Limits on the Adiabatic Index in Static Stellar Models

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1 SUMMARY

Dieter Brill has made important contributions to the study of the positive mass theorem in general relativity theory (Brill 1959) and to the analysis of the conformal properties of asymptotically flat space-times (Cantor & Brill 1981). It is a pleasure to honor him on this occasion by applying these analytical tools to the study of static stellar models. We use the positive mass theorem and the conformal properties of these spacetimes to deduce an interesting constraint on the allowed values of the adiabatic index in the static stellar models of general relativity theory.

It is well known that non-singular polytropic stellar models in the Newtonian theory can exist only if the adiabatic index of the fluid is sufficiently large, $\gamma \geq 6/5$ (or equivalently if the polytropic index is sufficiently small, $n \leq 5$; see e.g., Chandrasekhar 1939). In this paper we derive two limits on the allowed values of the adiabatic index in the stellar models of general relativity theory. Our limits apply to static stellar models that have a finite radius and an equation of state which satisfies certain fairly weak smoothness assumptions. Our first bound on the adiabatic index guarantees that

$$\gamma \equiv \frac{\rho + p}{p}\frac{dp}{d\rho} > 1, \tag{1}$$

at some points in every neighborhood of the surface of such stars. This limit, although weaker than the traditional Newtonian limit, is simply a consequence of the regularity of the spacetime at the surface of these stars. This result makes no assumption about the high-density portion of the equation of state. Thus it applies to (essentially) every static stellar model in general relativity theory. Our second bound on the adiabatic index guarantees that

$$\gamma > \frac{6}{5}\Big(1 + \frac{p}{\rho}\Big)^2. \tag{2}$$

in some portion of every static stellar model with finite radius. This result is the direct general-relativistic analogue of the Newtonian limit $\gamma > 6/5$ for spherical polytropic stars with finite radius. Our result is considerably more general, however, because we do not assume that the equation of state is polytropic (i.e., of the form $p = \kappa\rho^{1+1/n}$ where κ and n are constants). Our result is also more general because it does not assume that the stellar model has spherical symmetry. Thus, our result compliments the work on the necessity of spherical symmetry in static general-relativistic stellar models by Masood-ul-Alam (1988) and by Beig & Simon (1991, 1992). Those proofs of spherical symmetry specifically exclude any equation of state which violates equation (2) at any point within the star. Thus, our second bound eliminates a large class of potential counterexamples to the spherical-symmetry conjecture. In particular, our result shows that no stellar model whose equation of state violates equation (2) everywhere can exist in general relativity theory.

2 THE SURFACE LIMIT

A stellar model in general relativity theory is an asymptotically-flat spacetime that satisfies Einstein's equation with a perfect-fluid source. A static (i.e., time independent and non-rotating) stellar model has a time-translation symmetry (by assumption) whose trajectories are hypersurface orthogonal. Thus, coordinates may be chosen in which the spacetime metric tensor has the representation

$$ds^2 = -V^2 dt^2 + g_{ab} dx^a dx^b, \tag{3}$$

where V and the three-dimensional spatial metric g_{ab} are independent of t. The topology of the constant-t hypersurfaces must be $\mathbf{R}^3$ (Lindblom & Brill 1980, Masood-ul-Alam 1987a). Einstein's equation for static stellar models reduces in this representation to the pair of equations

$$D^a D_a V = 4\pi V(\rho + 3p), \tag{4}$$

$$R_{ab} = V^{-1} D_a D_b V + 4\pi(\rho - p) g_{ab}. \tag{5}$$

The density and pressure of the fluid are denoted ρ and p; and the density is assumed to be a given function of the pressure, $\rho = \rho(p)$ (the equation of state), which is non-negative and non-decreasing. The spatial covariant derivative compatible with g_{ab} is denoted D_a, and its Ricci curvature is denoted R_{ab}. The Bianchi identity for the three-dimensional spatial geometry may be reduced to the form

$$D_a p = -V^{-1}(\rho + p) D_a V, \tag{6}$$

with the use of equations (4)–(5).

The surface of a stellar model is the boundary between the interior fluid region where the pressure is positive, and the exterior region where the vacuum Einstein equation is satisfied. We limit our consideration here to stars whose surfaces occur at a finite radius (i.e., to stars which have non-trivial exterior vacuum regions). The pressure must vanish on the surface of the star in order to insure that the spacetime is non-singular there. We first prove that the ratio p/ρ must also vanish on this surface: Consider a smooth curve $x^a(\lambda)$ that lies inside the star for $\lambda_0 \leq \lambda < \lambda_S$, while the point $x^a(\lambda_S)$ lies on the surface. Let $n^a = dx^a/d\lambda$ denote the tangent to this curve. We evaluate the integral of $(\rho + p)^{-1} n^a D_a p$ along this curve using equation (6):

$$\int_{p(\lambda_0)}^{p(\lambda)} \frac{d\hat{p}}{\rho(\hat{p}) + \hat{p}} = \int_{\lambda_0}^{\lambda} \frac{n^a D_a p \, d\lambda}{\rho + p} = -\int_{\lambda_0}^{\lambda} \frac{n^a D_a V \, d\lambda}{V} = \log\left[\frac{V(\lambda_0)}{V(\lambda)}\right]. \tag{7}$$

In the limit $\lambda \to \lambda_S^-$ the expression on the right is well behaved, thus, the integral on the left must also be well behaved. Consequently the function,

$$h(p) \equiv \int_0^p \frac{d\hat{p}}{\rho(\hat{p}) + \hat{p}}, \tag{8}$$

is well defined for all p within any static stellar model in general relativity theory. The equation of state, $\rho(p)$, is a non-decreasing function; thus, we may estimate $h(p)$ as follows,

$$h(p) = \int_0^p \frac{d\hat{p}}{\rho(\hat{p}) + \hat{p}} \geq \int_0^p \frac{d\hat{p}}{\rho(p) + \hat{p}} = \log\left(1 + \frac{p}{\rho}\right). \tag{9}$$

Since $\lim_{p\to 0^+} h(p) = 0$, we conclude from equation (9) that

$$0 = \lim_{p\to 0^+} e^{h(p)} - 1 \geq \lim_{p\to 0^+} \frac{p}{\rho}. \tag{10}$$

Thus, the limit of p/ρ must vanish. So, we have established the desired result:

Lemma. Consider a static stellar model in general relativity theory whose surface occurs at a finite radius. Assume that the equation of state $\rho = \rho(p)$ is a positive and non-decreasing function in some open neighborhood of the surface of the star: i.e., for pressures $0 < p < \epsilon$. Then $\lim_{p\to 0^+}(p/\rho) = 0$.

We now turn to the derivation of our first limit on the adiabatic index,

$$\gamma(p) \equiv \frac{\rho + p}{p}\frac{dp}{d\rho}. \tag{11}$$

We assume that the equation of state, $\rho(p)$, is a positive and non-decreasing C^1 function in the open interval, $0 < p < \epsilon$ (for some $\epsilon > 0$). Thus, $1/\gamma(p)$ is a continuous and non-negative function there. Let $\Gamma(\epsilon)$ be the least upper bound of $\gamma(p)$ for $0 < p < \epsilon$: i.e., let $\Gamma(\epsilon) = \sup_{0<p<\epsilon} \gamma(p)$. If $\Gamma(\epsilon)$ is not finite, then there exist points in the open interval $0 < p < \epsilon$ where $\gamma > 1$, trivially. We turn our attention, therefore, to the case where $\Gamma(\epsilon)$ is finite. From the definition of the adiabatic index, we obtain the following inequality

$$\Gamma(\epsilon) \geq \gamma = \frac{\rho + p}{p}\frac{dp}{d\rho} \geq \frac{\rho}{p}\frac{dp}{d\rho}. \tag{12}$$

This inequality may be integrated to obtain an upper bound for the density function on the open interval $0 < p < \epsilon$:

$$\rho \leq \rho(\epsilon)\left(\frac{p}{\epsilon}\right)^{1/\Gamma(\epsilon)}. \tag{13}$$

This upper bound and the Lemma may then be used to obtain the following condition on $\Gamma(\epsilon)$,

$$\lim_{p\to 0+} \frac{p}{\rho} \geq \lim_{p\to 0+} \frac{\epsilon}{\rho(\epsilon)}\left(\frac{p}{\epsilon}\right)^{1-1/\Gamma(\epsilon)}. \tag{14}$$

The limit on the right vanishes only if $\Gamma(\epsilon) > 1$. Since $\lim_{p\to 0+}(p/\rho) = 0$ as a consequence of the Lemma, it follows that $\Gamma(\epsilon) > 1$. Thus there exist points in the open interval $0 < p < \epsilon$ where $\gamma > 1$. The constant ϵ may be chosen to be arbitrarily small, so there must exist points in every open neighborhood of the surface of the star where $\gamma > 1$. In summary then our first limit on the adiabatic index is:

Theorem 1. Consider a static stellar model in general relativity theory whose surface occurs at a finite radius. Assume that the equation of state $\rho = \rho(p)$ is a positive and non-decreasing C^1 function of the pressure in some neighborhood of the surface of the star, $0 < p < \epsilon$. Then the adiabatic index $\gamma(p) > 1$ at some point in every open neighborhood of the surface of the star.

3 THE MAXIMUM LIMIT

Our method for deriving a limit on the maximum value of the adiabatic index is very similar to the technique developed by Masood-ul-Alam (1987b) for proving the necessity of spherical symmetry in certain static stellar models. We construct a particular conformal factor which is used to transform the spatial metric g_{ab}. This conformal factor scales the mass associated with the spatial geometry to zero while leaving the transformed scalar curvature non-negative—unless the adiabatic index of the fluid is sufficiently large. For stellar models composed of fluid with small adiabatic index, then, the positive mass theorem implies that the transformed geometry is flat. We show, however, that this is inconsistent with our assumption that the stellar model has a finite radius. Thus, we conclude that the maximum value of the adiabatic index must exceed a certain lower bound in every static stellar model with a finite radius.

The Bianchi identity, equation (6), for static fluid spacetimes determines a functional relationship between the fluid variables, ρ and p, and the potential V. In particular, with the aid of equations (7)–(8), the density and pressure may be expressed as explicit functions of V: $p(V) = h^{-1}[\log(V_S/V)]$, and $\rho(V) = \rho[p(V)]$, where V_S is the value of the potential on the surface of the star. We use these functions to define a conformal factor ψ in the interior region of the star (where $0 < V \leq V_S$):

$$\psi(V) = \frac{1}{2}(1+V_S)\exp\left\{-\frac{V_S}{1+V_S}\int_V^{V_S}\frac{\rho(\hat{V})\,d\hat{V}}{\hat{V}[\rho(\hat{V})+3p(\hat{V})]}\right\}. \tag{15}$$

If the interior of the star is composed of more than one connected region, then $\psi(V)$ is defined in each region by equation (15) with the constant V_S taken to be the surface value of V for that region. In the exterior of the star (e.g., where $V_S \leq V \leq 1$) we define the conformal factor to be,

$$\psi(V) = \frac{1}{2}(1+V). \tag{16}$$

It is easy to see that this conformal factor is continuous across each component of the surface of the star by setting $V = V_S$ in equations (15) and (16). It is also straightforward to verify that $d\psi/dV$ is continuous at the surface of the star. The derivative of ψ in the interior of the star satisfies the equation

$$\frac{d\psi}{dV} = \frac{V_S\psi}{V(1+V_S)(1+3p/\rho)}. \tag{17}$$

Using the result of the Lemma, $\lim_{V\to V_S}(p/\rho) = 0$, it is easy to take the limit of the right side of equation (17) to verify that $d\psi/dV = 1/2$ at the surface of the star. The second derivative $d^2\psi/dV^2$ vanishes in the exterior of the star, while it is given by the expression

$$\frac{d^2\psi}{dV^2} = \frac{V_S\psi}{V^2(1+V_S)(1+3p/\rho)^2}\left[\frac{2+3V_S}{1+V_S} - \frac{3}{\gamma}\left(1+\frac{p}{\rho}\right)^2\right], \tag{18}$$

in the interior of the star. The function $d^2\psi/dV^2$ is continuous and has a finite upper bound for $V < V_S$ whenever $1/\gamma$ is bounded. Thus, ψ is $C^{1,1}$ (i.e, its first derivative satisfies a Lipshitz condition) even across the surface of the star as long as $1/\gamma$ is bounded. Note that $1/\gamma(p)$ is finite for any $p > 0$ as a consequence of our assumption that $\rho(p)$ is a C^1 function. Thus, our assumption that $1/\gamma$ is bounded is a restriction on the behavior of the equation of state only near $p = 0$. The boundedness of $1/\gamma$ follows automatically for equations of state which are smooth enough that the limit of $\gamma(p)$ is well defined (in the sense that $\lim\inf\gamma = \lim\sup\gamma$) as $p \to 0^+$. In this case Theorem 1 implies that $\lim\inf\gamma \geq 1$, and so $1/\gamma$ is bounded as $p \to 0^+$.

We use the function ψ given in equations (15) and (16) to define a conformally transformed spatial metric tensor $\bar{g}_{ab}$:

$$\bar{g}_{ab} = \psi^4 g_{ab}. \tag{19}$$

This transformed metric has two important properties: a) the mass associated with $\bar{g}_{ab}$ is zero; and b) the scalar curvature associated with $\bar{g}_{ab}$ is non-negative unless the adiabatic index is too large. The first of these properties can be deduced by examining the asymptotic boundary conditions on the stellar model. In general relativity theory a stellar model is a non-singular asymptotically-flat solution to equations (4)–(5). The appropriate asymptotic forms for V and g_{ab} are therefore,

$$V = 1 - \frac{M}{r} + O(r^{-2}), \tag{20}$$

$$g_{ab} = \left(1 + \frac{2M}{r}\right)\delta_{ab} + O(r^{-2}), \tag{21}$$

where the constant M is the mass of the star, δ_{ab} is the flat Euclidean metric, and r is a spherical radial coordinate associated with the metric δ_{ab}. These asymptotic forms imply that the conformal factor defined in equation (16) has the asymptotic form

$$\psi = 1 - \frac{M}{2r} + O(r^{-2}). \tag{22}$$

It follows, then, that the asymptotic form of the transformed metric is

$$\bar{g}_{ab} = \delta_{ab} + O(r^{-2}). \tag{23}$$

Thus, the mass associated with $\bar{g}_{ab}$ is zero. The second important property of $\bar{g}_{ab}$, the sign of its scalar curvature, is easily deduced. The general expression for the scalar curvature $\bar{R}$ associated with the metric $\bar{g}_{ab}$ is

$$\bar{R} = 8\psi^{-5}\left(2\pi\rho\psi - D^a D_a\psi\right). \tag{24}$$

The term $D^aD_a\psi$ can be evaluated by using equations (4) and (15)–(18). The result can be expressed in the form

$$\psi^4(1+V_S)\bar{R} = 16\pi\rho(1-V_S) + \frac{8\rho^2 V_S D^aVD_aV}{\gamma V^2(\rho+3p)^2}\left[3\Big(1+\frac{p}{\rho}\Big)^2 - \gamma\,\frac{2+3V_S}{1+V_S}\right] \qquad (25)$$

in the interior of each connected component of the star, while $\bar{R}$ vanishes in the exterior of the star. The potential on the surface of the star is strictly less than one, $V_S < 1$, in any stellar model that has a finite radius. Thus, the conformally transformed scalar curvature is non-negative as long as the condition,

$$\gamma \le 3\Big(1+\frac{p}{\rho}\Big)^2\frac{1+V_S}{2+3V_S}, \qquad (26)$$

is satisfied by the adiabatic index.

The positive mass theorem (Schoen & Yau 1979, Parker & Taubes 1982) implies that a $C^{1,1}$ asymptotically-flat Riemannian three-metric is flat if its scalar curvature is non-negative and its mass is zero. The needed smoothness of $\bar{g}_{ab}$ follows from the smoothness of ψ established above. Thus, the spatial metric $\bar{g}_{ab}$ is flat if the adiabatic index satisfies equation (26). But, if $\bar{g}_{ab}$ is flat, then the scalar curvature $\bar{R}$ vanishes. This implies, from equation (25), that $V_S = 1$ and that equality must hold in equation (26). But $V_S < 1$ in any star with a finite radius, and so we conclude that $\bar{R}$ must be negative somewhere in such stars. Thus equation (26) must be violated. Consequently, at some point in every static star with a finite radius the adiabatic index must satisfy

$$\gamma > 3\Big(1+\frac{p}{\rho}\Big)^2\frac{1+V_S}{2+3V_S} > \frac{6}{5}\Big(1+\frac{p}{\rho}\Big)^2 \ge \frac{6}{5}. \qquad (27)$$

In summary, then, we have established the following lower bound on the maximum value of the adiabatic index:

Theorem 2. Consider a static stellar model in general relativity theory whose surface occurs at a finite radius. Assume that the equation of state $\rho = \rho(p)$ is a positive and non-decreasing C^1 function of the pressure. Assume that $1/\gamma(p)$ is bounded as $p \to 0^+$. Then the adiabatic index must satisfy the inequality

$$\gamma > \frac{6}{5}\Big(1+\frac{p}{\rho}\Big)^2 \ge \frac{6}{5}, \qquad (28)$$

at some point within the star.

We point out that the extreme case in the argument that leads to Theorem 2—when equality holds in equation (26) and $V_S = 1$—is not vacuous. This case constrains the equation of state of the fluid to be the one used by Buchdahl (1964) to construct a family of non-singular but infinite-radius stellar models. The existence of these asymptotically flat, albeit infinite radius, stellar models suggests that our limit on the maximum value of the adiabatic index is the strongest possible limit of this kind.

4 CONCLUDING REMARKS

It is instructive to examine the implications of Theorem 2 for stellar models constructed from fluid that has a polytropic equation of state:

$$p = \kappa \rho^{1+1/n}, \tag{29}$$

where κ and n are constants. The constraint on the adiabatic index, equation (28), is equivalent for polytropes to a constraint on the polytropic index n:

$$n < 5(1 + 6p/\rho)^{-1} \leq 5. \tag{30}$$

Thus, $n < 5$ for general-relativistic stellar models that have a polytropic equation of state. This shows that equation (28) is the direct general-relativistic analogue of the familiar Newtonian limit on the polytropic index $n < 5$ (see e.g., Chandrasekhar 1939).

We have chosen to express the limit on the adiabatic index, equation (28), in a form which involves only the equation of state without making any reference to the macroscopic properties of the particular stellar model. In fact, a somewhat stronger limit was obtained in equation (27). That limit on the adiabatic index can be expressed in another form that involves the surface value of the potential V_S in a simple way:

$$\gamma > 3\frac{1 + V_S}{2 + 3V_S} > \frac{6}{5}. \tag{31}$$

Thus, for a spherical star of mass M and radius R the adiabatic index must satisfy the inequality

$$\gamma > 3\frac{1 + (1 - 2M/R)^{1/2}}{2 + 3(1 - 2M/R)^{1/2}} > \frac{6}{5}, \tag{32}$$

at some point within the star. For polytropic stars this condition is equivalent to the bound

$$n < 2 + 3V_S = 2 + 3(1 - 2M/R)^{1/2} < 5, \tag{33}$$

on the polytropic index.

The limits on the adiabatic index that we derive in this paper are necessary conditions for the existence of a static stellar model in general relativity theory. Our limits are weaker, therefore, than the bounds needed to guarantee the stability of these models. Glass and Harpaz (1983) have shown, for example, that the adiabatic index must exceed 4/3 in a stable relativistic polytrope by an amount that depends on the ratio p/ρ at the center of the star. A similar limit on a suitably averaged value of the adiabatic index, $\bar{\gamma} > 4/3$, is necessary for the stability of nearly Newtonian stars with any equation of state (see e.g., Misner, Thorne & Wheeler 1973).

The bound on the adiabatic index derived in Theorem 2 does not assume that the stellar model has spherical symmetry. Thus, our result compliments the work on the necessity of spherical symmetry in static general-relativistic stellar models by Masood-ul-Alam (1988) and by Beig & Simon (1991, 1992). Those proofs of spherical symmetry apply only to equations of state for which the adiabatic index satisfies the inequality (28) at every point. Thus, Theorem 2 eliminates a large class of potential counterexamples to the spherical-symmetry conjecture. In particular, our result shows that no stellar model whose equation of state violates equation (28) everywhere can exist in general relativity theory. Of course many equations of state exist, including many realistic models of neutron-star matter, which satisfy equation (28) for some values of the pressure and violate it for others.

This research was supported by the grant PHY-9019753 from the National Science Foundation, and grant NAGW-2951 from the National Aeronautics and Space Administration.

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