On the symmetries of equilibrium stellar models

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The equilibrium states of isolated self-gravitating fluids (i.e. stellar models) always have a high degree of spatial symmetry. This paper reviews the current understanding of these symmetries in both the newtonian and the general relativistic theories. The arguments that establish (under various appropriate conditions) the necessity of spherical symmetry, axisymmetry, and reflection symmetry are summarized. The limitations of the present results and suggestions for further progress are outlined in the concluding remarks.

1. Introduction

The equilibrium states of isolated, self-gravitating fluid matter are of considerable interest in astrophysics because they serve as our basic models of stars. These stellar models are the solutions of the equations for the hydrodynamic and thermodynamic properties of the fluid, together with the equations for the gravitational field. It is of fundamental importance to determine everything about the nature of the equilibrium solutions to these equations. In the traditional discussions (see, for example, Chandrasekhar 1939; Tassoul 1978) it is always assumed *a priori* that the equilibrium fluid matter has some degree of spatial symmetry: e.g. that the matter is spherically symmetric, or in rotating configurations axisymmetric. Are these symmetry assumptions made merely for analytical simplicity? Do there exist non-symmetric equilibrium solutions to these equations which might serve as models for interesting astrophysical phenomena? Or, are these assumptions in fact unnecessary? Do all solutions to the isolated, self-gravitating fluid equations have a high degree of spatial symmetry? I find the answer to these questions to be remarkable: all equilibrium stellar models must have spatial symmetries. This paper summarizes the arguments that establish the necessity of these 'intrinsic' symmetries in equilibrium stellar models.

Before beginning the discussion of the symmetries of stellar models, it is appropriate to review briefly what is meant by a stellar model and to introduce some notation. The discussion here will include results from the newtonian theory (where these symmetries were first analysed, and where our understanding is more complete) and from the general relativistic theory of equilibrium stellar models. The non-singular, time-independent, physically isolated solutions to the self-gravitating fluid equations are referred to as equilibrium stellar models. In the newtonian theory a fluid is described by its mass density ρ, pressure p, and velocity field v^a. These fields satisfy the usual hydrodynamic equations,

$$\partial_t \rho + \nabla_a(\rho v^a) = 0, \tag{1}$$

$$\rho(\partial_t v^a + v^b \nabla_b v^a) = -\nabla^a p + \rho \nabla^a \Phi, \tag{2}$$

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together with the gravitational field equation,

$$\nabla^a \nabla_a \Phi = -4\pi\rho, \tag{3}$$

where Φ is the gravitational potential. (Units are used in which $G = c = 1$.) The equilibrium solutions to these equations are (by definition) time independent: $\partial_t \rho = 0$ and $\partial_t v^a = 0$. And, the physically isolated solutions are those in which the fluid does not extend to spatial infinity (i.e. ρ has compact spatial support) and the gravitational potential Φ falls to zero at spatial infinity.

In general relativity theory a stellar model is a non-singular spacetime that satisfies Einstein's equation with the stress–energy tensor of a perfect fluid. The topology of this space-time manifold is assumed to be $\boldsymbol{R}^4$. (This assumption is probably not necessary. Several arguments have shown that $\boldsymbol{R}^4$ is the only possible space-time topology for non-singular isolated stellar models under various reasonable assumptions (see, for example, Lindblom & Brill 1980; Masood-ul-Alam 1987*a*). The state of a relativistic fluid is determined by its mass-energy density ρ, pressure p, and four velocity u^μ (a unit timelike vector). The fluid equations in this case are

$$\nabla_\mu[(\rho+p)\, u^\mu u^\nu + p g^{\mu\nu}] = 0, \tag{4}$$

where ∇_μ is the covariant derivative associated with the space-time metric $g_{\mu\nu}$. The gravitational field is described in general relativity by $g_{\mu\nu}$ which is determined by Einstein's equation

$$R_{\mu\nu} = 8\pi[(\rho+p)\, u_\mu u_\nu + \tfrac{1}{2}(\rho-p)\, g_{\mu\nu}], \tag{5}$$

where $R_{\mu\nu}$ is the Ricci curvature of the space-time. The equilibrium solutions to these equations are (by definition) those which admit a time translation symmetry. That is, any equilibrium fluid space-time admits (by definition) a timelike vector field t^μ that satisfies Killing's equation: $\nabla_\mu t_\nu + \nabla_\nu t_\mu = 0$. The physically isolated solutions to these equations have (by assumption) spatially bounded fluid sources, and metric tensors that approach the flat metric of special relativity at spatial infinity.

Of course, many non-symmetric solutions exist to the fluid equations (1)–(3) or (4) and (5). If the fluid has time dependence, or if it is not physically isolated (e.g. if the gravitational field is required arbitrarily to have some fixed value on an irregular surface that encloses the fluid), then the generic solution to the structure equations, (1)–(3) or (4) and (5), has no spatial symmetry. And, (obviously) non-fluid objects (like books) need not have any spatial symmetry even if they are in equilibrium and are physically isolated. The remarkable fact is that the time-independent, physically isolated fluid solutions not only can but must have spatial symmetries. The arguments that demonstrate the necessity of these spatial symmetries are summarized in the following sections. For static (that is time-independent and non-rotating) stellar models, the necessity of spherical symmetry is discussed in §2. For stationary (time-independent and rotating) stellar models, the necessity of axisymmetry about the rotation axis is discussed in §3. The necessity of reflection symmetry about a plane perpendicular to the rotation axis of rotating stars is discussed in §4. The final section, §5, discusses the limitations of these results and suggests directions for further progress.

2. Spherical symmetry

It seems almost obvious that non-rotating stellar models must be spherically symmetric. In fact, an intuitive argument is the basis of a rigorous proof of the necessity of spherical symmetry in non-rotating newtonian stellar models. The

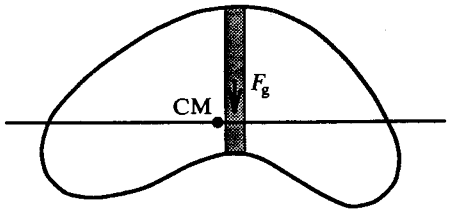

Figure 1. An illustration of the net gravitational force on an asymmetric fluid 'element'.

intuitive argument goes like this. Spherical symmetry about a point is equivalent to reflection symmetry about every plane passing through that point. Consider any one of the planes that passes through the centre of mass of a star. If the star failed to have reflection symmetry through this plane, then some 'element' of the fluid must be displaced asymmetrically above this plane farther than any other. The gravitational attraction of this asymmetric element by the rest of the star would result in a net force on this element toward the plane, as illustrated in figure 1. Fluid matter is incapable of exerting shear stresses to balance this force. Therefore in equilibrium the star must have reflection symmetry about this plane, and hence spherical symmetry about the centre of mass. The rigorous version of this argument (Lichtenstein 1918, 1933; Carleman 1919; Wavre 1932) was the first example of a theorem showing that symmetries are an inherent characteristic of equilibrium stellar models. An alternative argument proving the necessity of reflection symmetries in a much larger class of newtonian stellar models is summarized in §4.

The extension of this 'obvious' result to stars in general relativity theory has proven to be surprisingly difficult. Despite numerous studies of the problem over the years (see, for example, Avez 1964; Künzle 1971; Lindblom 1978, 1980, 1981) only 'technical' progress was made until quite recently. Probably the most meaningful of the early results is a proof that there are no 'almost spherical' static stars in general relativity theory (Künzle & Savage 1980). The real breakthrough came with Masood-ul-Alam's (1987*b*) elegant application of the positive mass theorem to the study of static stars. Although Masood-ul-Alam's (1987*b*) original analysis required unphysical assumptions about the equation of state of the fluid, his method was easily adapted by Lindblom (1988) to prove the necessity of spherical symmetry for the more physical case of uniform density stars. And, this analysis has since been generalized further to prove the necessity of spherical symmetry for a much larger and more realistic class of equations of state by Masood-ul-Alam (1988) and Beig & Simon (1991, 1992). A summary of the important features of this beautiful analysis is given in the remainder of this section.

In general relativity theory a static (i.e. time-independent and non-rotating) stellar model is a space-time whose metric tensor may be represented in the form

$$ds^2 = -V^2\,dt^2 + g_{ab}\,dx^a\,dx^b, \tag{6}$$

where g_{ab} is the positive definite three-metric of the constant-t surfaces; and, V and g_{ab} are independent of t: $\partial_t V = 0$ and $\partial_t g_{ab} = 0$. Einstein's equation (5) for this geometry reduces to

$$D^a D_a V = 4\pi V(\rho + 3p), \tag{7}$$

and

$$R_{ab} = V^{-1} D_a D_b V + 4\pi(\rho - p)\, g_{ab}, \tag{8}$$

where D_a and R_{ab} are the three-dimensional covariant derivative and its Ricci curvature tensor; and, ρ and p are the energy density and pressure of the fluid. The physically isolated solutions to these equations are those in which the density and pressure functions have compact spatial support, and in which the metric potentials V and g_{ab} asymptotically approach their flat-space values: $V = 1-m/r+O(r^{-2})$ and $g_{ab} = (1+2m/r)\,\delta_{ab}+O(r^{-2})$ where δ_{ab} is the flat euclidean metric, m is the mass of the star, and r is the asymptotic radial coordinate.

The problem is to show that the only solutions to (7) and (8) with the appropriate asymptotically flat boundary conditions are spherically symmetric. The first step is to show that spherical symmetry is equivalent to spatial conformal flatness for these solutions. The proof of this equivalence was the culmination of the early 'technical' results of Avez (1964), Künzle (1971) and Lindblom (1980). The proof uses Einstein's equation to relate the conformal structure of the three-metric g_{ab} to certain geometrical properties of the constant-V two-surfaces that are known to imply spherical symmetry (e.g. the constancy of the gaussian curvature on each two-surface). This first step shows, then, that the solutions to (7) and (8) must be spherically symmetric if and only if a conformal factor Ω can be found that makes the conformally transformed spatial metric,

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \tag{9}$$

identical to the flat euclidean metric: $\hat{g}_{ab} = \delta_{ab}$. To determine which conformal factor to choose for a particular static stellar model, we consider the 'reference' spherical model having the same equation of state, the same mass, and the same surface potential $V = V_s$ as the given star. In a spherical star the conformal factor Ω depends only on the radial coordinate r, and hence only on the value of the potential V. A natural candidate for the appropriate conformal factor to use in (9) is $\Omega = \Omega(V)$, where $\Omega(V)$ is the conformal factor which transforms the reference spherical metric to the flat metric. In the exterior of the star, $V > V_s$, this conformal factor has the simple functional form

$$\Omega = \tfrac{1}{4}(1+V)^2. \tag{10}$$

In the interior of the star $V < V_s$ the functional form of the conformal factor will depend on the equation of state of the fluid, and in general will have no simple analytic representation. In the interior, therefore, $\Omega(V)$ is defined only as the appropriate solution to the differential equation which it satisfies in the reference spherical star (Beig & Simon 1992).

The second step in the proof involves an elegant application of the positive mass theorem (Schoen & Yau 1979). The needed corollary to the positive mass theorem states that a complete, asymptotically flat riemannian metric must be flat if its scalar curvature is everywhere non-negative and if its mass vanishes. It is easy to see that the metric $\hat{g}_{ab}$ of equation (9) has vanishing mass as a consequence of the asymptotic form of the conformal factor (10). Thus all that remains to prove that $\hat{g}_{ab}$ is flat, and consequently that g_{ab} is spherically symmetric, is to show that the scalar curvature $\hat{R}$ is everywhere non-negative. A straightforward calculation shows that $\hat{R}$ is given by

$$\hat{R} = P(V)\,[W_0(V)-W], \tag{11}$$

where $P(V) = 2\Omega^{-2}[2\Omega^{-1}\Omega''-\Omega^{-2}(\Omega')^2]$ with the prime denoting differentiation with respect to V. The function W in (11) is defined as $W = D_a V D^a V$, while $W_0(V)$ is the function to which the W of the reference spherical model is equal. Beig & Simon

(1991, 1992) show that $\hat{R}$ is indeed non-negative as long as the equation of state of the fluid has a non-negative adiabatic index, $\gamma = p^{-1}(\rho+p)\,\mathrm{d}p/\mathrm{d}\rho \geqslant 0$, which satisfies the additional inequality

$$\frac{\mathrm{d}\gamma}{\mathrm{d}p} \geqslant \frac{6}{5p} + \frac{8}{5(\rho+3p)} - \gamma\frac{\rho^2+3\rho p-6p^2}{p(\rho+p)(\rho+3p)}. \tag{12}$$

The inequality (12) is expressed here in terms of the adiabatic index rather than the analytically simpler, but physically less transparent, expression given by Beig & Simon (1991, 1992). This restriction on the equation of state is sufficient to guarantee that each of the factors on the right side of (11) is separately non-negative. The non-negativity of $P(V)$ is demonstrated by constructing estimates of $\Omega(V)$ and its derivatives from the differential equation that defines Ω. The proof that the other factor, $W_0 - W$, is non-negative is the most subtle and difficult part of the proof. A rather complicated differential inequality that involves $W - W_0$ is constructed from Einstein's equations (7) and (8) (see Beig & Simon (1992) for the details). This identity shows that an elliptic operator acting on $W - W_0$ is non-negative whenever (12) is satisfied. As a simple example, for uniform density fluids this complicated identity reduces to the inequality: $D^a[V^{-1}(W-W_0)] \geqslant 0$. Similar inequalities have also been found for the exterior region of the star (Robinson 1977). When the maximum principle for elliptic differential operators is applied to these identities, it follows that the maximum of $W - W_0$ must occur at infinity (where it vanishes). Thus $W_0 - W$ is non-negative everywhere when (12) is satisfied. Since these functions (e.g. V and W) are not generally smooth at the surface of the star, the required application of the maximum principle is rather delicate there. The details are contained in the paper of Beig & Simon (1992). This argument completes the proof that $\hat{R}$ is non-negative. The final step in the proof that g_{ab} is spherical is to invoke the positive mass theorem. Since $\hat{R}$ is non-negative, and since $\hat{g}_{ab}$ has vanishing mass, then it follows that $\hat{g}_{ab}$ is flat. Thus g_{ab} is conformally flat, and so from the first step in the proof discussed above, g_{ab} is spherically symmetric.

3. Axisymmetry

The traditional models of rotating stars (Tassoul 1978) are assumed to be axisymmetric. Is this assumption necessary, or are there non-axisymmetric equilibrium models? The answer to this question depends rather delicately on ones definition of an equilibrium stellar model. There are solutions to the time-independent self-gravitating perfect-fluid equations (1)–(3) that are not axisymmetric. The Dedekind ellipsoids are uniform density fluid solutions which are time independent but not axisymmetric (see, for example, Chandrasekhar 1969). These solutions may be thought of as 'standing waves' superimposed on the steady rotation of the fluid. When the star rotates in the direction opposite the wave's propagation at just the right rate, the form of the wave and hence the structure of the star remains independent of time. General arguments have shown that rotating stars with any equation of state admit time-independent non-axisymmetric linear perturbations (Friedman & Schutz 1978; Friedman 1978). Thus it is generally believed that non-axisymmetric solutions analogous to the Dedekind ellipsoids exist in both the newtonian and the general relativistic theories. To confuse the situation further, there exists another class of non-axisymmetric solutions to the newtonian fluid equations that could also be considered 'equilibrium' stellar models. These are non-axisymmetric solutions of (1)–(3) that are time independent in some uniformly rotating frame of reference, although they are time dependent in the inertial frame. The Jacobi (or more generally the Riemann S) ellipsoids are examples of this kind of solution (Chandrasekhar 1969).

Given the plethora of non-axisymmetric solutions to the 'time-independent' fluid equations, is there any truth at all then in the claim: 'equilibrium stellar models must be axisymmetric?' The answer (perhaps surprisingly at this point) is yes. The non-axisymmetric solutions described above are all solutions to the non-dissipative fluid equations. And none of these examples is a time-independent solution to the equations for 'real' fluids including the dissipative effects of viscosity, thermal conductivity, and gravitational radiation. The Dedekind-like objects have fluid flows that are strongly sheared. Such flows are damped by viscosity and thus cannot be true equilibrium states. The Jacobi-like objects all have time-dependent mass densities as seen in the inertial frame. Such configurations emit gravitational radiation and thus cannot be true equilibrium states either. The remainder of this section summarizes the proof that the equilibrium states of rotating, self-gravitating, physically isolated, dissipative fluids must be axisymmetric. This proof was first given for general relativistic stellar models by Lindblom (1976). The proof for newtonian stellar models is very analogous and will not be discussed here (see Lindblom 1978).

The first step in the proof is to establish that two independent Killing vector fields exist within any stationary rotating stellar model composed of a dissipative fluid. Since the space-time is stationary, there exists an everywhere timelike Killing vector field t^μ that generates the time translation symmetry. The existence of this Killing vector field, and hence the time translation symmetry, ensures that the fluid must be in a non-dissipating state. A general feature of dissipative relativistic fluid theories is that the four-velocity of the fluid divided by the temperature, $\xi^\mu = u^\mu/T$, is a Killing vector field in any non-dissipating state (see Geroch & Lindblom 1991). In the Eckart (1940) theory, for example, this comes about because in a non-dissipating state the entropy production vanishes. The vanishing of the entropy production implies in the Eckart theory that the fluid four-velocity must be shear free,

$$0 = (g^{\mu\alpha} + u^\mu u^\alpha)(g^{\nu\beta} + u^\nu u^\beta)(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha), \tag{13}$$

to prevent viscous dissipation. Also, the temperature gradient in the Eckart theory must satisfy

$$0 = (g^{\mu\alpha} + u^\mu u^\alpha)(\nabla_\alpha T + T u^\beta \nabla_\beta u_\alpha) \tag{14}$$

in any equilibrium state to prevent dissipation due to thermal conduction. These two equations (13) and (14) imply that $\xi^\mu = u^\mu/T$ satisfies Killing's equation. This establishes that ξ^μ and t^μ are each Killing vector fields within the star. If the star were non-rotating so that the space-time was static, then these two Killing vectors would be proportional to each other. However, Lichnerowicz (1955) showed that if the star is rotating (so that the space-time is stationary but not static), then the four-velocity of the fluid cannot be proportional to t^μ. Thus, in rotating stars ξ^μ and t^μ are independent Killing vector fields.

The second step in the proof is to establish that the Killing vector field ξ^μ also exists in the vacuum region of the space-time outside of the star. Any Killing vector field in a vacuum space-time satisfies the 'wave' equation

$$0 = \nabla^\alpha \nabla_\alpha \xi^\mu. \tag{15}$$

Thus it is natural to define the extension of ξ^μ into the exterior of the star as a vector field that satisfies (15). Choose the solution to (15) whose value and derivative agree on the surface of the star with those of the interior field ξ^μ. The surface of the star is timelike, however, and therefore is not normally an appropriate Cauchy surface on which to specify initial data for equation (15). However, the timelike Killing vector field t^μ guarantees that the exterior geometry is analytic (Müller zum Hagen 1970). Thus the Cauchy–Kowalewsky theorem guarantees the existence of a solution to (15) with the given initial data specified on the surface of the star. To determine whether this extension of ξ^μ into the exterior of the star is a Killing vector field, consider the tensor

$$t_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{16}$$

It is straightforward to show that any vector field ξ^μ that satisfies (15) in a vacuum space-time ($R_{\mu\nu} = 0$) also satisfies the identity

$$0 = \nabla^\alpha \nabla_\alpha t_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta t_{\alpha\beta}, \tag{17}$$

where $t_{\mu\nu}$ is given in (16) and $R_\mu{}^\alpha{}_\nu{}^\beta$ is the Riemann curvature tensor. This is a linear homogeneous 'wave' equation for $t_{\mu\nu}$ whose solution is unique (due to a theorem of Holmgren) for given initial data on a non-characteristic surface. Since ξ^μ is a Killing vector field within the star, it follows that $t_{\mu\nu}$ and its derivative vanish there. The extension of ξ^μ via (15) is smooth enough to guarantee that $t_{\mu\nu}$ and its derivative are continuous across the surface of the star. Since the initial data for $t_{\mu\nu}$ are zero on the surface of the star, the unique solution to (17) is $t_{\mu\nu} = 0$. Thus the vector field ξ^μ extended into the exterior of the star via (15) is a Killing vector field. The third and final step in the proof is to note that any stationary asymptotically flat space-time with a second independent Killing vector field is axisymmetric (Ashtekar & Xanthopolous 1978). The symmetries of any asymptotically flat space-time must be a subset of the Poincaré group to preserve the asymptotic structure of the space-time. Thus, it follows that a space-time containing a star may only have time translation and/or rotation symmetries.

4. Reflection symmetry

The symmetries discussed in the last two sections, spherical symmetry and axisymmetry, are continuous symmetries. Those are infinite collections of symmetry transformations (e.g. rotations through any angle about the rotation axis) each of which leaves the structure of the star unchanged. Are there in addition any discrete symmetries, such as reflection through a plane, which all equilibrium stellar models must possess? The answer (in the newtonian theory at least) is yes; the answer is not yet known in general relativity theory. In the newtonian theory all equilibrium stars must have reflection symmetry through a plane (the 'equator') which is perpendicular to the rotation axis of the star. The proof of this theorem in the newtonian theory (Lichtenstein 1918, 1933; Carleman 1919; Wavre 1932) was the first example of a symmetry theorem for equilibrium stellar models. And as discussed briefly in §2, this symmetry implies spherical symmetry in non-rotating stars. Reflection symmetry is a necessary feature of a very large class of equilibrium stellar models, including many which do not admit either of the continuous symmetries discussed earlier. In particular, reflection symmetry is a necessary feature of any time-independent, physically isolated, perfect fluid solution whose velocity field is 'stratified' in the sense that $v^a\hat{z}_a = 0$ for some constant vector field $\hat{z}^a$. This includes any stellar model whose velocity field is purely rotational (including differential rotation) about some axis; and it includes stellar models having more complicated velocity fields such as those found in the non-axisymmetric Dedekind ellipsoids. The remainder of this section presents a proof of the necessity of reflection symmetry under these conditions. The proof summarized here uses a simpler method of identifying the appropriate reflection plane, but is otherwise based on the analysis of Lindblom (1977).

Consider a solution to the newtonian stellar-structure equations (1)–(3) whose density and pressure functions have compact spatial support. Assume that the equation of state is barotropic, $\rho = \rho(p)$, with $\mathrm{d}\rho/\mathrm{d}p \geqslant 0$. And, assume that the velocity of the fluid v^a is everywhere orthogonal to $\hat{z}^a$, the unit vector in the z direction: $v^a\hat{z}_a = 0$. Define the thermodynamic function

$$h(p) = \int_0^p \frac{\mathrm{d}\hat{p}}{\rho(\hat{p})}. \tag{18}$$

(This function is always well defined for an equation of state that has finite-sized stellar models.) Under these assumptions the fluid equation (2) reduces to

$$\partial_t v^a + v^b \nabla_b v^a = \nabla^a(\Phi - h). \tag{19}$$

The right side of (19) is a gradient, thus the velocity field for these solutions must be one for which the left side is also a gradient. Let $\partial_t v^a + v^b\nabla_b v^a = \nabla^a \Lambda$, where Λ is a scalar field determined entirely by v^a. We assume that Λ is at least a C^1 function (i.e., having continuous first derivatives); and Λ must be independent of the cartesian coordinate z because v^a is assumed to be orthogonal to $\hat{z}^a$. With these definitions, equation (19) may be integrated to obtain the 'Bernoulli' equation

$$0 = h(p) - \Phi + \Lambda, \tag{20}$$

for points within the support of the density. We assume that h is a C^1 function of position within the fluid, but it needs only to be C^0 (i.e. continuous) at the surface of the star where $h = 0$. The gravitational potential Φ is assumed to be a C^2 function (i.e. having continuous second derivatives) except at the surface of the star where it needs only to be C^1 for derivatives normal to the surface.

To investigate the reflection symmetries of these stellar models, we define the odd part of a function with respect to reflections about the plane $z = z_0$:

$$f^-(\boldsymbol{x}, z_0) = f(\boldsymbol{x}) - f(\boldsymbol{x} + 2[z_0 - z]\hat{\boldsymbol{z}}), \tag{21}$$

where $\boldsymbol{x} = (x, y, z)$ and $\hat{\boldsymbol{z}}$ is the unit vector with components $(0, 0, 1)$. A stellar model has reflection symmetry if there exists a constant z_0 such that

$$0 = \Phi^-(\boldsymbol{x}, z_0) = h^-(\boldsymbol{x}, z_0) = \rho^-(\boldsymbol{x}, z_0), \tag{22}$$

for all $\boldsymbol{x}$. The odd parts of these functions are determined by the odd parts of equations (3) and (20):

$$\nabla^a \nabla_a \Phi^- = -4\pi\rho^- \tag{23}$$

and

$$h^- = \Phi^-. \tag{24}$$

The domain of validity of (23) is the entire space; however, the domain of (24) is limited to points $\boldsymbol{x}$ that lie within the star whose reflections $\boldsymbol{x} + 2[z_0 - z]\hat{\boldsymbol{z}}$ are also within the star. While equations (23) and (24) are valid for any choice of z_0, the

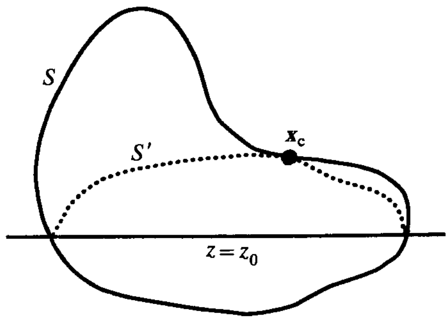

Figure 2. The critical point x_c where $h^-(x_c, z_0) = 0$. In this illustration the critical point is located on the surface of the star S.

solution will have reflection symmetry, equation (22), for only one value of z_0. The basic outline of the proof that these solutions must have reflection symmetry is as follows. Choose z_0 to be the largest value for which $\rho^- \geqslant 0$ for all $z \geqslant z_0$. For this choice of z_0 the odd part of the gravitational potential has two special properties: (*i*) Φ^- satisfies the inequality,

$$\nabla^a \nabla_a \Phi^- \leqslant 0, \tag{25}$$

on the domain $z \geqslant z_0$; and (*ii*) there is a critical point x_c in the domain $z \geqslant z_0$ (or on its boundary) where Φ^- has a minimum where its gradient vanishes. (These properties are derived in the following paragraphs.) The maximum principle for functions that satisfy elliptic differential inequalities is now used to deduce that Φ^- must vanish. The maximum principle states that any C^2 function Φ^- which satisfies (25) in some domain with smooth boundary must vanish if Φ^- has a minimum (in the domain or on the boundary) where the gradient of Φ^- vanishes. Thus, $\Phi^- = 0$ and so $\rho^- = h^- = 0$ as a consequence of (23) and (24). Thus the stellar model has reflection symmetry.

The appropriate value for z_0 is chosen in the following way. The function $h(x)$ vanishes except on a compact set of points, where it is non-negative. Thus, for sufficiently small values of z_0, $h^-(x, z_0)$ is non-negative for all $z \geqslant z_0$. Similarly, for sufficiently large z_0, $h^-(x, z_0)$ is non-positive for all $z \geqslant z_0$. We choose z_0 to be the largest value for which $h^-(x, z_0) \geqslant 0$ for all $z \geqslant z_0$. It follows that there exists a critical point x_c within the support of h where $h^-(x_c, z_0) = 0$. This critical point has the property that every neighbourhood of x_c of radius 2ϵ (for sufficiently small ϵ) contains points where $h^-(x, z_0 + \epsilon) < 0$ and $z > z_0 + \epsilon$. Such points must exist because z_0 was chosen to be the largest value for which $h^- \geqslant 0$. This critical point is a place where $h^-(x_c, z_0)$ is a minimum. If x_c lies within the support of h and has $z > z_0$, then h^- is C^1 and its gradient vanishes at this minimum point: $0 = \nabla_a h^-(x_c, z_0)$. If x_c lies on the plane $z = z_0$, then the gradient $\nabla_a h^-(x_c, z_0)$ must also vanish. Otherwise $h^-(x, z_0 + \epsilon)$ would not be negative in a neighbourhood of x_c for very small ϵ. If, however, x_c lies on the surface of the star then h^- is only C^0 there. In this case its gradient (computed from the interior of the fluid) need not vanish, but must be orthogonal to the surface of the star at that point. Figure 2 illustrates some of the important features of the critical point x_c. The boundary of the star is denoted S and the reflection the portion of the boundary with $z < z_0$ is denoted S'. The reflected boundary S' must lie within S for $z > z_0$, since otherwise h^- would be negative for points outside S but inside S'. The critical point must always lie within S' because

$h^- = h$ for points outside S'. In figure 2 the critical point is located on the boundary of the star where S is tangent to S'. The gradient $\nabla_a h^-(\boldsymbol{x}_c, z_0)$ must therefore be orthogonal to the surface S at this point.

We have assumed that the equation of state satisfies the condition $d\rho/dp \geqslant 0$. Thus, both ρ and h are increasing functions of p. It follows that the choice of z_0 which makes $h^- \geqslant 0$ will also make $\rho^- \geqslant 0$. Thus for this choice of z_0 the inequality (25) is satisfied for $z \geqslant z_0$. We first show that $\Phi^- > 0$ for all $z > z_0$ unless $\Phi^- = 0$ everywhere. If there were points with $\Phi^- \leqslant 0$ then Φ^- would have a minimum at one of these points. Since Φ^- is at least C^1 its gradient would vanish at this minimum. However, this contradicts the maximum principle unless Φ^- vanishes identically. Thus $\Phi^- > 0$ for all $z > z_0$ unless Φ^- vanishes identically. Now consider the critical point $\boldsymbol{x}_c$ where h^- vanishes. From equation (24) it follows that Φ_- also vanishes at $\boldsymbol{x}_c$. Thus $\boldsymbol{x}_c$ must lie on the plane $z = z_0$ unless Φ^- vanishes everywhere. The gradient $\nabla_a h^-(\boldsymbol{x}_c, z_0)$ vanishes and so $\nabla_a \Phi^-(\boldsymbol{x}_c, z_0) = 0$ as well. A straightforward application of the boundary version of the maximum principle rules out this possibility except for the singular case where x_c lies on the intersection of the surface of the star and the plane $z = z_0$. In this singular case the minimum of Φ^- occurs at a corner in boundary of the region where Φ^- is C^2 and (25) is satisfied. Since the boundary is not smooth at this point, a slightly more delicate argument is needed. Note that $\Phi^- \geqslant 0$ for all $z \geqslant z_0$ and consequently $\Phi^- \leqslant 0$ for all $z \leqslant z_0$. Thus, $\partial\Phi^-/\partial z \geqslant 0$ in some neighbourhood of the $z = z_0$ plane. Now since $\boldsymbol{x}_c$ lies on the surface of the star the gradient $\nabla_a h^-(\boldsymbol{x}_c, z_0)$ must be orthogonal to the surface at this point. Thus $\nabla_a h^-(\boldsymbol{x}_c, z_0)$ must be tangent to the plane $z = z_0$. Since h^- vanishes identically on this plane, the gradient $\nabla_a h^-(\boldsymbol{x}_c, z_0)$ and so $\nabla_a \Phi^-(\boldsymbol{x}_c, z_0)$ must vanish identically there as well. Thus, $\partial\Phi^-/\partial z$ vanishes and so is a local minimum at $\boldsymbol{x}_c$. At this critical point $\hat{z}$ is tangent to the surface of the star and so $\partial\Phi^-/\partial z$ is C^1 there. Thus, the gradient of $\partial\Phi^-/\partial z$ also vanishes at $\boldsymbol{x}_c$. In the exterior of the star $\partial\Phi^-/\partial z$ is a solution of Laplace's equation. Thus the maximum principle applied in the exterior of the star to the equation $\nabla^a\nabla_a \partial\Phi^-/\partial z = 0$ implies that $\partial\Phi^-/\partial z = 0$ everywhere since its gradient vanishes at the point $\boldsymbol{x}_c$ where it is a local minimum. So, we conclude finally that Φ^- vanishes everywhere and hence the star must have reflection symmetry about the plane $z = z_0$.

5. Concluding remarks

Although quite a lot is now known about the necessity of symmetries in equilibrium stellar models, many obvious questions remain unanswered. For example, it seems unlikely that the restriction on the equation of state (12) needed to prove the necessity of spherical symmetry in relativistic stellar models is really fundamental. It seems more likely that a slightly more powerful variation on the Masood-ul-Alam argument will succeed in proving the necessity of spherical symmetry without this condition. For example, the Masood-ul-Alam argument might be generalized by allowing the conformal factor Ω in equation (9) to depend on both V and W instead of V alone. Another possibility might be to construct a proof that the entire conformal scalar curvature $\hat{R}$ is non-negative rather than proving that its factors given in (11) must be individually non-negative. Perhaps a differential inequality acting directly on $\hat{R}$ could be constructed for this purpose.

I think there are a number of interesting issues regarding the necessity of axisymmetry of equilibrium stellar models that remain to be understood. The present version of the axisymmetry proof requires a rather awkward extension of the internal Killing field ξ^μ across the surface of the star. This extension depends on the analyticity of the exterior geometry and uses the Cauchy–Kowalewsky theorem to do the extension. Is there a more elegant way to do this? The present version of the proof also requires the assumption that the fluid is dissipative. Is this assumption necessary? The heuristic arguments presented in §3 suggest that the assumption is necessary and that stationary, non-axisymmetric, perfect-fluid models exist in both the newtonian and the general relativistic theories. However, no examples have actually been found except for the uniform density newtonian ellipsoids. Finding such examples would be quite interesting. Do such non-axisymmetric models have any other symmetries? For example, do such models always possess discrete symmetries such as reflections about an equatorial plane, or perhaps discrete rotations? The stationary solutions to the linear perturbation equations have angular dependence $e^{in\phi}$, and therefore are symmetric with respect to discrete rotations through the angle $2\pi/n$. Must the stationary, non-axisymmetric, nonlinear solutions share these discrete symmetries?

The necessity of reflection symmetry has only been established for newtonian stellar models. Must stationary general relativistic stellar models have a reflection symmetry (i.e. a discrete isometry with fixed points on a timelike three-surface)? Unfortunately, the newtonian proof of reflection symmetry relies very heavily on the linearity of the equations. Therefore, the proof for general relativistic models is likely to be very different. Before attempting the full problem for general relativistic models, it might be enlightening to attempt to answer some simpler questions. For example, must stationary and axisymmetric stellar models have reflection symmetry in general relativity theory? Or easier still, must all perturbations of a stationary, axisymmetric, and reflection symmetric stellar models have reflection symmetry? It seems likely to me that the answer to all of these questions is yes: stationary general relativistic stellar models must have a reflection symmetry.

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