus physically possible) stars. This paper reviews the theory of the stability of rapidly rotating stellar models in both the Newtonian theory and general relativity. The emphasis here is on the effects of dissipation on the stability of these stars. Instabilities driven by dissipative processes may well determine the maximum rotation rates of neutron stars. The discussion here also attempts to point out issues and questions on which further analysis is needed.

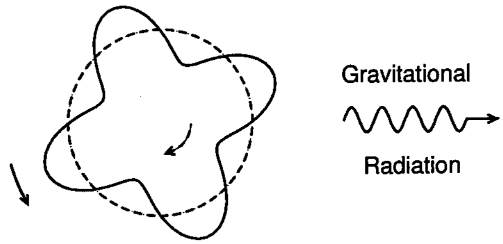

Figure 1 Non-radial modes of rotating stars emit gravitational radiation. These modes are driven unstable in stars rotating sufficiently rapidly that the pattern speed changes from counter-rotating to co-rotating.

Our understanding of the stability of rotating stars was (until quite recently) based entirely on the analysis of the uniform density rigidly rotating stellar models: the Maclaurin spheroids. It has been known for over a century that rapidly rotating Maclaurin spheroids are subject to an instability driven by viscosity (Thompson and Tait 1883). This instability causes a rapidly rotating Maclaurin spheroid to evolve into a rigidly rotating but non-axisymmetric configuration such as a Jacobi ellipsoid (Roberts and Stewartson 1963, Press and Teukolsky 1973). This type of instability is referred to as *secular* since it is driven by dissipative forces in the star. The Maclaurin spheroids are also subject to a second type of secular instability that is driven by gravitational radiation (Chandrasekhar 1970a,b). This instability causes the Maclaurin spheroid to evolve into a stationary but non-axisymmetric configuration such as a Dedekind ellipsoid (Detweiler and Lindblom 1977). Maclaurin spheroids with very large angular momenta (about 1.7 times that required to trigger the viscous secular instability) are also subject to a *dynamical* instability that is driven by purely hydrodynamical forces (Chandrasekhar 1969). Secular instabilities grow on time scales proportional to the strength of the dissipative process. These secular time scales are generally much longer than the characteristic hydrodynamic time scale of the system. Dynamical instabilities grow on time scales that are comparable to the hydrodynamic time scale.

The gravitational radiation driven secular instability is of particular interest in the study of neutron stars. Neutron stars have comparatively strong gravitational fields and many of their modes couple strongly to gravitational radiation. Thus, the time scale on which the gravitational radiation instability can act in these stars is relatively short. Further, the gravitational radiation instability was shown by Friedman (1978) and Friedman and Schutz (1978) to be generic: every rotating perfect fluid star has some mode that is driven unstable by this mechanism. The physical nature of this instability mechanism can be visualized as follows. Consider the perturbation of a slowly rotating neutron star depicted in Fig. 1. The surface of the unperturbed star (viewed from above the rotation axis) is depicted as a dashed line and the perturbed surface by a solid line in this figure. The star rotates in a clockwise direction while the perturbation—rather like a wave that propagates along the surface of the star—moves in the counter-clockwise direction. This perturbation carries negative (relative to the direction of the unperturbed star's rotation) angular momentum, and emits negative angular momentum gravitational radiation to infinity. As gravitational radiation removes energy and angular momentum from the perturbed fluid, the fluid motion is damped away.

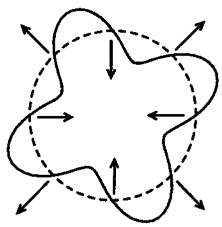

Figure 2 Non-radial modes are strongly sheared and thus strongly damped by internal fluid dissipation (e.g. viscosity).

Consider now the analogous mode in a very rapidly rotating neutron star. If the star rotates rapidly enough the pattern speed of the mode as seen from infinity would change from negative (counter-clockwise) to positive (clockwise). Under this circumstance the mode would emit positive angular momentum gravitational radiation, since it would be seen from infinity to propagate in the same direction as the star's rotation. The hydrodynamic waves themselves, however, carry negative angular momentum since they still propagate (relative to the fluid in the star) in the direction opposite the star's rotation. Angular momentum can only be conserved for this perturbation by increasing its amplitude in order to decrease its angular momentum. Thus any counter-rotating mode becomes unstable to this mechanism when the star's angular velocity reverses its propagation direction as seen from infinity. Friedman and Schutz (1978) and Friedman (1978) have shown that this happens to some mode in every rotating perfect fluid star.

Contrary to the argument above, however, all rotating neutron stars are not unstable. Figure 2 illustrates why. It depicts the same perturbation as shown in Fig. 1 with the velocities of selected fluid elements near the surface indicated. At any given instant of time, those fluid elements on the leading edge of the perturbation must move radially outward as the wave travels under them. Similarly those fluid elements on the trailing edge of the wave move radially inward as the wave passes them by. This motion is highly sheared and consequently internal fluid dissipation (e.g. viscosity) tends to damp out this type of motion. Those modes which are unstable to gravitational radiation emission in very slowly rotating stars are very high order multipole modes. (Only in these modes is the pattern speed slow enough to be overcome by the star's rotation.) These modes are very strongly sheared and couple strongly to viscosity. In contrast, gravitational radiation couples only weakly to the higher order multipole moments. Thus, the presence of any viscosity will completely suppress the gravitational radiation instability in sufficiently slowly rotating stars (Lindblom and Hiscock 1983).

The secular instabilities of primary interest here all involve non-axisymmetric perturbations of rotating stars. While there exist interesting axisymmetric instabilities (e.g. those that determine the maximum and minimum masses of neutron stars) these have been relatively well understood for some time now. The interested reader is referred to the literature for a discussion of the specialized analytical techniques developed to study this type of instability [e.g. for a recent review see Lindblom (1996)]. The primary focus here is on the study of the stability of non-axisymmetric perturbations of rapidly rotating stars—a subject that is far from completely understood even now.[1] During the past

[1] Most of the analysis described here also applies to the particular case of axisymmetric perturbations, but that special case is not emphasized.

two decades new mathematical techniques have been developed that make it possible to analyze the stability of such perturbations in rapidly rotating stellar models composed of fluid with any realistic equation of state. These recent developments are described and reviewed in this paper. Section 2 describes certain general criteria for evaluating the stability of rotating stars with respect to general non-axisymmetric perturbations. These criteria can be applied without any specific knowledge about the properties of the normal modes of these stars. Section 3 presents the analytical techniques needed to analyze the non-axisymmetric normal modes of rotating stars in the Newtonian theory and in general relativity. For certain types of instabilities the only tools presently available for analyzing instability are the normal modes. Section 4 completes the discussion of the normal modes by showing how dissipation effects their evolution and stability.

2 General Stability Criteria

The stability of stars with respect to non-axisymmetric perturbations is an interesting and difficult subject, and this has been the focus of most of the research effort in this area in recent years. For the case of non-rotating stars, the situation turned out to be remarkably simple. In this case the stability of the star is determined completely by the quantity $S(r)$ defined by

$$S(r) = \frac{dp}{dr} - \left(\frac{\partial p}{\partial \rho}\right)_s \frac{d\rho}{dr}, \tag{1}$$

where ρ, p, and s are the density, pressure, and specific entropy of the fluid in the star, and r is the radial spherical coordinate. When S is positive the adiabatic exchange of fluid masses at different "elevations" within the star requires the addition of energy to the system (Schwarzschild 1958). When S is negative in some region, however, the energy of the configuration can be lowered by re-arranging the fluid. In this region, consequently, the stellar fluid is unstable to convection. It has been shown that the condition $S > 0$ everywhere within the star is the necessary and sufficient condition for the stability of the non-radial modes of Newtonian stellar models (Lebovitz 1965). In general relativity theory it has also been shown that the non-radial outgoing modes are stable if $S > 0$ throughout the star (Detweiler and Ipser 1973). The proof that this is also a necessary condition for stability in general relativity theory has not been completed to date.
Simple local stability conditions analogous to eq. (1) have not been found and probably do not exist for rotating stars. A few global conditions have been found, however, and these have been extremely helpful in understanding a number of interesting instabilities in rotating stars. These global conditions determine the stability of rotating stars from the properties of certain non-local functionals of the perturbations. Perhaps the most important example of such a functional is the energy E. This functional can be expressed as a Hermitian quadratic form in the perturbation fields integrated over the volume of the star.[2] For perturbations that satisfy dissipation-free (e.g. no viscosity) evolution equations the energy E is conserved for all perturbations. Thus E is not a useful tool for diagnosing the presence of dynamical instabilities. When the effects of dissipation are considered, however, the energy functional E evolves with time; and in

[2] The explicit expression for this functional is long, complicated, and not particularly enlightening. The interested reader is referred to the literature: Friedman and Schutz (1978) and Friedman (1978).

some circumstances it decreases monotonically for all fluid perturbations. Under these conditions E can be used to diagnose secular instabilities. If E is positive for all possible perturbations then the star is stable. The evolution equations in this case may only change E by decreasing its value toward its lower bound, zero. This ensures that the perturbation remains bounded (at least in an $\mathcal{L}^2$ sense). If the energy E were negative for some perturbation, however, then E would have no lower bound. The evolution equations would cause a perturbation with negative E to decrease without bound and the star would be unstable. The emission of gravitational radiation by a stellar perturbation causes the energy functional E to decrease. Thus E can be used to test the secular stability of rotating stars with respect to the emission of gravitational radiation. When the functional E for rotating stars is examined in detail a remarkable fact emerges: every rotating star is unstable to the emission of gravitational radiation (Friedman and Schutz 1978, Friedman 1978). That is, there exists some perturbation in every rotating star for which E is negative. An example of such a perturbation is illustrated in Fig. 1 above. When the star rotates sufficiently rapidly that the pattern speed of the wave becomes co-rotating, then the energy functional E becomes negative. The analysis of Friedman and Schutz (1978) and Friedman (1978) shows that a perturbation with negative energy E can be found for any rotating star simply by choosing the wavelength of the perturbation to be sufficiently small. Thus, every rotating star is unstable to the emission of gravitational radiation.

A closely related functional $\tilde{E}$, which represents the energy of a perturbation as measured in the co-rotating reference frame of the star, has also been useful for diagnosing instabilities in rotating stars. For Newtonian stellar models this functional has an extremely simple form:

$$\tilde{E} = \tfrac{1}{2}\int \left(\rho\,\delta v_a^* \delta v^a + \frac{\delta\rho^*\delta p}{\rho} - \delta\rho^*\delta\Phi\right) d^3x, \tag{2}$$

where ρ is the mass density, and δv^a, $\delta\rho$, δp, and $\delta\Phi$ are the perturbations in the fluid velocity, density, pressure, and gravitational potential respectively. An analogous functional is also known in the general relativistic case (Lindblom and Hiscock 1983). $\tilde{E}$ is conserved for fluid perturbations that satisfy dissipation-free evolution equations, hence it is not a useful diagnostic of dynamical instabilities. Internal fluid dissipation causes $\tilde{E}$ to decrease with time. Thus, $\tilde{E}$ can be used to diagnose secular instabilities that are driven by viscous forces in rotating stars. The study of this functional has revealed that thermal conductivity and bulk viscosity can cause the same type of secular instability as shear viscosity in rotating stars (Lindblom 1979).

The use of these energy functionals to diagnose instabilities is based on the expectation that any negative energy perturbation will grow without bound and thus represent an instability. While this is believed to be the case for each of the energy functionals discussed above, the careful mathematical analysis needed to establish this has only been completed to date for the Newtonian $\tilde{E}$ in a star having viscosity and thermal conductivity but no interaction with gravitational radiation. In this case it has been shown that $\tilde{E}$ is strictly decreasing with time unless $\tilde{E}$ vanishes (Lindblom 1983). This shows that a necessary condition for stability is that $\tilde{E} \geq 0$ for all fluid perturbations.

The effects of gravitational radiation cause the functional E to decrease with time while viscous effects cause $\tilde{E}$ to decrease. Unfortunately, neither functional is decreasing for every perturbation when both viscous and gravitational radiation effects are considered simultaneously. Thus in general neither functional (nor any known combination of them) can be used to diagnose these secular instabilities except in special cases. For very slowly rotating stars the waves with negative E that are subject to the gravitational radiation driven secular instability have very short wavelengths. These waves couple only weakly to gravitational radiation but very strongly to viscosity. Under these conditions it has been shown that the functional $\tilde{E}$ is a decreasing function of time while E is not. Thus, $\tilde{E}$ may be used to evaluate the secular stability of these perturbations while E may not. This analysis reveals that any amount of viscosity suppresses the gravitational radiation driven secular instability in sufficiently slowly rotating stars (Lindblom and Hiscock 1983).

3 Normal Modes

The analysis of the energy functional stability criteria discussed in Sect. 2 has revealed that gravitational radiation tends to make all rotating stars unstable, while viscous forces tend to suppress this instability. Unfortunately there is no known functional that always decreases with time when all of the relevant dissipative forces are present together. Thus no generally applicable test for the stability of rotating stars is presently available at all. The study of the stability of rotating stars has been directed therefore toward the study of the normal modes of rotating stars: solutions of the perturbation equations having time dependence $e^{i\omega t}$. This analysis provides a sufficient test for instability: the instability of one mode proves that the star is unstable.[3] Even the analysis of the normal modes of rotating stars turns out to be a rather difficult and interesting subject however. Considerable progress has been made in transforming this problem into a more tractable form in recent years. The analysis that leads to this simplification is simple and elegant, and so it is presented here in some detail for the simplest case of Newtonian stellar models.

In real stars the effects of dissipation are rather weak in that dissipative effects occur on time scales that are much longer than the dynamical time scale. Under these conditions it is possible to ignore the effects of dissipation as a first approximation. In this section the discussion is confined therefore to the simpler problem of the dissipation-free modes of rotating stars. The techniques for evaluating the effects of dissipation are discussed in Sect. 4.

The equations that govern the perturbations of a dissipation-free self-gravitating Newtonian fluid are given by

$$\partial_t \delta\rho + v^a \nabla_a \delta\rho + \nabla_a(\rho\delta v^a) = 0, \tag{3}$$

$$\partial_t \delta v^a + v^b \nabla_b \delta v^a + \delta v^b \nabla_b v^a = -\nabla^a \left(\frac{\delta p}{\rho} - \delta\Phi\right), \tag{4}$$

and

$$\nabla^a \nabla_a \delta\Phi = -4\pi G \delta\rho, \tag{5}$$

[3] Lacking a proof of the completeness of the normal modes, however, stability of all normal modes does not prove that the star is stable.

where any quantity preceded by δ represents the (Eulerian) perturbation of that quantity, while those not preceded by δ represent equilibrium values. In these equations ρ, p, Φ, and v^a represent the mass density, pressure, gravitational potential, and the fluid velocity. This system of equations is completed by specifying the thermodynamic relationship between the perturbed pressure and density. For simplicity here the equation of state is taken to be barotropic so that

$$\delta p = \frac{dp}{d\rho}\delta\rho. \tag{6}$$

The unperturbed equilibrium stellar model is assumed here to be rigidly rotating, i.e. $v^a = \Omega\varphi^a$ where Ω is the (constant) angular velocity and φ^a is the vector field representing rotations about the z^a axis.

The equations (3)–(6) that describe the perturbations of rotating stars constitute a complicated sixth-order system for the five independent components of the perturbation fields $(\delta\rho, \delta v^a, \delta\Phi)$. The solutions to these equations are known analytically only for the perturbations of uniform density stars (Bryan 1889) and have only been directly solved numerically for more realistic models quite recently (Yoshida and Eriguchi 1995). Rather than attempt to solve these equations directly, two different approaches have been devised to reduce the complexity of the equations by analytical means. The first approach introduces a potential ξ^a, the Lagrangian displacement, for the velocity perturbation:

$$\delta v^a = \partial_t\xi^a + v^b\nabla_b\xi^a - \xi^b\nabla_b v^a. \tag{7}$$

Using this potential the perturbed continuity equation (3) can be solved analytically: $\delta\rho = -\nabla_a(\rho\xi^a)$. This substitution reduces the number of independent perturbation fields to four, $(\xi^a,\delta\Phi)$, and reduces the equations that must be solved to the system (4)–(6). One nice feature of this representation of the equations is the existence of a Lagrangian from which the equations in this form may be derived (Lynden-Bell and Ostriker 1967). Unfortunately this representation also increases the order of the system of differential equations from sixth to eighth. For the purposes of actually solving the equations, this transformation does not offer much simplification. The equations have only been solved in this form (to my knowledge) numerically for the special case of axisymmetric normal modes (Clement 1981).

A second analytical transformation has been found that does significantly simplify the perturbation equations (Ipser and Managan 1985). This transformation is limited to perturbations which are normal modes with angular dependence $e^{im\varphi}$, where φ is measured about the rotation axis of the star. For this case eq. (4) reduces to

$$\Big[i(\omega + m\Omega)\delta_{ab} + 2\nabla_b v_a\Big]\delta v^b = -\nabla_a\left(\frac{\delta p}{\rho} - \delta\Phi\right), \tag{8}$$

where δ_{ab} represents the three-dimensional Euclidean metric (i.e., the identity matrix in Cartesian coordinates). This equation is *algebraic* in the velocity perturbation δv^a and can be solved analytically:

$$\delta v^a = iQ^{ab}\nabla_b\delta U, \tag{9}$$

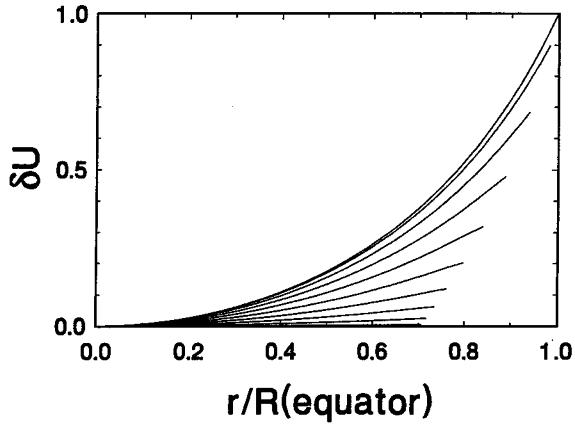

Figure 3 The eigenfunction δU for the $l = m = 3$ mode of a rapidly rotating Newtonian stellar model. Each curve represents the radial dependence of the eigenfunction along one angular spoke.

where δU is defined as

$$\delta U = \frac{\delta p}{\rho} - \delta\Phi \tag{10}$$

and Q^{ab} is the tensor

$$Q^{ab} = \frac{1}{(\omega + m\Omega)^2 - 4\Omega^2}\left[(\omega + m\Omega)\delta^{ab} - \frac{4\Omega^2}{\omega + m\Omega} z^a z^b - 2i\nabla^a v^b\right]. \tag{11}$$

Using equation (9) to replace δv^a in the remaining perturbation equations reduces the system to a pair of second-order equations for the scalar potentials δU and $\delta\Phi$:

$$\nabla_a\left(\rho Q^{ab}\nabla_b \delta U\right) = -(\omega + m\Omega)\rho\frac{d\rho}{dp}(\delta U + \delta\Phi), \tag{12}$$

$$\nabla^a\nabla_a\delta\Phi = -4\pi G\rho\frac{d\rho}{dp}(\delta U + \delta\Phi). \tag{13}$$

This transformation has reduced the equations for the modes of rotating stars to this relatively simple fourth-order system for the two scalar potentials (δU, $\delta\Phi$). These equations constitute a reasonably standard eigenvalue problem with eigenvalue ω. The tensor Q^{ab} in eq. (12) is positive definite if $(\omega + m\Omega)^2 > 4\Omega^2$, so the equation is elliptic for sufficiently slowly rotating stars. These equations can be solved for the eigenfunctions δU and $\delta\Phi$ and the eigenvalue ω using fairly standard numerical techniques (Ipser and Lindblom 1989, 1990). Figure 3 illustrates a typical eigenfunction δU for an $m = 3$ mode of a rapidly rotating Newtonian stellar model. Figure 4 illustrates the angular velocity dependence of the eigenvalue ω for two different sets of modes (Ipser and Lindblom 1990, Skinner and Lindblom 1996). The frequencies in Fig. 4 are displayed in terms of the dimensionless function $\alpha_m(\Omega)$,

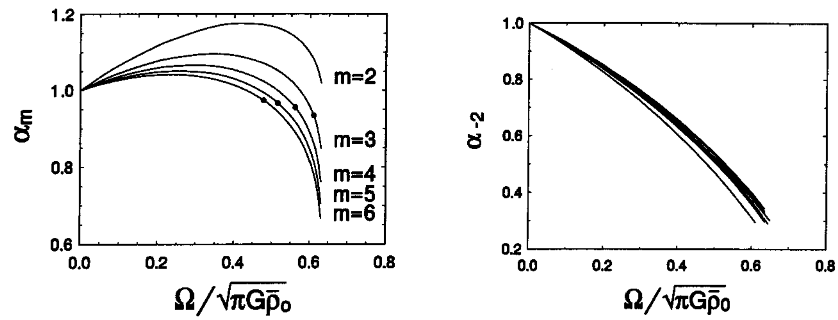

Figure 4 The angular velocity dependence of the frequencies of the modes of rapidly rotating Newtonian stellar models. The graph on the left gives the frequencies of the $l = m$ modes for $n = 1$ polytropic stellar models. The graph on the right gives the frequencies of the $l = -m = 2$ modes for stellar models constructed from thirteen realistic equations of state.

$$\alpha_m(\Omega) = \frac{\omega(\Omega) + m\Omega}{\omega(0)}, \tag{14}$$

which is normalized so that $\alpha_m(0) = 1$ for non-rotating stars.

Once the eigenfunctions δU and $\delta\Phi$ are determined, then every other physical property of the stellar oscillation may be determined from them. Equation (9) gives the velocity perturbation δv^a in terms of δU, while the density perturbation $\delta\rho$ is given by

$$\delta\rho = \rho\frac{d\rho}{dp}(\delta U + \delta\Phi), \tag{15}$$

and the Lagrangian displacement ξ^a by

$$\xi^a = \frac{Q^{ab}\nabla_b\delta U}{\omega + m\Omega}. \tag{16}$$

The particular version of the equations presented here, eqs. (12)–(13), is for the special case of barotropic perturbations of rigidly rotating stellar models. This approach can also be used to reduce the equations for the general adiabatic perturbations of differentially rotating stellar models without any restriction (e.g. barotropic) on the equation of state (Ipser and Lindblom 1991a). The equations in the general case are somewhat more complicated but remain, like eqs. (12)–(13), a fourth-order system for the two functions δU and $\delta\Phi$.

The problem of evaluating the modes of rapidly rotating stars has been rendered considerably simpler by the transformation that leads to eqs. (12)–(13). Nevertheless, there are some interesting questions that remain unresolved. The tensor Q^{ab} that appears in eq. (12) is positive definite whenever $(\omega + m\Omega)^2 > 4\Omega^2$. In this case eq. (12) is elliptic and can be solved using standard numerical techniques (Ipser and Lindblom 1990). This condition is always satisfied in non-rotating stars; however, in more rapidly rotating models it may be violated. When this condition is violated eq. (12) becomes hyperbolic yet the physical solutions must still satisfy Dirchlet boundary conditions. Little appears to be known about hyperbolic eigenvalue problems of this kind. Numerical techniques

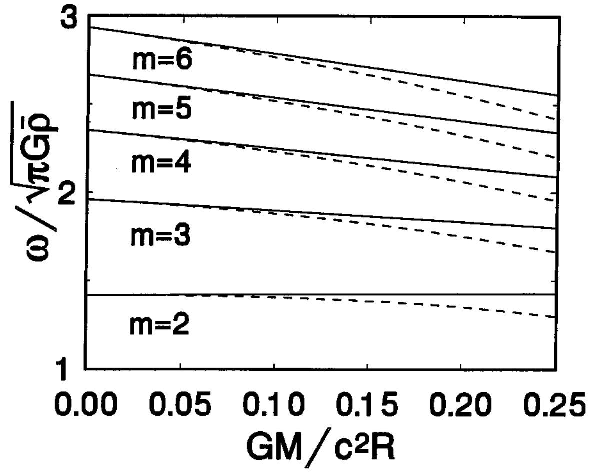

Figure 5 The post-Newtonian values for the frequencies of the modes of non-rotating stars (solid lines) are compared with the exact general relativistic values (dashed lines) for stars with different values of GM/c^2R.

based on a variational principle have been devised which give solutions to the equations even in this case however (Skinner and Lindblom 1996, Managan 1986). The change in signature of this equation does not appear to be connected to the onset of a physical instability. The physical significance of this change and the meaning of the characteristic surfaces that appear in eq. (12) are presently unknown.

In neutron stars the gravitational fields are rather strong and general relativistic effects significantly influence the structures and the dynamics. Thus it is of considerable interest to extend the analysis of the modes of rotating stars into the domain of general relativity theory. Unfortunately, this problem is extremely difficult. The chief obstacle is the coupling of these modes to gravitational radiation. In general relativity theory a star may oscillate at any frequency at all! If gravitational radiation of a given frequency were directed toward a star, then the star would oscillate at that frequency. The definition of normal modes for general relativistic stars must be refined therefore to include as an additional boundary condition that there be no incoming gravitational radiation. These solutions are referred to as the outgoing modes. This boundary condition is difficult to enforce because it must be done far away from the star in the wave zone of the gravitational radiation. This is reasonably easy to deal with in the case of non-rotating stars where the spacetime outside the unperturbed star is simply the Schwarzschild geometry (Thorne 1969, Lindblom and Detweiler 1983, Detweiler and Lindblom 1985). In rotating stars, however, the spacetimes outside the stars are only known numerically and only on rather small numerical grids. A practical method for imposing the outgoing radiation boundary condition on such spacetimes has not yet been devised. Fortunately there is a middle ground. The post-Newtonian approximation to general relativity provides a reasonably accurate description of the spacetimes associated with neutron stars. At the lowest orders the dynamics in the post-Newtonian approximation does not couple to gravitational radiation. Thus the problems associated with the outgoing radiation boundary condition does not arise in a post-Newtonian description of the modes of rotating stars.

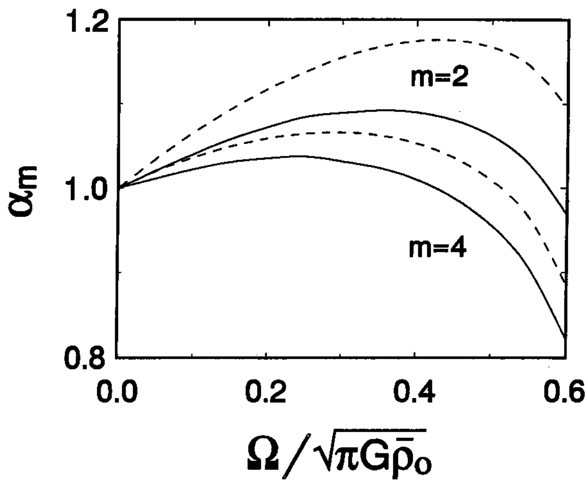

Figure 6 The angular velocity dependence of the frequencies of the modes of rotating stars. The post-Newtonian values for the frequencies (solid lines) are compared with Newtonian values (dashed lines) for stars with different angular velocities.

It is reasonably straightforward to extend the Newtonian analysis of the modes of rotating stars to the post-Newtonian theory (Cutler 1991, Cutler and Lindblom 1992). The oscillations of post-Newtonian stars are determined completely by the post-Newtonian corrections to the mode functions δU and $\delta\Phi$. These post-Newtonian eigenfunctions are determined by solving a pair of second-order equations having the same differential structures as eqs. (12)–(13) plus inhomogeneous terms that depend on δU and $\delta\Phi$ (and on the Newtonian and post-Newtonian structures of the equilibrium star). The post-Newtonian corrections to the frequency of a mode can be determined from the integrability condition for these pulsation equations, without solving the post-Newtonian pulsation equations at all! There exists an explicit formula for the post-Newtonian frequency that depends on δU and $\delta\Phi$ as well as the Newtonian and post-Newtonian structures of the star (Cutler and Lindblom 1992). As is typical of post-Newtonian analyses, this formula is extremely complicated (and unenlightening). However, it is straightforward to evaluate the needed integrals numerically and so determine the frequencies of the modes in this approximation. Figure 5 compares the frequencies of several modes of non-rotating stars computed in this post-Newtonian approximation with the exact general relativistic values. The post-Newtonian approximation for the frequencies of $1.4M_\odot$ neutron stars agree with the exact general relativistic frequencies to within about 4%. In comparison, the Newtonian frequencies agree with the exact values only to within about 12% for these same neutron stars. Figure 6 illustrates the angular velocity dependencies of the frequencies of the modes of rotating stars in both the Newtonian and post-Newtonian approximations for stars with $GM/c^2R = 0.2$. The post-Newtonian frequencies for these modes differ from the Newtonian values by about 10%.

The analysis of the modes of rotating stars in full general relativity theory is far less complete. But, the general equations for these modes have been derived and a certain amount of analysis has been done with them. The general relativistic version of the Lagrangian displacement has been used to transform the equations into a simpler and more canonical form (Friedman and Schutz 1975). These equations have been very useful for analyzing the effects of general relativity on the secular instabilities of rotating stars (Friedman 1978, Lindblom and Hiscock 1983). These equations have never been solved (even numerically), however, except in the case of non-rotating stars (Thorne 1969, Lindblom and Detweiler 1983). The general relativistic version of the transformation that leads to eq. (9) has also been found. For modes with angular dependence $e^{im\varphi}$ the perturbed conservation laws, $\delta(\nabla_a T^{ab}) = 0$, can be solved analytically for the perturbed four velocity δu^a in terms of a scalar potential δU, defined by

$$\delta U = \frac{\delta p}{\rho + p}, \tag{17}$$

and the perturbed metric tensor δg_{ab} (Ipser and Lindblom 1992). The resulting equation for δu^a,

$$\delta u^a = iQ^{ab}\nabla_b \delta U + \delta F^a(\delta g_{cd}), \tag{18}$$

is the relativistic analog of eq. (9). The vector δF^a that appears in eq. (18) depends on the metric perturbation δg_{ab} and the functions that describe the unperturbed star. The tensor Q^{ab} depends on the geometry of the unperturbed star and the frequency of the mode ω. This Q^{ab} is simply the relativistic generalization of eq. (11). There is also a general relativistic analog of eq. (12) which is derived by replacing the four-velocity perturbations in the energy conservation law using eq. (18). The resulting equation has the form

$$\nabla_a[(\rho + p)Q^{ab}\nabla_b \delta U] - Q^{ab}\nabla_a p \nabla_b \delta U + \Psi \delta U = \delta F(\delta g_{ab}) \tag{19}$$

where Ψ depends on the frequency of the mode and the unperturbed structure of the star, and δF depends on δg_{ab}. This equation is particularly useful when the dynamics of a mode is driven primarily by hydrodynamic rather than gravitational forces. Such is the case for the higher-order modes of stars (Lindblom and Splinter 1990), as well as the modes of objects like accretion disks where self gravitational effects are not important. Under these circumstances the metric perturbations may be ignored and the complete dynamics of the general relativistic mode is determined by eq. (19) with $\delta F = 0$. This equation is no harder to solve in the relativistic case than it is for Newtonian stellar models. The equation in this form has been used to determine the modes of relativistic accretion disks (Ipser 1996).

4 Dissipative Effects

Dissipation plays an important role in the stability of rotating stars. The general arguments outlined in Sect. 2 show that gravitational radiation tends to make all rotating stars unstable (Friedman and Schutz 1978, Friedman 1978) while internal fluid dissipation processes (e.g. viscosity) tend to suppress this instability and make sufficiently slowly rotating stars stable (Lindblom and Hiscock 1983, Lindblom and Detweiler 1977). In this section the techniques are described which have been used to evaluate the effects of dissipation on the stability of the normal modes of rotating stars. The principal tool that is used in this analysis is the equation that determines the evolution of the energy of the perturbation due to dissipative effects. For example, the evolution of $\tilde{E}$ defined in eq. (2) can be evaluated using the equations for a dissipative Newtonian fluid including the effects of gravitational radiation reaction forces (Ipser and Lindblom 1991b):

$$\begin{aligned}\frac{d\tilde{E}}{dt} &= -\int\left[2\eta\delta\sigma^{ab}\delta\sigma^*_{ab}+\zeta\delta\sigma\delta\sigma^*\right]d^3x \\ &\quad -(\omega+m\Omega)\sum_l N_l\omega^{2l+1}\delta D_{lm}\delta D^*_{lm}.\end{aligned} \tag{20}$$

The thermodynamic functions η and ζ that appear on the right side of eq. (20) represent the viscosities of the fluid. The viscous forces in a fluid are driven by the shear $\delta\sigma_{ab}$ and the expansion $\delta\sigma$ of the perturbation:

$$\delta\sigma_{ab}=\tfrac{1}{2}\left(\nabla_a\delta v_b+\nabla_b\delta v_a-\tfrac{2}{3}\delta_{ab}\nabla_c\delta v^c\right), \tag{21}$$

$$\delta\sigma=\nabla_c\delta v^c. \tag{22}$$

The gravitational radiation reaction force couples to the evolution of the fluid via the mass multipole moments of the perturbation δD_{lm},

$$\delta D_{lm}=\int\delta\rho\, r^l Y^*_{lm}d^3x, \tag{23}$$

with coupling constant N_l:

$$N_l=\frac{4\pi G}{c^{2l+1}}\frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \tag{24}$$

Now consider the normal modes of a rotating star that is subject to dissipative effects. Assume that the time dependence of the mode is $e^{i\omega t-t/\tau}$, where ω is the real part of the frequency and $1/\tau$ is the imaginary part. A mode is stable if $1/\tau$ is positive and unstable if negative. Thus the problem of evaluating the stability of a mode is reduced to determining the sign of the imaginary part of its frequency. Equation (20) provides a means of evaluating this quantity. The functional $\tilde{E}$ is real and quadratic in the perturbations, so its time dependence is $e^{-2t/\tau}$. It follows that the imaginary part of the frequency is given by

$$\frac{1}{\tau}=-\frac{1}{2\tilde{E}}\frac{d\tilde{E}}{dt}. \tag{25}$$

The right side of eq. (25) is, using eqs. (2) and (20), a functional of the eigenfunction of the mode. This is an exact identity which is not however particularly useful. If the exact dissipative eigenfunctions of the star were known, then the frequency of the mode could easily be evaluated in a number of ways. Equation (25) is nevertheless an extremely useful tool for evaluating $1/\tau$ approximately. Dissipation is a relatively weak force in stars: gravitational radiation and internal fluid dissipative processes effect the evolution of the fluid in a star on time scales that are much longer than the dynamical time scale. Thus the presence of dissipation has a relatively small effect on the evolution of the fluid in a star, and so the exact eigenfunctions of a mode (including the effects of dissipation) differ only slightly from the more easily evaluated eigenfunctions based on dissipation-free hydrodynamics. Thus, the functional on the right hand side of eq. (25) has essentially the same value whether evaluated using the exact or the dissipation-free eigenfunctions. This functional is straightforward to evaluate approximately, therefore, using the dissipation-free eigenfunctions as determined in Sect. 3. This approximation is expected to give values for the imaginary part of the frequency that have fractional errors of order $\tau\omega$, the ratio of the dissipative to the dynamical time scales. Studies have shown that this ratio is extremely small in neutron stars (Cutler et al. 1990).

The imaginary part of the frequency can be evaluated numerically using eq. (25). All that is needed is the dissipation-free eigenfunction of the mode, and the thermodynamic functions η and ζ that describe the viscous forces in the stellar fluid. The viscosity coefficients have been evaluated for neutron star matter (Sawyer 1989, Cutler and Lindblom 1987), and these quantities are given approximately by

$$\zeta = 6.0 \times 10^{-59} \left(\frac{\rho}{\omega}\right)^2 T^6, \tag{26}$$

$$\eta = 6.0 \times 10^{6} \left(\frac{\rho}{T}\right)^2 . \tag{27}$$

Note that these viscosities depend on the thermodynamic temperature T of the star. The bulk viscosity ζ is proportional to T^6 and becomes very large when the temperature of the star is high. The shear viscosity η is proportional to T^{-2} so it becomes large when the temperature is low. These two types of viscosity are comparable in neutron stars when $T \approx 10^9$K. Viscosity tends to suppress the gravitational radiation instability in rotating stars. Hence it is clear that these viscous forces will be very effective in suppressing this instability in very hot and very cool neutron stars.

To determine which rotating stars are unstable, the imaginary parts of the frequencies of their modes must be evaluated using eq. (25). The modes with the lowest values of l and m couple most strongly to gravitational radiation, while the viscous coupling increases as l and m increase. The viscous forces tend to suppress the gravitational radiation driven secular instability. Thus, the only modes that are likely to be unstable in these stars are those with relatively small values of l and m. In practice the viscous forces are always found to suppress the gravitational instability in modes with $m \geq 6$. In sufficiently slowly rotating stars all of the modes (that have been examined) are stable. It is useful therefore to define the critical angular velocity Ω_{crit} where some mode first becomes unstable, that is where

$$0 = \frac{1}{\tau(\Omega_{\text{crit}})}. \tag{28}$$

Figure 7 illustrates the critical angular velocities for a range of neutron star temperatures (Lindblom 1995). The critical angular velocity is displayed in units of Ω_{max} the maximum angular velocity for which there exists an equilibrium stellar model. Figure 7 reveals that in very cool neutron stars, $T < 10^7$K, the critical angular velocity is identical to Ω_{max}. Thus, the viscous forces completely suppress the gravitational radiation instability in these stars. Similarly in hot neutron stars, $T > 10^{10}$K, the bulk viscosity suppresses the

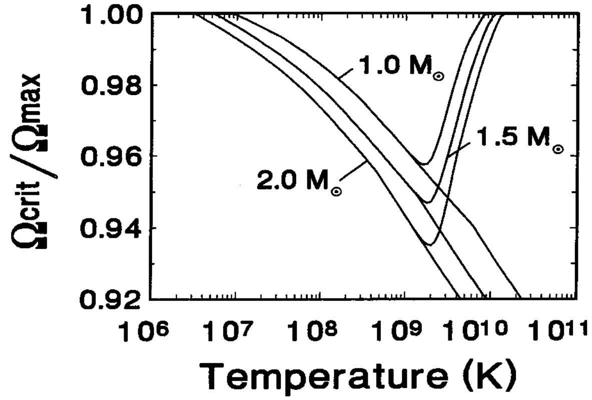

Figure 7 The temperature dependence of the critical angular velocities of neutron stars. The critical angular velocities $\Omega_{\rm crit}$ are expressed in terms of $\Omega_{\rm max}$ the maximum angular velocity for which there exists an equilibrium neutron star model.

instability. Only neutron stars with temperatures in the range $10^7 < T < 10^{10}$K are subject to the gravitational radiation driven secular instability. Further, this instability only occurs in the most rapidly rotating stars. Even for the most extreme case, $T \approx 2 \times 10^9$K, only those stars with angular velocities greater than about $0.96\Omega_{\rm max}$ may be subject to the gravitational radiation driven secular instability. Figure 7 illustrates that there is only a moderate dependence of $\Omega_{\rm crit}$ on the mass of the star. (More massive stars couple more strongly to gravitational radiation and hence have somewhat lower $\Omega_{\rm crit}$.) The discussion of the effects of dissipation up to this point has been based on Newtonian hydrodynamics, with the effects of gravitational radiation added as a small correction. Some work has been done, however, to estimate the effects of general relativistic dynamics on these results. Figure 8 illustrates the critical angular velocities based on a calculation

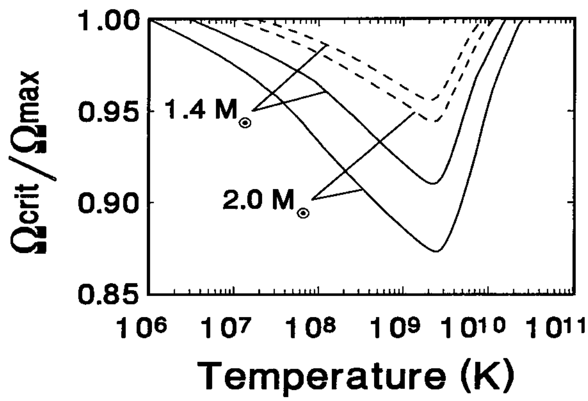

Figure 8 The temperature dependence of the critical angular velocities of neutron stars using Newtonian (dashed curves) and post-Newtonian (solid curves) gravitation and hydrodynamics.

that uses the post-Newtonian frequencies for the modes as described in Sect. 3 (Lindblom 1995). This calculation shows that post-Newtonian effects tend to enhance the gravitational radiation instability in these stars. This increases the range of temperatures where this instability may set in, and lowers the critical angular velocities to about $0.91\Omega_{\rm max}$ in the most extreme case for $1.4M_\odot$ stars. The effects of post-Newtonian hydrodynamics on these stability results are quite striking. It illustrates the need for us to press on to a more accurate fully relativistic analysis of this problem.

The earliest studies of the secular instabilities of rotating stars were concerned with the viscosity driven instability (Thompson and Tait 1883), rather than the gravitational radiation driven instability discussed extensively here. The viscosity driven instability occurs in a different set of modes, but the formalism described here can easily be turned to study it. Such studies reveal that the viscosity driven secular instability probably does not play any role in neutron stars at all. The principle reason is that the viscosity driven secular instability only occurs in stars with very stiff equations of state. In stars with polytropic equations of state, $p = \kappa\rho^\gamma$, the adiabatic index γ must exceed 2.237 for a viscosity driven secular instability to exist at all (James 1964). The equation of state of real neutron star matter appears to be not quite stiff enough. Analysis has shown that the viscosity driven instability does not occur in any of thirteen realistic equations of state for $1.4M_\odot$ neutron star models (Skinner and Lindblom 1996, Bonazzola et al. 1996). These realistic equations of state become stiffer at higher densities, however. In a few of the stiffest equations of state, it has been found that the most massive neutron star models are subject to this instability in the most rapidly rotating models. It remains to be seen whether the actual equation of state in neutron stars is stiff enough to allow this viscosity driven instability, and whether this instability plays any role in the astrophysics of real neutron stars.

Acknowledgments: I thank James Ipser for several helpful discussions. This research was supported by NSF grant PHY-9796079 and NASA grant NAG5-4093.

References

Backer, D.C., Kulkarni, S.R., Heiles, C., Davis, M.M., and Goss, W.M. (1989): A Millisecond Pulsar. *Nature*, **300**, 615–618

Bonazzola, S., Frieben, J., and Gourgoulhon, E. (1996): Spontaneous Symmetry Breaking of Rapidly Rotating Stars in General Relativity. *Astrophys. J.*, **460**, 379–389

Bryan, G.H. (1889): The Waves on a Rotating Liquid Spheroid of Finite Ellipticity. *Phil. Trans. Roy. Soc. London*, **A180**, 187–219

Chandrasekhar, S. (1969): *Ellipsoidal Figures of Equilibrium.* Yale University Press, New Haven

Chandrasekhar, S. (1970a): Solutions of Two Problems in the Theory of Gravitational Radiation. *Phys. Rev. Letters*, **24**, 611–615

Chandrasekhar, S. (1970b): The Effect of Gravitational Radiation on the Secular Stability of the Maclaurin Spheroid. *Astrophys. J.*, **161**, 561–569

Clement, M.J. (1981): Normal Modes of Oscillation for Rotating Stars. I. The Effect of Rigid Rotation on Four Low-Order Pulsations. *Astrophys. J.*, **249**, 746–760

Cook, G.B., Shapiro, S.L., and Teukolsky, S.A. (1994): Rapidly Rotating Neutron Stars in General Relativity— Realistic Equations of State. *Astrophys. J.*, **424**, 823–845

Cutler, C. (1991): Post-Newtonian Effects on the Modes of Rotating Stars. *Astrophys. J.*, **374**, 248–254

Cutler, C. and Lindblom, L. (1987): The Effect of Viscosity on Neutron Star Oscillations. *Astrophys. J.*, **314**, 234–241

Cutler, C. and Lindblom, L. (1992): Post-Newtonian Frequencies for the Pulsations of Rapidly Rotating Neutron Stars. *Astrophys. J.*, **385**, 630–641

Cutler, C., Lindblom, L., and Splinter, R.J. (1990): Damping Times for Neutron Star Oscillations. *Astrophys. J.*, **363**, 603–611

Detweiler, S.L. and Ipser, J.R. (1973): A Variational Principle and a Stability Criterion for the Non-Radial Motions of Pulsation of Stellar Models in General Relativity. *Astrophys. J.*, **185**, 685–707

Detweiler, S.L. and Lindblom, L. (1977): On the Evolution of the Homogeneous Ellipsoidal Figures. *Astrophys. J.*, **213**, 193–199

Detweiler, S. and Lindblom, L. (1985): On the Nonradial Pulsations of General Relativistic Stellar Models. *Astrophys. J.*, **292**, 12–15

Friedman, J. (1978): Generic Instability of Rotating Relativistic Stars. *Comm. Math. Phys.*, **62**, 247–278

Friedman, J.L. and Schutz, B.F. (1975): On the Stability of Relativistic Systems. *Astrophys. J.*, **200**, 204–220

Friedman, J.L. and Schutz, B.F. (1978): Secular Instability of Rotating Newtonian Stars. *Astrophys. J.*, **222**, 281–296

Friedman, J.L., Ipser, J.R., and Parker, L. (1986): Rapidly Rotating Neutron Star Models. *Astrophys. J.*, **304**, 115–139

Ipser, J.R. (1996): Relativistic Accretion Disks: Low Frequency Modes and Frame Dragging. *Astrophys. J.*, **458** 508–513

Ipser, J.R. and Lindblom, L. (1989): Oscillations and Stability of Rapidly Rotating Neutron Stars. *Phys. Rev. Letters*, **62**, 2777–2780; Errata, **63**, 1327

Ipser, J.R. and Lindblom, L. (1990): The Oscillations of Rapidly Rotating Newtonian Stellar Models. *Astrophys. J.*, **355**, 226–240

Ipser, J.R. and Lindblom, L. (1991a): On the Adiabatic Pulsations of Accretion Disks and Rotating Stars. *Astrophys. J.*, **379**, 285–289

Ipser, J.R. and Lindblom, L. (1991b): The Oscillations of Rapidly Rotating Newtonian Stellar Models II. Dissipative Effects. *Astrophys. J.*, **373**, 213–221

Ipser, J.R. and Lindblom, L. (1992): On the Pulsations of Relativistic Accretion Disks and Rotating Stars: The Cowling Approximation. *Astrophys. J.*, **389**, 392–399

Ipser, J.R. and Managan, R.A. (1985): An Eulerian Variational Principle and a Criterion for the Occurrence of Nonaxisymmetric Neutral Modes Along Rotating Axisymmetric Sequences. *Astrophys. J.*, **292**, 517–521

James, R.A. (1964): The Structure and Stability of Rotating Gas Masses. *Astrophys. J.*, **140**, 552–582

Lebovitz, N.L. (1965): On Schwarzschild's Criterion for the Stability of Gaseous Masses. *Astrophys. J.*, **142**, 229–242

Lindblom, L. (1979): On the Secular Instability from Thermal Conductivity in Rotating Stars. *Astrophys. J.*, **233**, 974–980

Lindblom, L. (1983): Necessary Conditions for the Stability of Rotating Newtonian Stellar Models. *Astrophys. J.*, **267**, 402–408

Lindblom, L. (1992): Determining the Nuclear Equation of State from Neutron Star Masses and Radii. *Astrophys. J.*, **398**, 569–573

Lindblom, L. (1995): Critical Angular Velocities of Rotating Neutron Stars. *Astrophys. J.*, **438**, 265–268

Lindblom, L. (1996): Stellar Stability According to Newtonian Theory and General Relativity. In *Proceedings of the 14th International Conference on General Relativity and Gravitation* (eds. M. Francavagilia, G. Longhi, L. Lussana, and E. Sorace). World Scientific, Singapore, in press

Lindblom, L. and Detweiler, S.L. (1977): On the Secular Instabilities of the Maclaurin Spheroids. *Astrophys. J.*, **211**, 565–567

Lindblom, L. and Detweiler, S.L. (1983): The Quadrupole Oscillations of Neutron Stars. *Astrophys. J. Supp.*, **53**, 73–92

Lindblom, L. and Hiscock, W.A. (1983): On the Stability of Rotating Stellar Models in General Relativity Theory. *Astrophys. J.*, **267**, 384–401

Lindblom, L. and Splinter, R.J. (1990): The Accuracy of the Relativistic Cowling Approximation. *Astrophys. J.*, **348**, 198–202

Lynden-Bell, D. and Ostriker, J. (1967): On the Stability of Differentially Rotating Bodies. *Mon. Not. Roy. Astron. Soc.*, **136**, 293–310

Managan, R. (1986): Nonaxisymmetric Normal Modes and Secular Instabilities of Rotating Stars. *Astrophys. J.*, **309**, 598–608

Press, W.H. and Teukolsky, S.A. (1973): On the Evolution of the Secularly Unstable, Viscous Maclaurin Spheroids. *Astrophys. J.*, **181**, 513–517

Roberts, P.H. and Stewartson, K. (1963): On the Stability of a Maclaurin Spheroid of Small Viscosity. *Astrophys. J.*, **137**, 777–790

Sawyer, R.F. (1989): Bulk Viscosity of Hot Neutron-Star Matter and the Maximum Rotation Rates of Neutron Stars. *Phys. Rev. D*, **39**, 3804–3806

Schwarzschild, M. (1958): *Structure and Evolution of the Stars.* Dover Publications, New York

Skinner, D. and Lindblom, L. (1996): On the Viscosity Driven Secular Instability in Rotating Neutron Stars. *Astrophys. J.*, **461**, 920–926

Thompson, W. and Tait, P.G. (1883): *Principles of Natural Philosophy.* Clarendon Press, Oxford

Thorne, K.S. (1969): Nonradial Pulsation of General-Relativistic Stellar Models. III. Analytic and Numerical Results for Neutron Stars. *Astrophys. J.*, **158**, 1–16

Thorsett, S.E., Arzoumanian, Z., McKinnon, M.M., and Taylor, J.H. (1993): The Masses of Two Binary Neutron Star Systems. *Astrophys. J. Letters*, **405**, L29–L32

Yoshida, S. and Eriguchi, Y. (1995): Gravitational Radiation Driven Secular Instability of Rotating Polytropes. *Astrophys. J.*, **438**, 830–840