FUNDAMENTAL PROPERTIES OF EQUILIBRIUM STELLAR MODELS

by

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ABSTRACT

Title of Dissertation: Fundamental Properties of Equilibrium Stellar Models

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This dissertation discusses the final equilibrium states of stars which result from the completion of the thermonuclear evolution process. The stellar models, based on the Newtonian and the general relativistic theories of gravitation, are analyzed to determine what predictions they make about the properties of the final equilibrium configurations of stars. Since it is very difficult to solve the equations of stellar structure (especially when the angular momentum is non-zero), the equations themselves are analyzed to determine general properties which any particular solution to the equations (a stellar model) must possess.

The equations of thermodynamics, fluid mechanics and gravitation are presented for both the Newtonian and general relativistic theories. The concepts of thermodynamic equilibrium and stationarity ("time independence") of the stellar models computed from these theories are defined and discussed. The relationships between stationarity and thermodynamic equilibrium are derived.

It is shown that the final equilibrium state of a star must be a highly symmetric object. Stellar models composed of viscous, heat-conducting fluids must be axisymmetric in their final stationary states (in either the Newtonian or the general relativistic theories). Newtonian stellar models must have a plane of mirror symmetry if the velocity of the fluid satisfies certain (fairly weak) assumptions. Newtonian stellar models which are static (i.e. stationary with no fluid motion) must be spherical. A review

is given of attempts to demonstrate that "static implies spherical" for general relativistic stellar models also.

In addition to the results on the symmetries of stellar models, a number of other properties of the structure of these models are presented. For Newtonian stellar models, it is shown that a barotropic model in differential rotation must have the angular velocity constant on cylinders concentric with the rotation axis; the shape of the constant density surfaces must satisfy certain convexity properties; the angular velocity of a rotating star cannot exceed a certain limit; a stellar model whose density is constant on the surfaces of ellipsoids is either spherical or has uniform density; there are no pressureless fluid stellar models.

Since general relativity is a more complicated theory, the properties of equilibrium stellar models which have been derived to date tend to be more technical and less complete than the properties derived for Newtonian models. It is shown that general relativistic stellar models must be described by analytic functions if they satisfy certain minimal differentiability criteria; a large number of identities involving the Killing vector fields which describe the symmetries of stellar models are derived; the general relativistic generalizations of the rotation on cylinders theorem and the upper limit on the angular velocity theorem are presented; the equivalence of the material and the metric staticity conditions is derived; the equivalence of orthogonal transitivity and convection free flow is derived; a number of inequalities which must be satisfied by the "gravitational potentials" in a stellar model are presented.

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§1 INTRODUCTION

What are the properties of the final state of thermonuclear evolution of a star? Consider a star which has N baryons and a certain amount of angular momentum, J. Let all nuclear and chemical reactions proceed to their final endproducts. Let radiation carry off any excess heat, and leave the star as a cold burned out cinder. What are the possible forms that such an endproduct of stellar evolution may assume? What is the constitution of the material in the star? What spatial configuration does the matter in the star assume in its final equilibrium configuration? The purpose of this dissertation is to seek answers to this kind of question through the rigorous application of our current physical theories.

The physical interactions in a star occur on two widely differing distance scales: the short scale nuclear and chemical interactions and the long scale gravitational interaction. The distance scales of these two types of interactions are so different that it is possible to separate the study of stellar models into two parts. One part studies the inter-particle interactions on a small scale to determine the constitution of cold catalyzed matter, while neglecting the gravitational interaction. The second part studies the large scale structure of the matter and the gravitational field, while treating the matter as a smoothed out fluid distribution. In this dissertation we will concentrate on investigating the second part of this problem. A review of the small scale properties of matter, relevant to the study of the endpoint of stellar evolution, is given by Zeldovitch and Novikov (1971).

According to our present understanding, a star may approach as its final equilibrium configuration either an "ordinary" star containing degenerate matter (i.e. a white dwarf or neutron star) or if it has too much mass it may become a black hole. At the present time a great deal is known about the final configurations which may be described as stationary black holes. It is now known that the Kerr family of black hole solutions provide the complete description of this possible endpoint of stellar evolution (see Carter 1973 and Robinson 1975). In comparison, much less is known about the properties of the non-singular endpoint which consists of an "ordinary"star. We will concentrate here on the study of these "ordinary" non-singular stellar models.

To study the properties of equilibrium non-singular stellar models, we should like to take our favorite theory of gravitation, impose the condition of equilibrium on the matter and then find the most general solution for the configuration of the matter and gravitational field. Unfortunately, even for a relatively simple theory of gravity like Newton's theory, it has only been possible to find very simplistic solutions to the equations of stellar structure. For example, the uniform density ellipsoidal models are essentially the only non-zero angular momentum solutions which are known (except for approximate or numerical solutions). The prospect of determining <u>all</u> equilibrium solutions seems quite hopeless. Our approach will be: attempt to extract from the theory the general properties which <u>any</u> solution of the equations must possess, but do this <u>without</u> actually writing down all of the possible solutions. We are able to determine in this way, a great deal about the symmetries which an equilibrium stellar model must possess. Further, we are able to determine a large number of very interesting properties which the functions describing a stellar model must possess. These symmetries and properties of stellar models will be derived and discussed at length in the following chapters.

Our discussion of the fundamental properties of equilibrium stellar models has two parts. We discuss separately the stellar models of the Newtonian and the general relativistic theories of gravitation. Newtonian stellar models are considered because the theory is relatively simple, and it is easier to determine what kinds of results are possible. Newtonian stellar models are discussed in Part I of this dissertation. The matter in a burned out, equilibrium stellar model must be in a relatively high density state where the Fermi repulsion of the electrons or neutrons can provide the pressure needed to keep the star from collapsing. Consequently the gravitational fields in these stars will be rather strong. Therefore, it is important to study the properties of these stellar models using the general theory of relativity, which is believed to be a more accurate theory in strong field situations. Since general relativity is a considerably more complicated theory, our results are incomplete and tend to be of a more technical nature than those discussed for the Newtonian stellar models. General relativistic stellar models are discussed in Part II of this dissertation. Complete summaries of the contents of each Part are given in sections 2.1 and 6.1 respectively.

Part I.

Newtonian Stellar Models

§2 DESCRIPTION OF THE NEWTONIAN MODELS

2-1. Introduction to Part I

In the first part of this dissertation we discuss the fundamental properties of equilibrium stellar models, within the context of the Newtonian theory of gravitation. Our approach will be to write down the thermodynamic, gravitational, and fluid equations of motion in a form which is very general. These equations will therefore be adequate to describe the most complicated, chaotic time dependent motions of the fluid in a star. From these general equations we will attempt to deduce the properties of a stellar model which has come to an equilibrium state. Earlier reviews of this type of work are given by Jardetzky (1958), Lebowitz (1967) and Roxburgh (1970).

The assumption that a stellar model is in a state of equilibrium is a very stringent one. We would not expect to find any real stars in a state of equilibrium. However, we might expect a large number of stars to be in "nearly equilibrium" states which can be described adequately as perturbations of some equilibrium model. To determine the way in which a stellar model evolves toward the equilibrium state, and to compute the rate at which this occurs is an extremely difficult task. Appendix I describes the evolution of the simplistic "homogeneous ellipsoidal figures" under the influence of viscosity and gravitational radiation reaction. Simple calculations of this sort, and calculations based on the perturbations of equilibrium models give some information about the evolution of a stellar model toward the equilibrium state. A comprehensive review of this kind of calculation is beyond the scope of the present work however. We will concentrate instead on determining as much as possible about the equilibrium state.

Chapter 2 describes the general laws of thermodynamics, fluid mechanics and Newtonian gravitation on which the stellar models are based. We also describe the set of boundary conditions which we put on the solutions of the fluid equations. These boundary conditions distinguish our solutions as being stellar models, rather than other possible solutions of the fluid equations, such as flow through a pipe.

Chapter 3 discusses two different notions of the concept of "equilibrium": thermodynamic equilibrium and stationarity. We show that stationarity implies thermodynamic equilibrium (Theorem 3.1) and that thermodynamic equilibrium implies stationarity with respect to some uniformly rotating reference frame (Theorem 3.4). We prove that stationary viscous stellar models must be axisymmetric (Theorem 3.5). And, we prove that the angular velocity of a stationary axisymmetric ideal fluid model is constant on cylinders coaxial with the rotation axis, if and only if the fluid is barotropic (Theorem 3.7).

Chapter 4 discusses the mirror symmetry of stellar models. We prove that an ideal fluid stellar model with stratified flow, must necessarily have a plane of mirror symmetry (Theorem 4.6). Two corollaries of this result are also presented. We show that the shape of a star must be in a certain sense convex (Theorem 4.7). And, we show that static stars must be spherically symmetric (Theorem 4.8).

Chapter 5 derives three results which delineate properties which a stellar model may not have. We derive Poincare's limit on the angular velocity of a stellar model (Theorem 5.1). We prove that the level surfaces of the density of the fluid cannot have a certain form unless the density of the fluid is uniform (Theorems 5.2 and 5.3). These theorems show (as a special case) that the ellipsoidal stratification of the density of the fluid is not possible except for uniform density models: the Maclaurin spheroids. We also prove that no pressureless fluid (dust) stellar models exist (Theorem 5.4).

2-2. Newtonian Fluid Mechanics

The thermodynamic properties of a fluid will be described by the following functions:

ρ ··· mass density,

ε ··· internal energy density,

p ··· pressure,

s ··· entropy density,

T ··· temperature,

n ··· particle number density,

μ ··· chemical potential,

m ··· particle rest mass,

η, ζ ··· coefficients of viscosity,

κ ··· coefficient of heat conduction, and

q^i ··· the heat flow vector.

The motion of the fluid is described by the functions:

v^i ··· velocity field of the fluid,

a^i ··· acceleration,

θ ··· expansion, and

σ^{ij} ··· the shear tensor.

Finally, the Newtonian gravitatioal field is described by

ϕ ··· the gravitational potential.

These functions are related to each other by the laws of thermodynamics, the equations of motion of the fluid and by the Newtonian law of gravitation. Let us begin by describing the laws of thermodynamics. Fluids which are composed of different types of matter will have somewhat different physical properties. For example, a fluid of hydrogen atoms will behave somewhat differently then a fluid of helium atoms. These individual characteristics of the fluid are described mathematically by the equations of state of the fluid. The equation of state, as used in this work, will be represented by taking ε, η, ζ and κ to be known functions of the particle number density and the entropy density

$$\varepsilon = \varepsilon(n,s), \ \eta = \eta(n,s), \ \zeta = \zeta(n,s) \text{ and } \kappa = \kappa(n,s); \tag{2.1}$$

and by specifying $m > 0$, the constant rest mass of a particle of fluid. We do not require that these functions take any particular form, except a) each function must be sufficiently smooth, C^2; and b) the functions η, ζ and κ must be non-negative

$$\eta \geq 0 \ , \ \zeta \geq 0 \text{ and } \kappa \geq 0. \tag{2.2}$$

From the equation of state, and using the first law of thermodynamics, the four remaining thermodynamic potentials can be expressed as functions of n and s:

$$T = \partial\varepsilon/\partial s \ , \tag{2.3}$$

$$\mu = \partial\varepsilon/\partial n \ , \tag{2.4}$$

$$p = Ts + \mu n - \varepsilon \ , \text{ and} \tag{2.5}$$

$$\rho = mn. \tag{2.6}$$

The heat flow for these fluids, will be assumed to proceed according to the usual law,

$$q^i = - \kappa \nabla^i T. \tag{2.7}$$

These conditions are sufficient to completely determine all of the thermodynamic potentials in terms of the two functions n and s. To ensure that n, s, p and T represent the number density, the entropy density, the pressure and the temperature of a realistic fluid, we require:

$$n \geq 0, \quad s \geq 0, \quad p \geq 0 \quad \text{and} \quad T \geq 0. \tag{2.8}$$

We require the third law of thermodynamics: if T = 0 then s = 0. We will also impose certain theromodynamic stability criteria on the fluids of our stellar models (see Callen 1960, p. 194). These conditions are given by

$$\partial p/\partial n > 0, \quad \partial T/\partial s > 0 \text{ and } \quad \partial \mu/\partial n > 0. \tag{2.9}$$

Imposing these conditions ensures that the speed of sound of the material is real, and that phase transitions do not occur.

The system of thermodynamics which we use here is discussed at length by Landau and Lifshitz (1959) in §§2, 6, 15, 49 and 63.

The motion of the particles in the fluid are described by the velocity vector field v^i. The acceleration, expansion and shear tensor are related to v^i by

$$a^i = \partial_t v^i + v^j \nabla_j v^i , \tag{2.10}$$

$$\theta = \nabla_i v^i , \text{ and} \tag{2.11}$$

$$\sigma^{ij} = \nabla^i v^j + \nabla^j v^i - (2/3)\delta^{ij}\theta . \tag{2.12}$$

Consequently, the state of the fluid can be completely specified by giving the functions n, s, v^i and ϕ . An equation of motion is needed for each of these functions. According to the theory of Newtonian hydrodynamics (see Landau and Lifshitz [1959]) these equations of motion are given by: the conservation of particle number

$$\partial_t n + \nabla_i (n\, v^i) = 0 \ , \tag{2.13}$$

the equation of entropy production

$$T[\partial_t s + \nabla_i (s v^i)] = \nabla_i (\kappa \nabla^i T) + \zeta \theta^2 + \frac{1}{2}\, \eta\, \sigma_{ij}\, \sigma^{ij} \ . \tag{2.14}$$

Newton's law of gravitation

$$\nabla^i \nabla_i \phi = - 4\pi\rho \ , \tag{2.15}$$

and Euler's equation (also called the Navier-Stokes equation)

$$\rho\, a_i = - \nabla_i p + \rho\, \nabla_i \phi \ + \nabla^k (\eta \sigma_{ki}) + \nabla_i (\zeta\theta) . \tag{2.16}$$

A fluid which satisfies the above system of equations (2.1)-(2.16) will be called a *viscous heat-conducting Newtonian fluid.* Two special cases of these equations are of particular interest in the study of stellar structure. Any fluid having vanishing coefficients of viscosity and heat condition,

$$\eta = \zeta = \kappa = 0 \ , \tag{2.17}$$

will be called an *ideal fluid.* The equations of motion for an ideal fluid take on a particularly simple form. The conservation of particle number, and Newton's law of gravitation remain unchanged, however the entropy production equation and Euler's equation simplify to

$$T[\partial_t s + \nabla_i (s\, v^i)] = 0, \text{ and} \tag{2.18}$$

$$\rho a_i = - \nabla_i p + \rho\, \nabla_i \phi\ . \tag{2.19}$$

Another interesting special case occurs when the level surfaces of the density function coincide with the level surfaces of the pressure function,

$$\nabla_i p\, \nabla_j \rho \ - \nabla_j p\, \nabla_i \rho \ = 0. \tag{2.20}$$

When eq. (2.20) is satisfied the fluid is called *barotropic*.

2-3. Boundary Conditions

To distinquish the solutions of the fluid equations which we will call *stellar models*, from other possible solutions, we must impose certain boundary conditions on the various functions in the model. These boundary conditions will be imposed both at infinity, and at the interface between the fluid and the vacuum exterior: the *surface of the star*. We impose the following conditons:

a) The gravitational potential, ϕ, vanishes as $x^2 + y^2 + z^2 \to \infty$.

b) The support of the thermodynamic variables is bounded.

c) The pressure, p, vanishes on the surface of the star.

d) The heat flow vector, q^i, is tangent to the surface of the star.

We also require the functions in the model to be reasonably smooth. In particular, we require:

e) The solutions must be non-singular, therefore we require the functions to be bounded.

f) Each function must be at least C^2, except at the surface of the star, where discontinuities in some functions may occur.

g) The gravitational potential must be at least C^3 except at the surface of the star where it must be at least C^1 with respect to normal derivatives and C^2 with respect to tangential derivatives.

§3 STATIONARITY AND THERMAL EQUILIBRIUM

3-1. Introduction

In this chapter we begin to investigate the properties of stellar models which are assumed to be stationary, or in a state of thermal equilibrium. (The precise meanings of these terms are given below.) We show that these two concepts are in a certain sense equivalent. In particular we show that a stationary stellar model must be in thermal equilibrium (Theorem 3.1); and we show that a viscous fluid stellar model which is in thermal equilibrium must appear stationary in some rotating frame of reference (Theorem 3.4). We also derive a remarkable result for stationary viscous fluid stellar models; these models must necessarily be axisymmetric (Theorem 3.5). It does not follow, however, that all stellar models which are in thermal equilibrium are necessarily axisymmetric; a counter-example is discussed. To conclude this chapter, we prove one of the classic results in this field: that the angular velocity of the fluid must be constant on cylinders (which are coaxial with the rotation axis), in a stationary axisymmetric ideal fluid stellar model (Theorem 3.7).

We begin by defining the concepts of thermodynamic equilibrium, and stationarity. For our purposes, we will say that a particular solution of the fluid equations is in a state of *thermodynamic equilibrium* if the entropy of each particle of fluid is constant along the trajectory of that particle. The precise mathematical statement of this condition is given by

$$\partial_t(s/n) + v^i\nabla_i(s/n) = 0 \ . \qquad (3.1)$$

A somewhat more useful, equivalent form, can be derived from eq. (3.1) by using the conservation of particle number eq. (2.13):

$$\partial_t s + \nabla_i (s v^i) = 0 \ . \tag{3.2}$$

A solution of the fluid equations is called *stationary* if all of the functions n, s, ϕ and v^i are independent of time. This concept is complicated somewhat by the Galilean invarience of the fluid equations. That is, if we make a coordinate transformation of the form

$$t' = t \quad , \; x'^i = x^i - t v^i \ , \tag{3.3}$$

with constants v^i the equations of the fluid are invariant. However, as we can see from the expression for the transformed time differential

$$\partial_{t'} = \partial_t + v^i \nabla_i \ , \tag{3.4}$$

a fluid solution which is stationary according to one reference frame is not necessarily stationary according to another reference frame. Thus, we will call a solution stationary if it is stationary according to some Galilean frame. Another useful concept is that of a stellar model which is stationary in a uniformaly rotating frame of reference; this has the obvious interpretation.

3-2. Stationarity and Thermal Equilibrium are Equivalent (Almost)

The concepts of thermal equilibrium and stationarity are closely related physically. Our next objective is to establish that relationship through two theorems. The first (Theorem 3.1) shows that stationarity implies thermal equilibrium while the second (Theorem 3.4) establishes the extent to which thermal equilibrium implies stationarity. The proofs which we give here apply only to the nonzero temperature case. The results are still valid for the zero temperature case, and the proofs are fairly straightforeward generalizations of those given here.

THEOREM 3.1 - *A Newtonian fluid stellar model which is stationary must be in a state of thermal equilibrium.*

PROOF: From eq. (2.14) it follows that

$$\partial_t s + \nabla_i (sv^i - \kappa\nabla^i \log T) = [\kappa T^{-1}\nabla^i T\nabla_i T + \zeta\theta^2 + \frac{1}{2}\eta\sigma_{ij}\sigma^{ij}]/T. \qquad (3.5)$$

In the stationary case $\partial_t s = 0$, therefore it follows that

$$\nabla_i (sv^i - \kappa\nabla^i \log T) \geq 0. \qquad (3.6)$$

Integrating the divergence on the left hand side of eq. (3.6) over the interior of the star yields an integral over the surface of the star. This surface integral vanishes because a) the heat flow vector is assumed to be tangent to the surface of the star (see §2.3) and b) in the stationary case the velocity vector is also tangent to the surface of the star. Thus the integral of $\nabla_i (sv^i - \kappa\nabla^i \log T)$ vanishes. This fact togeather with eq. (3.6) implies that the integrand itself vanishes:

$$\nabla_i (sv^i - \kappa\nabla^i \log T) = 0. \qquad (3.7)$$

The right hand side of eq. (3.5) is the sum of three non-negative terms, therefore each term must vanish separately whenever the left hand side vanishes. In particular it follows that $\nabla_i T = 0$, and from eq. (3.7) that

$$\partial_t s = \nabla_i (sv^i) = 0 \qquad (3.8)$$

Therefore eq. (3.2) is satisfied and the star must be in thermodynamic equilibrium. ∎

Before proceeding to establish the extent to which thermal equilibrium implies stationarity, it is convenient to prove two preliminary results.

LEMMA 3.2 - *The only solutions of the equation* $\nabla_i v_j + \nabla_j v_i = 0$ *in a flat 3-dimensional space are constant linear combinations of the six vectors (expressed in Cartesian coordinates):*

$$\vec{T}_x = (1,0,0) \qquad \vec{R}_x = (0,-z,y)$$

$$\vec{T}_y = (0,1,0) \qquad \vec{R}_y = (z,0,-x)$$

$$\vec{T}_z = (0,0,1) \qquad \vec{R}_z = (-y,x,0)$$

PROOF: It is straightforeward to show that these six vectors do in fact solve the equation. That these are the only solutions is a special case of a well known theorem (see for example Weinberg 1972, p. 377) which shows that this equation admits exactly N(N+1)/2 solutions in an N dimensional symmetric space. ■

THEOREM 3.3 - *A viscous heat conducting Newtonian fluid stellar model which is in thermal equilibrium must be in a state of rigid rotation, the temperature must be uniform and the fluid must be barotropic.*

PROOF: Since the fluid is in thermal equilibrium it must satisfy (see eq. 2.14),

$$-\nabla_i(\kappa\nabla^i \log T) = [\kappa T^{-1} \nabla_i T\nabla^i T + \zeta\theta^2 + \frac{1}{2}\eta\sigma_{ij}\sigma^{ij}]/T. \tag{3.9}$$

The left hand side is a divergence while the right hand side is non-negative.

Consequently, by an argument similar to that given in the proof of Theorem 3.1, each side must vanish. Thus it follows that each of the positive terms on the right must vanish separately so that,

$$\nabla_i v_j + \nabla_j v_i = 0 \text{ , and} \tag{3.10}$$

$$\nabla_i T = 0 \text{ .} \tag{3.11}$$

Lemma 3.2 shows us that the independent solutions of eq. (3.10) are precisely the vectors $\vec{T}_x$, $\vec{T}_y$, $\vec{T}_z$ and $\vec{R}_x$, $\vec{R}_y$, $\vec{R}_z$. The first three represent uniform linear motions while the second three represent uniform rigid rotations. The velocity vector v^i must be a linear combination of these, where the coefficient of each of the vectors (e.g. $\vec{T}_x$) can depend only on time. To show that the motion is rigid, we must prove that the distance between two particles of the fluid does not change with time. Let $x_1(t)$ and $x_2(t)$ be the trajectories of two fluid particles. The distance between the two varies with time according to the formula

$$d[(\vec{x}_1-\vec{x}_2) \cdot (\vec{x}_1-\vec{x}_2)]/dt = 2(\vec{x}_1-\vec{x}_2) \cdot (\vec{v}_1-\vec{v}_2). \tag{3.12}$$

The right hand side of eq. (3.12) is a sum of terms of the form $(\vec{x}_1-\vec{x}_2) \cdot [\vec{T}_x(x_1) - \vec{T}_x(x_2)]$ and $(\vec{x}_1-\vec{x}_2) \cdot [\vec{R}_x(x_1) - \vec{R}_x(x_2)]$. It is easy to check that each of these types of terms vanish identically. Consequently the distance between particles is independent of time, so the motion is rigid. Equation (3.11) shows that the temperature of the fluid is uniform. To show that the fluid is barotropic, use equations (2.3)-(2.5):

$$\nabla_i p = s \nabla_i T + n \nabla_i \mu. \tag{3.13}$$

Since the temperature is uniform, it follows that

$$\nabla_i p \ \nabla_j n - \nabla_j p \ \nabla_i n = 0 \ . \tag{3.14}$$

That the fluid is barotropic, follows directly now from eq. (2.6). ∎

This last theorem (3.3) shows that the definition of thermal equilibrium used here implies other common notions of thermal equilibrium: uniform temperature and barotropic flow.

We proceed next to establish the extent to which thermal equilibrium implies stationarity.

THEOREM 3.4 - *A viscous, heat conducting, Newtonian fluid stellar model which is in thermal equilibrium must be stationary in the co-moving frame of the fluid, and the fluid must rotate uniformly.*

PROOF: From Theorem 3.3 we learned that the velocity of the fluid must satisfy eq. (3.10). Consequently the acceleration vector is given by

$$a^i = \partial_t \ v^i - \frac{1}{2} \nabla^i (v^j v_j) \ . \tag{3.15}$$

Since the fluid must be barotropic (Theorem 3.3), Euler's equation (2.16) for this system reduces to

$$\partial_t \ v^i = \nabla^i (\tfrac{1}{2} \ v^j \ v_j + \phi - W), \tag{3.16}$$

where W is defined by $\nabla_i W = \rho^{-1} \ \nabla_i p$. Consequently, it follows that

$$\nabla^i \ \partial_t \ v^j - \nabla^j \ \partial_t \ v^i = 0 \tag{3.17}$$

However, from eq. (3.10) it also follows that

$$\nabla^i \ \partial_t \ v^j + \nabla^j \ \partial_t \ v^i = 0 \tag{3.18}$$

The only vectors, $\partial_t v^j$, which satisfy (3.17) and (3.18) are linear combinations of $\vec{T}_x$, $\vec{T}_y$ and $\vec{T}_z$. Therefore the velocity vector must have the form (see Lemma 3.2),

$$\nabla = f_x(t)\vec{T}_x + c_x \vec{R}_x + \dots \quad . \tag{3.19}$$

where $f_x(t)$, $f_y(t)$ and $f_z(t)$ are function of time, and c_x, c_y, c_z are constants.

The equations of motion for n and s are given by eq. (2.13) and (3.2). For the velocity field in eq. (3.19) the expansion vanishes, $\theta = 0$. Therefore the equations for n and s reduce to

$$\partial_t n + v^i \nabla_i n = 0 \text{ , and} \tag{3.20}$$

$$\partial_t s + v^i \nabla_i s = 0 \text{ .} \tag{3.21}$$

We can derive a similar equation for the gravitational potential. We use the gravitational field equation (2.15) and the properties of the velocity field in eq. (3.19) to derive

$$\nabla_i \nabla^i (\partial_t \phi + v^i \nabla_i \phi) = - 4\pi (\partial_t \rho + v^i \nabla_i \rho). \tag{3.22}$$

The right hand side of eq. (3.22) vanishes by eq. (3.20). Since ϕ vanishes asymptotically, eq. (3.22) implies

$$\partial_t \phi + v^i \nabla_i \phi = 0 \text{ .} \tag{3.23}$$

Now, multiply eq. (3.16) by v^i and use eqs. (3.20), (3.21) and (3.23) to conclude that

$$\partial_t (\tfrac{1}{2} v^i v_i + \phi - W) = 0. \tag{3.24}$$

We can now derive a further simplification of the velocity field by taking the time derivative of eq. (3.16) and using eq. (3.24):

$$\partial_t \partial_t v^i = 0 . \tag{3.25}$$

Therefore, the velocity field reduces to the form

$$\vec{v} = (a_x t + b_x)\vec{T}_x + c_x \vec{R}_x + \cdots \tag{3.26}$$

where a_i, b_i and c_i are all constants.

To simplify the velocity field more, we must use the conservation laws for the mass and the momentum of the star. These conservation laws are derived from the following expressions which are derived from eqs. (2.15), (3.16) and (3.20):

$$\partial_t \rho = - \nabla_i (\rho v^i), \text{ and} \tag{3.27}$$

$$\partial_t (\rho v^i) = - \nabla_j \{\rho v^i v^j + p\delta^{ij} + [\nabla^i \phi \nabla^j \phi - \frac{1}{2} \delta^{ij} \nabla^k \phi \nabla_k \phi]/4\pi\}. \tag{3.28}$$

The right hand sides of eqs. (3.27) and (3.28) are divergences, whose integrals vanish because of the boundary conditions imposed in §2. Therefore we conclude that

$$\partial_t \int \rho \ d^3x = 0, \text{ and} \tag{3.29}$$

$$\partial_t \int \rho \ v^i \ d^3x = 0 . \tag{3.30}$$

When these two constraints are taken into account, it is always possible to perform a Galilean transformation of the coordinates which boosts to a frame of reference in which the constant a_x, a_y and a_z vanish. By performing another coordinate transformation which is a combination translation and boost, a frame of reference can be reached in which the constants b_x, b_y and b_z vanish. The velocity vector in this frame takes the form

$$\vec{v} = c_x \vec{R}_x + c_y \vec{R}_y + c_z \vec{R}_z , \qquad (3.31)$$

which represents a uniform rotation. We can, if we choose, perform a further single rigid rotation of the coordinates to align the rotation axis of the fluid with the z axis of the coordinates. Therefore, we can always choose coordinates in which the velocity field of the fluid takes the simple form,

$$\vec{v} = \Omega(-y, x, 0). \qquad (3.32)$$

Equations (3.20) and (3.21) demonstrate that all of the thermodynamic functions are stationary in the co-moving frame of the fluid. Equation (3.32) demonstrates that this co-moving frame is a uniformly rotating one. ∎

3-3. Stationary Stars are Axisymmetric

The next theorem is an example of the most fascinating property of equilibrium stellar models: they are necessarily highly symmetric objects. Anyone who has examined photographs of the nearby "stars" (the Sun, Jupiter, Saturn) cannot have failed to notice and wonder about the high degree of symmetry which these objects posess. It is gratifying, therefore, to discover that our models predict this very simple, highly symmetrical motion for the stationary state.

THEOREM 3.5 - *A viscous heat conducting Newtonian fluid stellar model which is stationary must also be axisymmetric.*

PROOF: From Theorem 3.1 we know that the stellar model must be in thermal equilibrium, and from Theorem 3.4 we know that the velocity field must be that of a uniform rotation eq. (3.32). Furthermore we know that the equation of motion for n and s are given by eqs. (3.20) and (3.21). In the stationary case, where $\partial_t s = \partial_t n = 0$, these equations reduce to

$$v^i \nabla_i n = 0, \quad \text{and} \tag{3.33}$$

$$v^i \nabla_i s = 0 \ . \tag{3.34}$$

Therefore, all of the thermodynamic variables must be constant along the integral curves of the vector field v^i. Thus, the stellar model must be axisymmetric. ∎

One might have guessed that one could derive a result analogous to Theorem 3.5 by replacing the stationarity assumption, by the requirement of thermal equilibrium. Such a result is false however; Newtonian stellar models which are in thermal equilibrium need not be axisymmetric. An example of a non-axisymmetric, uniformly rotating Newtonian fluid stellar model is provided by the Jacobi ellipsoids (see Chandrasekhar 1969, p. 101). These models are uniform density fluid models, which are triaxial ellipsoidal figures uniformly rotating about one of the principal axes of the ellipsoid.

3-4. Barotropes Rotate on Cylanders

We conclude this chapter by discussing some of the properties of stellar models constructed with ideal fluids, rather than the viscous heat conducting fluids discussed above. Even though realistic fluids all have non-vanishing coefficients of viscosity and heat conduction, the time-scales required to achieve equilibrium are often very large for stellar models. Therefore, it is in some sense more realistic to model stars by ideal fluids. Mathematically, however, this is a far more difficult job, because there is no longer the necessity of rigid motion in the equilibrium state. In fact for ideal fluid models, our requirement of thermal equilibrium (eq. 3.2) is always satisfied. Therefore there is a very large diversity of possible fluid motions, even in the stationary case.

In order to make some progress possible in the study of ideal fluid stellar models it is necessary to make some (admittedly arbitrary) assumptions about the motion of the fluid. One approach is to consider generalizations of the motion which is allowed by a "truely equilibrium" fluid stellar model. As shown in Theorem 3.4, the motion of a fluid in equilibrium is given by the vector field

$$\vec{v} = \Omega(-y, x, 0) \qquad (3.35)$$

where Ω is a constant. One possible generalization of this motion is to let Ω be an arbitrary function of position; this generalization is called *differential rotation.* Another possible generalization introduces yet another arbitrary function

$$v = (v_x, v_y, 0); \qquad (3.36)$$

this generalization has been called *stratified flow.* The Jacobi, Dedekind and Riemann S ellipsoids (see Chandrasekhar 1969) are interesting examples of stratified flow. We now present two results for ideal fluid stellar models having these specialized fluid flows. The first has been derived previously by Lindblom (1977a).

LEMMA 3.6 - *Euler's equation for an ideal barotropic Newtonian fluid stellar model with stratified flow may be written in the simplified form*

$$\nabla_i p = \rho \nabla_i \psi \tag{3.37}$$

where $\psi = \phi - T$ *and* T *is some function which is independent of* z.

PROOF: Euler's equation for an ideal fluid is given by eq. (2.10), which may be rewritten in the form

$$a^i = -\rho^{-1} \nabla_i p + \nabla_i \phi. \tag{3.38}$$

The right hand side of eq. (3.38) is a gradient whenever the fluid is barotropic, consequently the left hand side must also be a gradient.

$$a^i = \nabla^i T. \tag{3.39}$$

Equation (3.38) and (3.39) can be combined to give equation (3.37) if we define

$$\psi = \phi - T \quad . \tag{3.40}$$

Since the fluid is stratified (eq. 3.36) it follows that $v_z = 0$, and consequently $a_z = 0$ (using eq. [2.10]). Therefore $\partial_z T = 0$ from eq. (3.39) so that T is independent of z as required. ■

This lemma will be needed in the proof of the mirror symmetry theorem in the next chapter.

The final result presented in this chapter is one of the older known "fundamental" properties of equilibrium stellar models (see for example, Wavre 1932).

THEOREM 3.7 – *A stationary axisymmetric ideal Newtonian fluid stellar model which is in differential rotation is barotropic if and only if the angular velocity of the fluid is constant on cylinders coaxial with the rotation axis.*

PROOF: When the z axis is chosen as the rotation axis of the fluid, (as in eq. [3.35] with Ω an arbitrary function of $x^2 + y^2$ and z) the acceleration is given by

$$a^i = -\frac{1}{2}\ \Omega^2\ \nabla^i(x^2+y^2). \tag{3.41}$$

Euler's equation therefore becomes

$$-\frac{1}{2}\ \Omega^2\ \nabla_i(x^2+y^2) = -\rho^{-1}\ \nabla_i p + \nabla_i \phi\ . \tag{3.42}$$

Taking the curl of eq. (3.42) yields

$$-\Omega\ \nabla_{[i}\ \Omega\ \nabla_{j]}(x^2+y^2) = \rho^{-2}\ \nabla_{[i}\ \rho\ \nabla_{j]}p. \tag{3.43}$$

When the fluid is barotropic the right hand side of eq. (3.43) vanishes so that the angular velocity Ω must only be a function of $x^2 + y^2$. Consequently Ω must be constant on cylinders coaxial with the rotation axis. Whenever Ω depends only on $x^2 + y^2$ the left hand side of eq. (3.43) vanishes, so the fluid must be barotropic. ■

§4 MIRROR SYMMETRY:

4-1. Introduction and Summary

This chapter presents what is perhaps the most remarkable result in the theory of the fundamental properties of equilibrium stellar models: a rotating stellar model must have a plane of mirror symmetry which is perpendicular to the rotation axis of the star. This theorem, like the axisymmetry theorem (3.5) of the last chapter, demonstrates that equilibrium stellar configurations must be highly symmetrical. The first proof of this result was given by Lichtenstein (1918), (1933) for the case of uniform density stellar models in rigid rotation. The result was generalized by Wavre (1932) to the case of stationary axisymmetric barotropic ideal fluids in differential rotation. A further generalization has been given by Lindblom (1977a) for the case of barotropic ideal fluids which have stratified flows. This latter result does not assume that the fluid is either stationary or axisymmetric. Consequently, the result applies to interesting non-axisymmetric, non-stationary objects such as the Jacobi, Dedekind and the Riemann S ellipsoidal stellar models (see Chandrasekhar 1969).

The version of the proof due to Lindblom (1977a) is given here as Theorem 4.6. It is similar in many respects to the proof of Wavre (1932); it differs, however, in one important respect. Wavre's proof depends crucially on the use of the Green's function for the Laplace operator in the Newtonian gravitational field equation. Consequently Wavre's proof is completely unsuitable as a model for generalization to the case of general relativistic stellar models, where the gravitational field equations are non-linear. The proof presented here, however, does not make use of the Green's functions; instead it uses the maximum principle satisfied by solutions of certain elliptic differential equations. Consequently, the proof presented here may serve as a model for a generalization of this theorem to general relativistic stellar models.

We complete this chapter by proving two interesting corollaries of Theorem 4.6. The first corollary provides us with a qualitative picture of the structure of the stellar model, by proving that the star must be in a certain sense convex. We will call a stellar model *z-convex* if, for every two points (x,y,z_1) and (x,y,z_2) within the support of the density function ρ, it follows that all points (x,y,z) (with z between z_1 and z_2) are also within the support of ρ. Figure 4.1 illustrates this concept by showing several figures which violate it. Theorem 4.7 proves that stratified Newtonian stellar models must be z-convex.

The second important corollary of the mirror plane theorem is presented here for the case of static stellar models. A stellar model is called *static* if the velocity field vanishes,

$$v^i = 0. \tag{4.1}$$

In this special case Carleman (1919) (see also Lichtenstein 1933) has shown that the stellar model must also be spherically symmetric. This result is presented here as Theorem 4.8.

Since the proof of Theorem 4.6 is somewhat circuitous, it will be helpful to describe the proof qualitatively. We begin by considering the set of chords which are parallel to the z axis, and which have both endpoints on the same level surface of the gravitational potential function. Lemma 4.3 is used to show that every point is the endpoint of some such chord. Next we consider the set of midpoints of those chords. For this purpose we define a function m_ϕ, which maps the endpoints of chords into

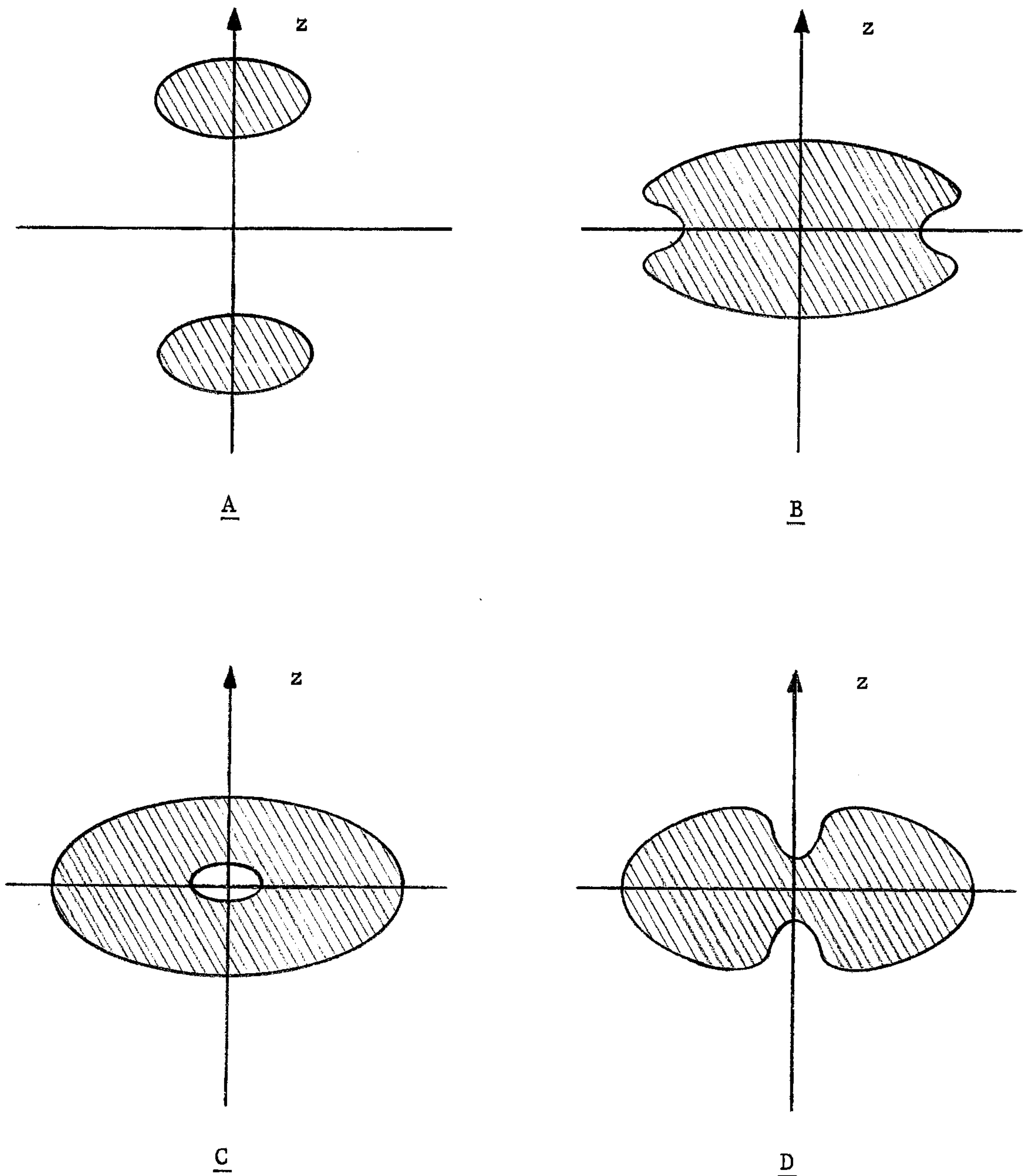

FIGURE 4.1: These figures represent possible sections of stratified Newtonian stellar models. Figures A, B and C are not z-convex and consequently cannot represent stellar models according to Theorem 4.7. Figure D is z-convex.

their midpoints. In Lemma 4.4 we show that there is a chord whose midpoint's z component, z_m, is larger than or equal to the z component of the midpoint of any other chord. We will decompose each of the functions into even and odd parts with respect to reflection about the plane $z = z_m$; and we will show that this plane is a mirror plane of the star. In Lemma 4.5 we derive the important fact that the odd part of the mass density, ρ^-, is negative for all z exceeding z_m. This brings us to the proof of Theorem 4.6. We show that the odd part of the gravitational potential, ϕ^-, must satisfy the differential equation $\nabla_i \nabla^i \phi^- = -4\pi\rho^- \geq 0$ for all $z \geq z_m$; this follows from Lemma 4.5. In addition we argue that ϕ^- must have a maximum in the half space $z > z_m$. The maximum principles for this type of differential equation are then invoked to show that in fact $\phi^- = 0$ everywhere. It follows that the odd parts of the mass density and pressure must vanish also. Thus the star must have a plane of mirror symmetry. Our rigerous derivation of this result follows next.

4-2. Preliminary Lemmas

Theorem 4.6 deals with a fluid having a velocity field which is stratified according to eq. (3.36), thus

$$v_z = 0 \quad . \tag{4.2}$$

This stratification picks out the preferred z coordinate which we use in the following discusion.

To construct the plane, z = constant, which we show in Theorem 4.6 is a plane of mirror symmetry of the stellar model, we classify the points in the star, based on the nearby behavior of the gravitational potential ϕ. We call a point (x,y,z) *normal* if $\partial\phi/\partial z(x,y,z) \neq 0$; and we call a point *special* if $\partial\phi/\partial z(x,y,z) = 0$. Furthermore we say that the point $(x,y,\bar{z})$

is *associated* with the point (x,y,z) if $\phi(x,y,z) = \phi(x,y,\bar{z})$ and $\phi(x,y,z) < \phi(x,y,z')$ for all z' between z and $\bar{z}$. Pairs of associated points form the endpoints of chords which are parallel to the z axis. If the set of midpoints of these chords were contained in some z=constant surface, then the stellar model would necessarily be mirror symmetric about this plane. Therefore, it is useful to consider the function m_ϕ which maps the endpoints of chords into their midpoints:

$$m_\phi(x,y,z) = (x,y,1/2[z + \bar{z}]) \tag{4.3}$$

For technical reasons we restrict the domain of m_ϕ to points lying in the support of the mass density function, ρ.

The proof of the mirror plane theorem presented here depends crucially on the maximum principles which must be satisfied by the solutions of certain elliptic differential equations. Therefore, we reproduce here (without proof) the versions of the maximum principle needed to prove our result. Proofs of these theorems, and more general versions of the maximum principle may be found in Bers, John and Schecter (1964).

THEOREM 4.1 - *Let B be an open ball, and x_o a point on its boundary. Assume that f is a C^2 function everywhere in B, and C^0 in the closure of B. Let $\nabla_i\nabla^i f \geq 0$ and $f \leq f(x_o)$ everywhere in B. Then the outward normal derivative $df/dn > 0$ at x_o, or $f = f(x_0)$ everywhere in B.*

THEOREM 4.2 - *Assume that f is a C^2 function everywhere in a bounded open neighborhood U, and that $\nabla_i\nabla^i f \geq 0$ everywhere in U. If there is a point x_o in U such that $f(x_o) \geq f(x)$ for all x in U, then $f(x_o) = f(x)$ for all x in U.*

We present next three lemmas which define the prospective mirror plane, and delineate some of its properties.

LEMMA 4.3 - *Let ϕ be the gravitational potential of a barotropic ideal Newtonian fluid stellar model which has stratified flow. For every point (x,y,z) there exists a unique associated point $(x,y,\bar{z})$.*

PROOF: Let us first show that $\phi(x,y,z) > 0$ everywhere. If there is a point with $\phi(x,y,z) \leq 0$, then we could find some point, say (x',y',z'), with $\phi(x',y',z') \leq \phi(x,y,z)$ for all points (x,y,z). By eq. (2.15) we have $\nabla_i \nabla^i \phi \leq 0$. Using Theorem 4.2 one can show that if the point (x',y',z') exists, then $\phi = 0$ everywhere. If the point (x',y',z') lies on the boundary of the star, a slightly different argument using Theorem 4.1 gives the same result, $\phi = 0$. Thus we can conclude that ϕ must be positive everywhere.

We next consider the normal point (x,y,z). One can start at (x,y,z) and proceed along the line (x,y) = constant in the direction of increasing ϕ. When one reaches points having sufficiently large values of $x^2 + y^2 + z^2$, the potential ϕ will become arbitrarily small. This guarantees that a point, say $(x,y,\hat{z})$, will be reached along the line at which $\phi(x,y,z) = \phi(x,y,\hat{z})$. If one takes the first such point reached along the line, say $(x,y,\bar{z})$, then $\phi(x,y,z') > \phi(x,y,z)$ for all z' between z and $\bar{z}$. Thus $(x,y,\bar{z})$ is associated with (x,y,z) and the lemma is proved. ∎

The next lemma will derive an important property of the function m_ϕ.

LEMMA 4.4 - *There exists a point (x_o,y_o,z_o) in the domain of m_ϕ, whose image $(x_o,y_o,z_m) \equiv m_\phi(x_o,y_o,z_o)$ is a least upper bound of the z component of the range of m_ϕ: i.e. for every point (x,y,z) in the range of m_ϕ, $z_m \geq z$.*

PROOF: Let us first argue that the z components of the range of m_ϕ are bounded. We can consider the total potential ψ, defined in Lemma 3.6. The function m_ψ, constructed using ψ rather than ϕ, is identical to the function m_ψ because $\phi-\psi = T$ is independent of z. By eq.(3.37) the level surfaces of ψ coincide with the level surfaces of the functions ρ and p. Therefore the points which are associated with normal points within the support of the density will also lie within the support of the density. Thus, the range of m_ϕ must be bounded since the domain, which is the support of the density, is bounded by assumption. Since the range of m_ϕ is bounded, the z component of the range must also be bounded and therefore must have a least upper bound, say z_m.

We will now show that z_m is the z component of some element in the range of m_ϕ. In any case, there must be a sequence of numbers ζ_n each of which is the z component of some element of the range of m_ϕ, and $\lim \zeta_n = z_m$. There must also be a corresponding sequence of points ξ_n in the domain of m_ϕ whose images have ζ_n as z components: $m_\phi(\xi_n) = (x_n, y_n, \zeta_n)$. The domain of m_ϕ is compact, therefore, there is a subsequence ξ_n' of ξ_n which converges to a point in the domain, say $\lim \xi_n' = (x_o,y_o,z_o)$. It follows that $\lim m_\phi(\xi_n') = (x_o,y_o,z_m)$. The prime will henceforth be dropped from the name of the sequence of points ξ_n'. If m_ϕ were a continuous function, it would follow that $m_\phi(x_o,y_o,z_o) = (x_o,y_o,z_m)$ and the proof would be complete. m_ϕ is not necessarily continuous however.

Let us first consider the case where there is a subsequence ξ_n'' of ξ_n which are all special points. At each of these points we have $\partial\phi/\partial z(\xi_n'') = 0$; and since $\partial\phi/\partial z$ is a continuous function, $\partial\phi/\partial z(x_o,y_o,z_o) = 0$. For special points $m_\phi(x,y,z) = (x,y,z)$, therefore $\lim m_\phi(\xi_n'') = (x_o,y_o,z_o) = (x_o,y_o,z_m)$. Therefore (x_o,y_o,z_m) must be an element of the domain of m_ϕ with the property

$m_\phi(x_o,y_o,z_m) = (x_o,y_o,z_m)$. Thus we have shown that the lemma follows if there exists a subsequence ξ''_n of special points.

The other case we need to consider is when ξ_n are all normal points when n becomes sufficiently large. To each of the normal points ξ_n (with z component ω_n) there is an associated point $\bar{\xi}_n$ (with z component $\bar{\omega}_n$). We also know that $\lim \omega_n = z_o$ and $\lim 1/2(\omega_n + \bar{\omega}_n) = z_m$, thus $\lim \bar{\omega}_n = 2z_m - z_o$. There are three possibilities: $z_o = z_m$, $z_o > z_m$ and $z_o < z_m$. We will consider first the case where $z_o = z_m$. The chord connecting each pair of points ξ_n to $\bar{\xi}_n$ in our sequence must contain a point ξ''_n where $\partial\phi/\partial z(\xi''_n) = 0$. Thus, the sequence ξ''_n are all special points. Furthermore $\lim \xi''_n = \lim \xi_n = \lim \bar{\xi}_n = (x_o,y_o,z_o)$. Thus, we have a sequence of special points whose limit point is (x_o,y_o,z_o). We have shown above that the lemma follows in this case. We next consider the case where $z_o > z_m$; then (x_o,y_o,z_o) must be a normal point with associated point $(x_o,y_o,\bar{z}_o)$. It follows that $\bar{z}_o < 2z_m - z_o$ because z_m is the least upper bound. Since ϕ is a continuous function $\lim \phi(\xi_n) = \phi(x_o,y_o,z_o) = \lim \phi(\bar{\xi}_n) = \phi(x_o,y_o,2z_m - z_o)$. Therefore the point $(x_o,y_o,2z_m - z_o)$ must be the point associated with (x_o,y_o,z_o) and as a result $m_\phi(x_o,y_o,z_o) = (x_o,y_o,z_m)$ and the lemma follows. The last possibility is that $z_o < z_m$. In this case the sequence of associated points $\bar{\xi}_n$ must converge to $(x_o,y_o,2z_m - z_o)$ and $2z_m - z_o > z_m$. The same argument as the one given for the case $z_o > z_m$ shows that (x_o,y_o,z_o) is the point associated with $(x_o,y_o,2z_m - z_o)$. In this case $m_\phi(x_o,y_o,2z_m - z_o) = (x_o,y_o,z_m)$ and the lemma follows. ■

We can now derive a very important inequality for the odd part of the density function, when it is taken with respect to the plane $z = z_m$.

LEMMA 4.5 - *Let ρ be the mass density function of a barotropic ideal Newtonian fluid stellar model which has stratified flow. Then,*

$$\rho^-(x,y,z) \equiv 1/2\ \rho(x,y,z) - 1/2\ \rho(x,y,2z_m - z) \leq 0 \text{ for all } z \geq z_m.$$

PROOF: Consider a point (x,y,z) with $z > z_m$. If (x,y,z) is not in the support of ρ, then $\rho^-(x,y,z) = -1/2\ \rho(x,y,2z_m - z) \leq 0$. Next suppose that (x,y,z) is in the support of ρ. Since z_m is the least upper bound of the midpoints, (x,y,z) must be a normal point and the associated point $(x,y,\bar{z})$ must satisfy $\bar{z} \leq 2z_m - z \leq z$. Lemma 4.3 implies $\phi(x,y,2z_m - z) \geq \phi(x,y,z)$ so that $\phi^-(x,y,z) = 1/2\ \phi(x,y,z) - 1/2\ \phi(x,y,2z_m - z) \leq 0$. The total potential ψ, defined in Lemma 3.6 satisfies $\psi^- = \phi^-$, because T is independent of z; consequently $\psi^-(x,y,z) \leq 0$. From eq. (3.37) it follows that the level surfaces of ρ, p and ψ all coincide. This fact and the requirement that $\rho \geq 0$, $p \geq 0$ and $dp/d\rho \geq 0$ from eq. (2.9) imply that $\rho^-(x,y,z) \leq 0$ for all $z \geq z_m$. ■

4-3. Stars Have Mirror Symmetry

Having established these preliminary results, we are ready to proceed with the main theorem.

THEOREM 4.6 - *Consider a barotropic ideal Newtonian fluid stellar model which has stratified flow. There exists a plane $z = z_m$, such that the odd parts of the functions φ, ρ, p vanish when taken with respect to the plane $z = z_m$. Thus, the star must have a plane of mirror symmetry.*

PROOF: From Lemma 4.4 we know that there is a point (x_o,x_o,z_o) such that $m_\phi(x_o,y_o,z_o) = (x_o,y_o,z_m)$. We will consider two separate cases. In the first case (x_o,y_o,z_o) is assumed to be a normal point, in the second case it is assumed to be a special point.

CASE 1: Associated with the point (x_o,y_o,z_o) is the point $(x_o,y_o,\bar{z}_o)$ with $\bar{z}_o = 2z_m - z_o$. Since $\phi^-(x_o,y_o,z_o) = 1/2\ \phi(x_o,y_o,z_o) - 1/2\ \phi(x_o,y_o,\bar{z}_o) = 0$, there exists a point [either (x_o,y_o,z_o) or $(x_o,y_o,\bar{z}_o)$] say (x_o,y_o,z_o) with $z_o > z_m$ where ϕ^- vanishes. The function ϕ^- vanishes on the boundary of the half space $z > z_m$. In the interior of this region ϕ^- is bounded by assumption; therefore there must exist a point (x,y,z) in this half space where ϕ^- is maximal. The odd part of eq. (2.15) is given by $\nabla_i \nabla^i \phi^- = -4\pi\rho^-$. From Lemma 4.5 we have $\nabla_i \nabla^i \phi^- \geq 0$ for all $z > z_m$. This inequality, the existence of a point where ϕ^- is maximal and Theorem 4.2 guarantee that $\phi^- = 0$ everywhere. That $\rho^- = p^- = 0$ follows trivially.

The argument given above is not strictly correct for the case where the maximum of ϕ^- lies on the boundary of the star. The density ρ need not be continuous at the surface of the star, and consequently the potential ϕ need not be sufficiently differentiable there to apply Theorem 4.2 Consider now the case where the maximum of ϕ^-, $(\hat{x},\hat{y},\hat{z})$ lies on the boundary of the star. Find an open ball B which has $(\hat{x},\hat{y},\hat{z})$ as a point on its boundary, which is tangent to the surface of the star at $(\hat{x},\hat{y},\hat{z})$ and which is sufficiently small that all of the points of B lie in the exterior of the star. Within B, ϕ^- will be C^3, and ϕ^- is C^1 at $(\hat{x},\hat{y},\hat{z})$. Furthermore $\phi^- \leq \phi^-(\hat{x},\hat{y},\hat{z})$ at all points in B and $\nabla_i\phi^-(\hat{x},\hat{y},\hat{z}) = 0$, since ϕ^- is a maximum at $(\hat{x},\hat{y},\hat{z})$. From Theorem 4.1 it follows that ϕ^- has the constant value $\phi^-(\hat{x},\hat{y},\hat{z})$ everywhere in B and consequently everywhere. This constant value must be zero since ϕ^- vanishes on the boundary of the half space $z > z_m$.

CASE 2: We now consider the case where (x_o,y_o,z_o) is a special point. We have shown that $\phi^- \leq 0$ and $\rho^- \leq 0$ for all $z \geq z_m$. Similarly $\phi^- \geq 0$ and $\rho^- \geq 0$ for all $z \leq z_m$. It follows that there is a neighborhood U of the plane $z = z_m$ in which the following inequalities must hold: $\partial\phi^-/\partial z \leq 0$, $\partial\rho^-/\partial z \leq 0$. From eq. (2.15) it follows that $\nabla_i \nabla^i(\partial\phi^-/\partial z) = -4\pi\partial\rho^-/\partial z$, hence $\nabla_i \nabla^i(\partial\phi^-/\partial z) \geq 0$ in U. At a special point $\partial\phi/\partial z = 0 = \partial\phi^+/\partial z + \partial\phi^-/\partial z$, but at $z = z_m$, $\partial\phi^+/\partial z$ vanishes, therefore $\partial\phi^-/\partial z(x_0,y_o,z_m) = 0 \geq \partial\phi^-/\partial z$ for all points in U. By Theorem 4.2 it follows that $\partial\phi^-/\partial z = 0$ everywhere in U, and consequently $\phi^- = 0$ everywhere in U, and as a result $\phi^- = 0$ everywhere.

As in case 1) special consideration must be given to the case that (x_o,y_o,z_m) is on the boundary of the star. By assumption (see Chapter 2) we know that ϕ must be at least C^1 in the normal direction, and C^2 in the tangential direction at the surface of the star. Therefore Theorem 4.2 cannot be applied and Theorem 4.1 must be used. Since (x_o,y_o,z_m) is a special point, it follows that $\partial\phi/\partial z = \partial\psi/\partial z = 0$. Therefore $\partial/\partial z$ is a tangential derivative to the surface at this point. Thus, $\partial\phi^-/\partial z$ is C^1 at (x_o,y_o,z_m). We have argued that $\partial\phi^-/\partial z \leq 0$ in the set U. Thus $\partial\phi^-/\partial z$ will be a maximum at (x_o,y_o,z_m) so that $\nabla_i(\partial\phi^-/\partial z) = 0$ there also. Construct an open ball B which contains (x_o,y_o,z_m) as one of its boundary points, which is tangent to the surface of the star at (x_o,y_o,z_m), and which is sufficiently small that B lies completely within U and completely within the exterior of the star. Within B, $\nabla_i \nabla^i(\partial\phi^-/\partial z) = 0$

and $\partial\phi^-/\partial z$ is C^2. Thus by Theorem 4.1, $\partial\phi^-/\partial z = 0$ in B, and therefore $\phi^- = 0$ in B (the plane $z = z_m$ intersects the center of B). It follows that $\phi^- = 0$ everywhere since it vanishes at an interior point of the half space $z > z_m$. ∎

We note that Theorem 4.6 is in a sense incomplete as a mirror plane theorem. We have shown that the functions ρ, p and ϕ must all have mirror symmetry. However, it appears that no simple analogous result exists for the velocity field of the fluid, v^i. For example, consider a stationary axisymmetric star with azimuthal velocity field. An infinite number of related stellar models may be constructed by keeping the functions ρ, p and Φ fixed while defining a new velocity field $v'^i = hv^i$, where h is an arbitrary function which is independent of aximuthal angle and $h^2 = 1$. Note that h may be discontinuous, so that parts of the fluid may rotate one direction while other parts rotate the other way. These related stellar models need not have simple mirror symmetry in the velocity field. A final point to note is that the assumption that the velocity field is stratified, is only used to prove Lemma 3.6. This assumption could be replaced by the weaker (but physically less transparent) assumption $0 = a_z = \partial v_z/\partial t + v^j \nabla_j v_z$.

4.4. Stars Are Z-Convex

We next consider an interesting corollary of the mirror plane theorem.

THEOREM 4.7. - *A barotropic ideal Newtonian fluid stellar model which has stratified flow (having vanishing z component of velcotiy) must be z-convex.*

PROOF: From Theorem 4.6 it follows that there is a plane $z = z_m$ which is a plane of mirror symmetry of the stellar model. We first argue that this must be the only plane z = constant which is a plane of mirror symmetry. If there were another plane of mirror symmetry $z = z_{m'}$, then clearly each of the planes $z = z_m + N(z_{m'} - z_m)$ for arbitrary integer N must also be a plane of mirror symmetry. But in this case, the density function could not have bounded support, contrary to assumption.

Consider two points (x,y,z_1) and (x,y,z_2) which are contained within the support of the density function (take $|z_1-z_m| \geq |z_2-z_m|$). If the point $m_\phi(x,y,z_1)$ does not lie in the plane $z = z_m$, then Theorem 4.6 proves the existence of an additional plane of mirror symmetry. Since this is forbidden, it follows that $\phi(x,y,z_1) = \phi(x,y,2z_m-z_1)$ and $\phi(x,y,z_1) < \phi(x,y,z)$ for all z between z_1 and $2z_m-z_1$, hence for all points with $|z_1-z_m| > |z-z_m|$. It also follows (using Lemma 3.6) that $\psi(x,y,z_1) < \psi(x,y,z)$ for all z with $|z_1-z_m| > |z-z_m|$. By using equation (3.37) it follows that $\rho(x,y,z_1) < \rho(x,y,z)$. Therefore all points having z satisfying $|z_1-z_m| > |z-z_m|$ are contained within the support of the density function. In particular all points between (x,y,z_1) and (x,y,z_2) are contained within the support of ρ; thus, the stellar model is z-convex. ■

4.5. Static Stars Are Spherical

The final result presented in this chapter, is also possibly the most well known of the fundamental properties of stellar models:

THEOREM 4.8 - *Static Newtonian fluid stellar models must be spherically symmetric.*

PROOF: For a static stellar model (see eq. 4.1) the Euler's equation (2.16) reduces to the simple form

$$\nabla_i p = \rho \nabla_i \phi. \tag{4.4}$$

Consequently the fluid must be barotropic. Also, since the fluid is static, it follows that the fluid velocity is stratified with respect to any chosen direction. Therefore from Theorem 4.6 it follows that for any choice of direction the stellar model has a plane of mirror symmetry orthogonal to that direction. Consider the three mirror planes which are orthogonal to the x, y and z coordinate axes. Choose the coordinates so that these planes are the planes $x = 0$, $y = 0$ and $z = 0$ respectively. Now consider any one of the mirror planes, say M. Let (n_x, n_y, n_z) denote a vector which is orthogonal to the plane M. Upon relecting the plane M through the mirror plane z=0, we will obtain another mirror plane M' with orthogonal vector $(n_x, n_y, -n_z)$. In the same way a mirror plane M" can be obtained by reflecting M' through the plane y=0, and then again through the plane x=0. The normal vector to M" will be $(-n_x, -n_y, -n_z)$. Hence M" is parallel to the plane M. As we showed in the proof of Theorem 4.7 two distinct parallel mirror planes are not possible, hence M=M". Consequently M (and therefore every mirror plane) must contain the point (0,0,0). Consider two points (x,y,z) and (x',y',z') which lie on the surface of a sphere centered at (0,0,0). Consider the plane, N, which bisects the chord which joins (x,y,z) and (x',z',z'). Since these points lie on the surface of a sphere centered at (0,0,0) it follows that N must intersect (0,0,0) also. Therefore N must be a plane of mirror symmetry so that $\phi(x,y,z) = \phi(x',y',z')$. Thus the gravitational potential is constant on spheres centered at (0,0,0), so the stellar model must be spherical.

■

§5 NEGATIVE RESULTS

5-1. Introduction and Summary

In the days of an era gone by (before the availability of large computers) a great deal of effort was spent in attempts to obtain exact analytic solutions to the equations for rotating Newtonian stars. Some of the more notable solutions which resulted from these efforts bear the names of their discoverers: Dedekind, Jacobi, Maclaurin and Riemann (see Chandrasekhar 1969). In the course of these efforts, several results of a very general nature were derived which showed that certain classes of stellar models could not exist. In this sense, these results were negative ones.

In this chapter we will recall those negative results, and reproduce their proofs here. We derive a generalization of Poincare's result which shows that no star can rotate faster than a certain limit (Theorem 5.1). We present another result which is the culmination of the work of Volterra, Wavre and Dive (see Dive 1952). Their analysis shows that the level surfaces of the density function cannot take a certain (fairly general) form, unless the density is uniform or the star is spherical (Theorems 5.2 and 5.3). These theorems show that no stellar model exists which has ellipsoidal density stratifications, unless the model is Maclaurin's constant density spheroid.

Finally, we show that there are no bounded stellar models composed of pressureless fluid (dust). This result (Theorem 5.4) is of interest because of recent speculations (see Bonner 1977) that general relativistic pressureless fluid stellar models might exist. The only negative result of this sort which we neglect here is von Zeipel's theorem that radiative equilibrium is incompatible with a barotropic fluid in purely rotational motion. We neglect this result because it requires a more complicated thermodynamics than that developed in §2. (We have not included the possibilities of energy sources or radiation flux out of the fluid.) See Eddington (1926) or Clayton (1968) for a discussion of von Zeipel's theorem.

5-2. Poincare's Limit

The first result of a negative type which we present here is Poincare's limit on the angular velocity of a star. Poincare's version of this theorem was for the case of rigid rotation (see Poincare 1903 or Lamb 1932, p. 597). We present here a slight generalization of this result to the case of an arbitrary ideal fluid.

THEOREM 5.1 - *The velocity of an ideal Newtonian fluid stellar model must be sufficiently small so that*

$$4\pi M \geq - \int \nabla_i a^i \, d^3x \quad , \tag{5.1}$$

where M is the total mass of the star. In the case of a star which is rotating rigidly with angular velocity Ω*, this limit can be written as*

$$2\pi \bar{\rho} \geq \Omega^2 , \qquad (5.2)$$

where $\bar{\rho}$ *is the average density of the star.*

PROOF: In the interior of an ideal Newtonian stellar model, the following equation must be satisfied (see eqs. 2.15 and 2.19):

$$\nabla^i(\rho^{-1}\nabla_i p) = -4\pi\rho - \nabla_i a^i . \qquad (5.3)$$

We integrate eq. (5.3) over the support of the pressure function. The left hand side is a divergence, consequently its integral reduces to an integral over the surface of the star.

$$\oint \rho^{-1} \nabla^i p \, d^2 s_i = -4\pi M - \int \nabla_i a^i \, d^3x \qquad (5.4)$$

The integrand on the left hand side of eq. (5.4) is proportional to the outward normal derivative of the pressure at the surface of the star. The pressure is positive within the star and zero on the surface, consequently the outward normal derivative is non-positive. It follows that

$$0 \geq -4\pi M - \int \nabla_i a^i \, d^3x \qquad (5.5)$$

For the special case of rigid rotation, the acceleration of the fluid is given by

$$a^i = -1/2 \quad \Omega^2 \nabla^i (x^2 + y^2). \tag{5.6}$$

In this case eq. (5.5) becomes

$$0 \geq -4\pi M + 2\Omega^2 \int d^3x \tag{5.7}$$

Eq. (5.7) is equivalent to eq. (5.2) when the average density $\bar{\rho}$ is defined in the usual way. ■

Poincare's limit on the angular velocity of a stellar model is not a very stringent one. In fact far more restrictive conditions have been derived (see Roxburgh 1969, p. 13; and Chandrasekhar 1969, Chapter 5). These more stringent conditions are based on the stability of the stellar models, rather than being conditions on the existence of equilibrium models. Appendix II describes the considerably more restrictive condition which applies to the Maclaurin spheroids when the effects of viscosity and gravitational radiation are taken into account. However, since the analysis of the stability of stellar models is beyond the scope of this work, we will not discuss the other conditions here.

5-3. Ellipsoidal Stratification Theorem

The following theorem shows that the density function of a stellar model cannot have level surfaces of a certain, fairly general form, unless the density is in fact uniform. These level surfaces include all cases of ellipsoidal stratification except the case where the ellipsoids all have the same eccentricity. Since we feel that rotating stars should look qualitatively like oblate spheroids, this theorem goes a long way in eliminating simple possible shapes.

THEOREM 5.2 - *Consider a stationary axisymmetric barotropic ideal Newtonian fluid stellar model whose velocity field is purely rotational. If the level surfaces of the density function coincide with the level surfaces of* β*, defined implicetly by*

$$\beta = z^2 + \sum_{k=1}^{n} \xi_k (x^2+y^2)^k \tag{5.8}$$

where ξ_k *is an arbitrary function of* β *[with* $\mu_n \neq 0$; $\mu_k = -\partial\xi_k/\partial\beta$*], then the density is uniform throughout the star.*

PROOF: The simplest case of this theorem is that of elliposidal stratification (n=1). We will work out completely the proof in this case, and then sketch more briefly the proof for general n. We set $\xi_1 = \xi$ and $\mu_1 = \mu$.

Euler's equation for this type of stellar model is given by eq. (3.33). We take the divergence of that expression to find

$$\nabla_i(\rho^{-1}\nabla^i p) = -4\pi\rho + 1/2\ \nabla_i[\Omega^2\ \nabla^i\ r^2], \tag{5.9}$$

where $r^2 = x^2+y^2$. Theorem 3.7 proves that $\Omega = \Omega\ (r^2)$. Consequently the term $\nabla_i[\Omega^2\ \nabla^i\ r^2]$ is only a function of r^2; we write

$$\zeta(r^2) = 1/2\ \nabla_i[\Omega^2\ \nabla^i\ r^2]. \tag{5.10}$$

To establish the theorem, we will assume that the level surfaces of the density function coincide with the surfaces of ellipsoids, and then we will show that the density must be uniform. Consider the family of ellipsoids defined by eq. (5.8) with n=1:

$$\beta = z^2 + \xi\ r^2 \tag{5.11}$$

where $\sqrt{\beta}$ is the semi-minor axis of the ellipsoid and $\xi = \xi(\beta)$ is the oblateness parameter. We assume that the density is constant on the surfaces of these ellipsoids, therefore we have

$$\rho = \rho(\beta) . \qquad (5.12)$$

We will now consider the left hand side of eq. (5.9). Using the fact that the fluid is barotropic, and eq. (5.12) we find that

$$\nabla_i(\rho^{-1}\nabla^i p) = W'(\beta)\nabla_i \nabla^i\beta + W''(\beta) \nabla_i\beta \nabla^i\beta , \qquad (5.13)$$

where the two functions of β, W' and W'' are defined by

$$W' = \rho^{-1} \partial_\beta p, \text{ and} \qquad (5.14)$$

$$W'' = \partial_\beta(\rho^{-1}\partial_\beta p). \qquad (5.15)$$

We now wish to compute the expressions $\nabla_i \nabla^i\beta$ and $\nabla_i\beta \nabla^i\beta$ in a system of coordinates using $\alpha = r^2$ and β as coordinates rather than the standard cylindrical coordinates r and z. These coordinate differentials are related by

$$\partial_r = 2\sqrt{\alpha}\, \partial_\alpha + 2\xi\sqrt{\alpha}\, (1+\alpha\mu)^{-1}\partial_\beta , \text{ and} \qquad (5.16)$$

$$\partial_z = 2(\beta-\alpha\xi)^{1/2}(1+\alpha\mu)^{-1} \partial_\beta . \qquad (5.17)$$

It follows that

$$\nabla_i\beta \nabla^i\beta = 4(\beta - \alpha\xi + \alpha\xi^2)(1 + \alpha\mu)^{-2} , \text{ and} \qquad (5.18)$$

$$\nabla^i\nabla_i\beta = 2(1 + \alpha\mu)^{-1} + 4\xi(1 - \alpha\mu)(1 + \alpha\mu)^{-2}$$
$$-4 \alpha \frac{d\mu}{d\beta} (\beta - \alpha\xi + \alpha\xi^2)(1 + \alpha\mu)^{-3} . \qquad (5.19)$$

We note that the right hand sides of eq. (5.18) and (5.19) are ratios of polynomials in α. Equation (5.9) can now be rewritten in the following way:

$$\zeta(\alpha) = W'(\beta)\nabla_i\nabla^i\beta + W''(\beta)\nabla_i\beta\ \nabla^i\beta + 4\pi\rho(\beta)\ . \tag{5.20}$$

Therefore $\zeta(\alpha)$ is a rational fraction in α. Since ζ is independent of β, it follows that the coefficients in the polynomials which make up ζ must also be independent of β. The polynomial in the denominator of ζ is precisely $(1+\mu\alpha)^3$, therefore we conclude that μ is independent of β; and hence it is a constant. The polynomial which makes up the numerator of ζ is given by:

$$(1+\mu\alpha)^3\ \zeta(\alpha) = 4\pi\rho(1+\mu\alpha)^3 + 2W'(1+\mu\alpha)^2 + 4W'\xi(1-\alpha^2\mu^2) + 4W''(\beta-\alpha\xi+\alpha\xi^2)(1+\alpha\mu). \tag{5.21}$$

In the case that $\mu \neq 0$ (so that the ellipsoids do not all have the same eccentricity) eq. (5.21) is a cubic polynomial. The coefficient of α^3 in this case is $4\pi\rho\mu^3$. It follows that ρ is independent of β, and therefore is a constant.

We return now to the more general case of a stratification defined by eq. (5.8). We compute $\nabla_i(\rho^{-1}\nabla^i p)$ according to eq. (5.13) where the relevant derivatives of β are now given by

$$\nabla_i\beta\ \nabla^i\beta = 4\{\beta-\xi_k\alpha^k + \alpha(k\xi_k\ \alpha^{k-1})^2\}/(1+\mu_k\alpha^k)^2\quad , \text{ and} \tag{5.22}$$

$$\nabla_i \nabla^i \beta = 2(1+2k^2 \xi_k \alpha^{k-1})/(1+\mu_k \alpha^k)$$

$$- 8(k\xi_k \alpha^{k-1})(k\mu_k \alpha^k)/(1+\mu_k \alpha^k)^2$$

$$- 4(\alpha^k \partial_\beta \mu_k)\{\beta - \xi_k \alpha^k + \alpha(k\xi_k \alpha^{k-1})^2\}/(1+\mu_k \alpha^k)^3 \quad , \qquad (5.23)$$

where k is summed from 1 to n within the first set of parenthesis. When these expressions are used in the right hand side of eq. (5.20) we again find a rational fraction in α. The polynomial in the denominator is $(1+\mu_k \alpha^k)^3$. Therefore the function μ_k must be independent of β. If $\mu_n \neq 0$ (as assumed) then the coefficient of the largest power of α in the numerator is $4\pi\rho(\mu_n)^3$. This coefficient must be independent of β, therefore ρ must be a constant. ■

Theorem 5.2 ruled out a large class of possible density stratifications for rotating barotropic stellar models. This theorem did not, however, rule out the possibility of stratification on concentric self-similar ellipsoids. The next theorem rules out this possibility, along with another large class of possible density stratifications. The proof of the n = 1 case of this theorem is given by Dive (1952) while the more general case is new.

THEOREM 5.3– *Consider a stationary axisymmetric barotropic ideal fluid stellar model whose velocity field is purely rotational. If the level surfaces of the density function coincide with the level surfaces of* β*, defined by*

$$\beta = z^2 + \sum_{k=1}^{n} \xi_k (x^2 + y^2)^k \qquad (5.24)$$

where ξ_k *are constants and for* $n = 1$, $0 < \xi_1 \leq 1$, *then the density is uniform throughout the star or the star is static and spherical.*

PROOF: When ξ_k are constants, the functions $\mu_k = \partial_\beta \xi_k$ vanish. In this case eqs. (5.22) and (5.23) simplify to the forms:

$$\nabla_i \beta \nabla^i \beta \;=\; 4\{\beta - \xi_k \alpha^k + \alpha(k\xi_k \alpha^{k-1})^2\}, \quad \text{and} \tag{5.25}$$

$$\nabla^i \nabla_i \beta \;=\; 2 + 4k^2 \xi_k \alpha^{k-1}. \tag{5.26}$$

Therefore, the function $\zeta(\alpha)$ from eq. (5.20) is given by:

$$\zeta(\alpha) \;=\; 4W''\beta + 2W' + 4\pi\rho - 4W''\xi_k \alpha^k + 4W'k^2 \xi_k \alpha^{k-1} + 4W''k^2 \xi_k^2 \alpha^{2k-1}. \tag{5.27}$$

The right hand side of eq. (5.27) is a polynomial in α, while the left hand side depends only on α. Therefore the coefficients in the polynomial must be constants. We consider the two cases $n = 1$ and $n > 1$ separately.

If $n > 1$, it follows that W'' is a constant, by requiring the coefficient of the highest power of α on the right hand side of eq. (5.27) to be a constant. It then follows that W' must be constant also by requiring the coefficients of the other powers of α to be constants. Therefore, W'' vanishes. Finally, by requiring the coefficient of α^o to be constant, it follows that the density, ρ, must be constant.

For the case $n = 1$, the argument is not as simple. In this case non-constant density solutions to the fluid equations exist, however, it is not possible to match on a properly behaved external gravitational field. We set $\xi = \xi_1$ in eq. (5.27) for the case $n = 1$ to find:

$$\zeta(\alpha) = [4\pi\rho + 2W'(1+2\xi) + 4W''\beta] + \alpha[4W''\xi(\xi-1)]. \tag{5.28}$$

The right hand side of eq. (5.28) is a linear equation in α, therefore each of the coefficients must be a constant. We take the constants K_1 and K_2:

$$K_1 = W'', \quad \text{and} \tag{5.29}$$

$$K_2 = 4\pi\rho + 2W'(1+2\xi) + 4W''\beta \ . \tag{5.30}$$

Equation (5.29) can be integrated to give W':

$$W' = K_1 \beta + K_3. \tag{5.31}$$

The density function ρ can be determined now by substituting eq. (5.31) and (5.29) into (5.30). Furthermore, the pressure p can be found by integrating eq. (5.31). The angular velocity function Ω can be determined by integration of eq. (5.28), when one recalls eq. (5.10). Finally, the gravitational potential is determined by integrating Euler's eq. (3.42). The resulting solution, in terms of the six constants K_1, K_2, K_3, K_4, K_5 and ξ is given as follows

$$4\pi\rho = - 2K_1(2\xi+3)\beta + K_2 - 2K_3(2\xi+1) \ , \tag{5.32}$$

$$4\pi p = - \frac{2}{3} K_1^2(2\xi+3)\beta^3 + \frac{1}{2} K_1[K_2 - 8K_3(\xi+1)]\beta^2 + K_3[K_2-2K_3(2\xi+1)]\beta + K_4 \ , \tag{5.33}$$

$$\Omega^2 = K_1\xi(\xi-1)\alpha + \frac{1}{2} K_2 \ , \tag{5.34}$$

$$\phi = \frac{1}{2} K_1 \beta^2 + K_3\beta - \frac{1}{4} K_1\xi(\xi-1)\alpha^2 - \frac{1}{4} K_2\alpha + K_5 \ . \tag{5.35}$$

We recall that the coordinates α and β are related to Cartesian coordinates by

$$\beta = z^2 + \xi(x^2+y^2) \ , \text{ and} \tag{5.36}$$

$$\alpha = x^2 + y^2 \ . \tag{5.37}$$

Equations (5.32)- (5.35) represent valid solutions to the fluid equations. To represent valid interiors of stellar models, however, the gravitational potential must be matched onto an exterior solution which has no singularities, and which falls off to zero at infinity. The gravitational potential will have the correct form only if it is the one produced by the asymptotically vanishing Green's function:

$$\phi = \int \frac{\rho(x')d^3x'}{|x - x'|} \quad . \tag{5.38}$$

Therefore, we must check the potential given in eq. (5.35) to see if it agrees with the potential given by eq. (5.38). The integral in eq. (5.38) is very difficult to perform. However, for the type of ellipsoidal stratification given in eq. (5.36) the integral can be transformed to a more managable form:

$$\phi(\alpha,\beta) = -\pi\beta_o^{3/2} \int_o^\infty \frac{\psi(u,\alpha,\beta)\ du}{(\beta_o+\xi u)(\beta_o+u)^{1/2}} \tag{5.39}$$

where β_o is the semi-minor axis of the surface of the star,

$$\psi(u,\alpha,\beta) = \int_1^{f(u,\alpha,\beta)} \rho(\beta')\ d\beta', \quad \text{and} \tag{5.40}$$

$$f(u,\alpha,\beta) = (\beta - \xi\alpha)/(\beta_o + u) \quad + \quad \xi\alpha/(\beta_o + \xi u). \tag{5.41}$$

The transformation of eq. (5.38) into this form is a lengthy derivation which may be found in Chandrasekhar (1969) p. 52.

The integrals in eqs. (5.39) and (5.40) can be explicetly evaluated when eq. (5.32) is used for the density of the star. The resulting expression has the form:

$$\phi(\alpha,\beta) = \Gamma_1\alpha^2 + \Gamma_2\alpha + \Gamma_3\beta^2 + \Gamma_4\beta + \Gamma_5\alpha\beta + \Gamma_6 . \tag{5.42}$$

The coefficients $\Gamma_1 - \Gamma_6$ are certain complicated functions of ξ, β_o, K_1, K_2, and K_3. Comparing the expression for ϕ given in eq. (5.35) with this expression, we see that the interior models will admit asymptotically regular gravitational potentials only if Γ_5 vanishes. The precise form of Γ_5 is computed from eqs. (5.39) and (5.40):

$$\Gamma_5 = \frac{1}{2}K_1\xi(2\xi+3)\beta_o^{3/2}\int_o^\infty (\beta_o+\xi u)^{-2}(\beta_o+u)^{-3/2} \times \times\{1 - (\beta_o+\xi u)(\beta_o+u)^{-1}\}\, du. \quad (5.43)$$

This integral vanishes only for $\xi = 1$: spherical stars. In this case it follows that the star must be static also. The only other way for Γ_5 to vanish is for K_1 to be zero. In this case the density must be constant from eq. (5.32). ■

5-4. No Dust Stars

Recent calculations indicate that there exists a gravitational spin-spin repulsion force in general relativity theory (e.g. Wald 1972). This has lead to speculation that perhaps a rapidly rotating star could counteract completely the attractive aspect of gravity via the spin-spin repulsion. Hence, it is conjectured that stationary axisymmetric pressureless fluid (dust) stellar models might exist within general relativity theory (see Bonner 1977). In this section we will reproduce Bonner's (1977) argument that no such Newtonian model can exist.

THEOREM 5.4 - *A stationary axisymmetric Newtonian fluid stellar model, whose velocity is purely rotational, cannot exist if the fluid is pressureless.*

PROOF: When the pressure of the fluid vanishes, Euler's equation becomes

$$-\frac{1}{2}\ \Omega^2\nabla_i(x^2+y^2) = \nabla_i\phi. \qquad (5.44)$$

The z component of eq. (5.44) reduces to $\partial_z\phi = 0$. Therefore the gravitational potential is independent of z, at least within the fluid. In the exterior of the star we have

$$\nabla_i\nabla^i(\partial_z\phi) = 0\ , \qquad (5.45)$$

from the Newtonian law of gravitation. The function $\partial_z\phi$ vanishes on both the surface of the fluid, and at infinity. Therefore, from eq. (5.45) it vanishes everywhere in the exterior of the fluid. Consequently $\partial_z\phi$ is zero everywhere so that $\partial_z\rho$ vanishes everywhere. Therefore the fluid does not have bounded support, and cannot be a stellar model by definition (see §2). ■

PART II.

GENERAL RELATIVISTIC STELLAR MODELS

§6 DESCRIPTION OF THE RELATIVISTIC MODELS

6.1 Introduction to Part II

The second portion of this dissertation discusses the properties of equilibrium stellar models within the general theory of relativity. Our approach to this problem will be similar to our approach to the study of Newtonian stellar models. We will review in great detail those results which may be deduced about the properties of equilibrium stellar models, without making a large number of unphysical assumptions. And, we will neglect a vast body of important work which deals with other aspects of relativistic stellar structure. We will not discuss perturbations of stationary stellar models. We will not discuss the many results which have been derived for spherical models. Nor will we discuss the important work on the equation of state of cold catalyzed matter or the work on the maximum mass of neutron stars. Instead we will concentrate on trying to answer such fundamental questions as, "Are non-rotating stellar models necessarily spherical?", and "What are the topology and symmetries of equilibrium stellar models?".

Chapter 6 reviews the general relativistic theory of fluid mechanics on which the theory of relativistic stellar models is based. We also discuss the boundary conditions on the fluid and gravitational fields which distinguish the solutions to the equations which we call stellar models, from other possible solutions.

Chapter 7 discusses two different notions of equilibrium for relativistic stellar models. We define stationary spacetimes by the presence of a timelike Killing vector field. We prove many useful identities which must be satisfied by any Killing vector field. A notion of thermodynamic equilibrium is defined for relativistic fluids, and the relationships between the notions of stationarity and thermodynamic equilibrium are derived.

Chapter 8 derives some useful technical properties of stationary stellar models. We demonstrate that if the stellar model is assumed to have certain minimal smoothness properties, then in fact the model must be analytic when written in appropriate coordinates. This technical result is used in our proofs in later chapters of this work.

Chapter 9 discusses the properties of static stellar models. Lichnerowicz's theorem is presented, which demonstrates the equivalence of the material and the metric staticity conditions. The progress which has been made on the proof of the conjecture that static stellar models must be spherical is reviewed.

Chapter 10 presents the proof that stationary viscous fluid stellar models must be axisymmetric.

Chapter 11 reviews the fundamental properties of stationary axisymmetric stellar models. We present a number of useful identities which involve two independent Killing vector fields. We present the proof of the equivalence of the orthogonal transitivity of the spacetime and the convection-free nature of the matter. We discuss two different ways of explicitly writing out Einstein's equations for stationary axisymmetric stellar models. And, we prove a number of properties which these models must possess including the relativistic versions of Poincare's limit on the angular velocity of a star and the rotation on cylinders theorem.

6.2 General Relativistic Fluid Mechanics

The thermodynamic properties of a fluid are local properties. Consequently the general relativistic treatment of thermodynamics is essentially identical to the Newtonian treatment, whenever quantities are measured with respect to the co-moving frame of the fluid. The following functions will represent the thermodynamic properties of the fluid as measured by a co-moving observer:

ρ	. . .	mass density,
ε	. . .	internal energy density,
p	. . .	pressure,
s	. . .	entropy density,
T	. . .	temperature,
n	. . .	particle number density,
μ	. . .	chemical potential,
η, ζ	. . .	coefficients of viscosity,
κ	. . .	coefficient of heat conduction,
q^{α}	. . .	heat flow vector, and
$T^{\alpha\beta}$	. . .	the stress energy tensor.

The motion of the fluid is described by the functions,

u^{α}	. . .	four-velocity of the fluid,
a^{α}	. . .	acceleration,
θ	. . .	expansion,
$\sigma^{\alpha\beta}$	. . .	shear tensor,
$\omega^{\alpha\beta}$	. . .	rotation tensor, and
$P^{\alpha\beta}$	. . .	the projection tensor.

The gravitational field in the general theory of relativity is represented by the geometry of the spacetime. The geometry is described by the following functions:

$g_{\alpha\beta}$ . . . metric tensor,

$\Gamma^{\alpha}{}_{\beta\gamma}$ . . . Christoffel symbol,

$R^{\alpha}{}_{\beta\mu\nu}$ . . . Riemann tensor,

$R_{\alpha\beta}$ . . . Ricci tensor, and

R . . . the scalar curvature.

These functions will be related to each other by the laws of thermodynamics, the fluid equations of motion, the general relativistic law of gravitation, and the equations of differential geometry.

The equation of state of the fluid will be represented by specifying ε, η, ζ and κ as given functions of the particle number density and the entropy density:

$$\varepsilon = \varepsilon(n,s), \quad \eta = \eta(n,s), \quad \zeta = \zeta(n,s) \quad \text{and} \quad \kappa = \kappa(n,s) \tag{6.1}$$

We require that these functions be C^2, and that they all be non-negative,

$$\varepsilon \geq 0, \quad \eta \geq 0, \quad \zeta \geq 0 \quad \text{and} \quad \kappa \geq 0. \tag{6.2}$$

The remaining thermodynamic quantities are defined as follows:

$$T = \partial\varepsilon/\partial s, \tag{6.3}$$

$$\mu = \partial\varepsilon/\partial n, \tag{6.4}$$

$$p = Ts + \mu n - \varepsilon, \tag{6.5}$$

$$\rho = \varepsilon, \tag{6.6}$$

$$q^{\alpha} = -\kappa P^{\alpha\beta}(\nabla_{\beta}T + Ta_{\beta}), \tag{6.7}$$

$$T^{\alpha\beta} = \rho u^{\alpha}u^{\beta} + (p-\zeta\theta)P^{\alpha\beta} - \eta\sigma^{\alpha\beta} + q^{\alpha}u^{\beta} + q^{\beta}u^{\alpha}. \tag{6.8}$$

The major differences between the Newtonian and the relativistic theories of thermodynamics come in eqs. (6.6) and (6.7). In the relativistic theory we equate the mass density with the internal energy density of the fluid, while the Newtonian theory identifies the mass density with the particle number density. We also point out the additional term in the heat flow equation (see Eckart 1940). This extra term takes into account the gravitational redshifts which occurr whenever energy is transported through a gravitational field.

As in the Newtonian thermodynamics, we require the number density, entropy density, pressure and the temperature to be non-negative functions:

$$n \geq 0, \quad s \geq 0, \quad p \geq 0 \quad \text{and} \quad T \geq 0. \tag{6.9}$$

We require the third law: if T = 0 then s = 0. And we require the stability conditions:

$$\partial p/\partial n > 0, \quad \partial T/\partial s > 0 \quad \text{and} \quad \partial \mu/\partial n > 0. \tag{6.10}$$

The motion of the relativistic fluid is described by the vector field u^α. The integral curves of u^α are the world lines of the particles in the fluid. We normalize u^α to unit length:

$$u^\alpha u_\alpha = -1. \tag{6.11}$$

In terms of u^α and the metric $g_{\alpha\beta}$, the other properties of the fluid motion are defined by:

$$a^\alpha = u^\beta \nabla_\beta u^\alpha, \tag{6.12}$$

$$\theta = \nabla^\alpha u_\alpha, \tag{6.13}$$

$$\sigma^{\alpha\beta} = P^{\alpha\mu}P^{\beta\nu}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) - (2/3)P^{\alpha\beta}\theta, \tag{6.14}$$

$$\omega^{\alpha\beta} = P^{\alpha\mu}P^{\beta\nu}(\nabla_\mu u_\nu - \nabla_\nu u_\mu), \tag{6.15}$$

$$P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta. \tag{6.16}$$

We note that these functions provide a useful decomposition of the gradient of u^α:

$$\nabla_\alpha u_\beta = -u_\alpha a_\beta + \frac{1}{2}\sigma_{\alpha\beta} + \frac{1}{2}\omega_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta}. \tag{6.17}$$

Furthermore, we see that the thermodynamic properties, and the motion of the fluid are completely determined by the functions n, s, u^α and $g_{\alpha\beta}$.

The geometry of a general relativistic spacetime is determined completely by the metric tensor $g_{\alpha\beta}$. The connection coefficients and the curvature tensors are related to $g_{\alpha\beta}$ by the formulae:

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma} - \partial_\mu g_{\beta\gamma}), \tag{6.18}$$

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\sigma\mu}\Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\beta\mu}, \tag{6.19}$$

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \tag{6.20}$$

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{6.21}$$

We use the notation ∂ for the partial derivative and ∇ for the covariant derivative. Our sign conventions are chosen so that the derivatives satisfy the relationships:

$$\nabla_\alpha u^\beta = \partial_\alpha u^\beta + \Gamma^\beta{}_{\alpha\mu}u^\mu, \tag{6.22}$$

$$\nabla_\alpha \nabla_\beta u^\mu - \nabla_\beta \nabla_\alpha u^\mu = R^\mu{}_{\nu\alpha\beta}u^\nu. \tag{6.23}$$

We also point out that the curvature tensors satisfy the following Bianchi identity,

$$\nabla_\alpha [R^\alpha{}_\beta - \frac{1}{2} \delta^\alpha{}_\beta R] = 0. \tag{6.24}$$

The equations of motion for the functions n, s, u^α and $g_{\alpha\beta}$ (which completely determine the state of the stellar model) are given by the conservation of particle number

$$\nabla_\alpha (n u^\alpha) = 0, \tag{6.25}$$

and Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{6.26}$$

The equations which correspond to Euler's equation and the entropy production equation of Newtonian hydrodynamics are implied by eq. (6.26). These equations can be derived by taking the divergence of eq. (6.26). The left hand side vanishes by the Bianchi identity eq. (6.24), which implies

$$\nabla_\alpha T^{\alpha\beta} = 0. \tag{6.27}$$

We project this equation parallel and orthogonal to u^α to obtain the expressions:

$$\nabla_\alpha (\rho u^\alpha + q^\alpha) = -p\theta + \frac{1}{2}\eta\sigma^{\alpha\beta}\sigma_{\alpha\beta} + \zeta\theta^2 - a_\alpha q^\alpha, \tag{6.28}$$

$$(\rho + p) a^\alpha = -P^{\alpha\beta}\nabla_\beta p + P^\alpha{}_\gamma \nabla_\beta \{\zeta\theta P^{\beta\gamma} + \eta\sigma^{\beta\gamma} - q^\beta u^\gamma - q^\gamma u^\beta\}. \tag{6.29}$$

These equations correspond to the Newtonian energy conservation law and Euler's equation respectively. Equation(6.28) can be transformed into the relativistic version of the entropy production equation by using eqs. (6.3)-(6.5) and (6.25). The resulting equation is given by

$$T\nabla_\alpha(su^\alpha + q^\alpha/T) = q^\alpha q_\alpha/\kappa T + \tfrac{1}{2}\eta\sigma^{\alpha\beta}\sigma_{\alpha\beta} + \zeta\theta^2. \qquad (6.30)$$

The equations of motion for the gravitational degrees of freedom are also contained in eq. (6.26).

Solutions to the system of fluid equations defined above will be called *viscous heat-conducting relativistic fluids*. Certain special cases of these equations will also be of interest. Whenever the coefficients of viscosity and heat-conduction vanish, the solutions to the fluid equations will be called *ideal relativistic fluids*. The fluid equations for this case take a particularly simple form. The stress energy tensor becomes

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + pP^{\alpha\beta}. \qquad (6.31)$$

Euler's equation and the entropy production equation simplify to

$$(\rho + p)a^\alpha = -P^{\alpha\beta}\nabla_\beta p, \text{ and} \qquad (6.32)$$

$$T\nabla_\alpha(su^\alpha) = 0. \qquad (6.33)$$

Another interesting special case occurrs when the level surfaces of the density coincide with the level surfaces of the pressure:

$$\nabla_\alpha p\nabla_\beta\rho - \nabla_\alpha\rho\nabla_\beta p = 0. \qquad (6.34)$$

When eq. (6.34) is satisfied the fluid will be called *barotropic*. We note that this definition of a barotropic fluid is formally the same as the definition used for a Newtonian fluid. In fact, however, the concepts are very different from a thermodynamic point of view, since the density is identified with the internal energy of the fluid in one case, and with the particle number density in the other case.

The fluid mechanics which we describe here is also discussed by Weinberg (1972) p. 53ff, Misner, Thorne and Wheeler (1972) §22, and by Landau and Lifshitz (1959) §§125-127.

6.3 Boundary Conditions

In this section we discuss the conditions which distinguish stellar models from all other possible solutions to the equations of relativistic fluid mechanics. We must specify boundary conditions at the *surface of the star*, which is the surface which separates the fluid *interior* of the stellar model from the vacuum *exterior* region. Furthermore, we must specify the asymptotic behavior of the gravitational field in the external vacuum region.

A spacetime $(\hat{M}, \hat{g}_{\alpha\beta})$ will be called *asymptotically flat at null infinity* if there exists a smooth manifold M (with boundary $\mathscr{I}$) which has a smooth non-degenerate metric $g_{\alpha\beta}$ and a diffeomorphism Ψ from $\hat{M}$ onto $M - \mathscr{I}$ satisfying the following conditions:

i) There exists a smooth function Ω on M such that $\Psi_*(g_{\alpha\beta}) = \Psi_*(\Omega^2)\hat{g}_{\alpha\beta}$ on $\hat{M}$, with $\Omega = 0$ and $\nabla_\alpha\Omega \neq 0$ on $\mathscr{I}$.

ii) There exists a neighborhood N of $\mathscr{I}$ in M such that $\hat{g}_{\alpha\beta}$ satisfies the vacuum Einstein equation in $N \cap \Psi(\hat{M})$.

iii) If Ω is so shosen that $\nabla^\alpha\nabla_\alpha\Omega = 0$ on $\mathscr{I}$ (where ∇ is the covariant derivative on $(M, g_{\alpha\beta})$),then the vector field $n^\alpha = g^{\alpha\beta}\nabla_\beta\Omega$ is complete on $\mathscr{I}$ and the space $\mathcal{S}$ of orbits of n^α on $\mathscr{I}$ is diffeomorphic to S^2 (where the differential structure on $\mathcal{S}$ is the one induced by that on $\mathscr{I}$).

This definition of asymptotic flatness has been introduced recently by Geroch and Horowitz (1978) and is used to prove the theorem which classifies the possible symmetries of asymptotically flat spaces by Ashtekar and Xanthopoulos (1978).

In addition to the above condition which characterizes the asymptotic behavior of the geometry at null infinity, we will need to use the properties of the asymptotic behavior in spacelike directions. We will call a spacetime *asymptotically flat at spacelike infinity* if there exist coordinates t, x, y, z such that the metric has the asymptotic form:

$$ds^2 = -[1 - \frac{2M}{r} + O(r^{-2})]dt^2 + [\delta_{ij} + O(r^{-1})]dx^i dx^j$$
$$+4[Jr^{-3}\varepsilon_{ijz}x^j + O(r^{-3})]dx^i dt, \qquad (6.35)$$

where $r^2 = x^2 + y^2 + z^2$. It may be that the two notions of asymptotic flatness, which we have introduced here, are not independent. However, since this relationship has not been clarified yet, we introduce both here. For simplicity, a spacetime will be called *asymptotically flat* if both the spacelike and the null asymptotic flatness conditions are satisfied.

A general relativistic *stellar model* is a solution of the equations of relativistic fluid mechanics, eqs. (6.25) and (6.26), which also satisfies the following conditions:

a) The solution must be asymptotically flat.

b) The solution must be globally hyperbolic.

c) The fluid must have compact support on each Cauchy surface.

d) The fluid variables are C^2 functions of position in the interior of the star.

e) The surface of the star is a C^2 surface which is the boundary of the region of positive pressure. The pressure is continuous at the surface of the star.

f) The spacetime manifold has a C^5 differentiable structure.

g) The metric tensor is C^3 except at the surface of the star. At the surface of the star the metric is C^2 for tangential derivatives and C^1 for normal derivatives.

h) The heat flow vector, at the surface of the star, is tangent to the surface of the star.

§7 STATIONARY SPACETIMES AND THERMODYNAMIC EQUILIBRIUM

7-1. Introduction

In this chapter we begin to investigate the final equilibrium configuration of general relativistic stellar models. We study the relationship between the relativistic notions of thermodynamic equilibrium and stationarity. We use a slightly stronger notion of thermodynamic equilibrium here then was used in the study of Newtonian stellar models (§3.1). We will say that a relativistic stellar model is in *thermodynamic equilibrium* if the entropy current, $su^\alpha + q^\alpha/T$, is divergenceless:

$$\nabla_\alpha (su^\alpha + q^\alpha/T) = 0. \tag{7.1}$$

This condition implies the constancy of the entropy per particle along the world lines of the fluid; however, the two conditions do not appear to be equivalent as they were for a Newtonian fluid.

The concept of stationarity is more complicated in the general relativistic case than it was in Newtonian mechanics. The curved spacetimes of general relativity have no preferred time coordinate which one might use to define stationarity. Instead, we will call a spacetime *stationary* (see Carter 1973) if there exists a global symmetry group action on the spacetime whose trajectories are timelike curves diffeomorphic to the real line, and under which all physical fields are invarient. The existence of such a group action implies the existence of a timelike vector field η^α whose integral curves are the trajectories of the group action, and along which all physical fields are Lie transported. In particular the metric has vanishing Lie derivative along η^α , which is equivalent to Killing's equation:

$$\nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha = 0 \quad . \tag{7.2}$$

The main purpose of this chapter is to establish the extent to which the concepts of stationarity and thermodynamic equilibrium are equivalent for relativistic fluids (Section 7.3). We will show that a fluid in a stationary spacetime is necessarily in thermodynamic equilibrium (Theorem 7.7). Furthermore, we will show that a fluid in thermodynamic equilibrium moves along the trajectories of a timelike Killing vector field, and is therefore in a certain sense "locally" stationary (Theorem 7.9). We are not able to determine at this time whether or not a global isometry is necessary however.

The other purpose of this chapter is to establish a number of "technical" results which are needed to study the properties of stationary spacetimes. Since the stationarity of a spacetime is defined by the presence of a Killing vector field, we will investigate the properties of Killing vector fields. Section 7.2 states and proves a number of useful properties which will be needed throughout the later chapters.

7-2. Killingvectorology

In this section we derive some useful properties of non-null Killing vector fields. We do not use any assumptions about the dimension or signature of the manifold for these results. We will assume that the Killing vector fields are sufficiently smooth, usually at least C^2. The identities which we present here can be found in the literature of Papapetrou (1966), Carter (1973) and Trautman (1974). The complete list of results preceeds the proofs.

LEMMA 7.1 - *Any Killing vector field,* η^{α} *, satisfies*

$$\nabla_{\alpha} \nabla_{\beta} \eta_{\gamma} = \eta_{\mu} R^{\mu}{}_{\alpha\beta\gamma} . \tag{7.3}$$

LEMMA 7.2 - *Any Killing vector field,* η^{α} *, satisfies*

$$\nabla^{\gamma}\{\eta_{[\gamma} \nabla_{\alpha} \eta_{\beta]}\} = (2/3)\eta_{[\alpha} R_{\beta]}{}^{\mu} \eta_{\mu} . \tag{7.4}$$

The vector field ω^{α} will be called the *twist* of the vector field η^{α} if

$$\omega^{\alpha} = \eta^{\alpha\beta\mu\nu} \eta_{\beta} \nabla_{\mu} \eta_{\nu} \tag{7.5}$$

where $\eta^{\alpha\beta\mu\nu}$ is the totally antisymmetric tensor field, having values $\pm 1/\sqrt{-g}$ or zero.

LEMMA 7.3 - *If* η^{α} *is a Killing vector field, and if* ω^{α} *is the twist of* η^{α} *, then*

$$\nabla_{[\alpha} \omega_{\beta]} = \eta_{\alpha\beta\mu\nu} \eta^{\mu} R^{\nu}{}_{\gamma} \eta^{\gamma}. \tag{7.6}$$

LEMMA 7.4 - *If* η^{α} *is a non-null Killing vector field, and if* ω^{α} *is the twist of* η^{α} *, then the gradient of* η^{α} *can be written in the form*

$$\nabla_{\alpha} \eta_{\beta} = -\{\eta_{[\alpha} \nabla_{\beta]}(\eta^{\mu} \eta_{\mu}) + \frac{1}{2} \eta_{\alpha\beta\mu\nu} \omega^{\mu} \eta^{\nu}\}/(\eta^{\sigma} \eta_{\sigma}). \tag{7.7}$$

LEMMA 7.5 - *If* η^{α} *is a non-null Killing vector field, and if* ω^{α} *is the twist of* η^{α} *, then*

$$\nabla_{\alpha}\{\omega^{\alpha}/(\eta^{\mu} \eta_{\mu})^{2}\} = 0. \tag{7.8}$$

LEMMA 7.6 - *Let* u^α *be a unit vector field,* $u^\alpha u_\alpha = \pm 1$. *The vector field* $\eta^\alpha = f\, u^\alpha$ *is a Killing vector field if and only if*

$$u^\alpha \nabla_\alpha u_\beta = \mp \nabla_\beta \log f, \text{ and} \tag{7.9}$$

$$P_\alpha{}^\mu P_\beta{}^\nu (\nabla_\mu u_\nu + \nabla_\nu u_\mu) = 0 , \tag{7.10}$$

where $P^{\alpha\beta} = g^{\alpha\beta} \mp u^\alpha u^\beta$.

The proofs of these lemmas are now given. Those not interested in the details of these proofs are advised to skip to section 7.3.

PROOF OF LEMMA 7.1: The Ricci identity for η^α gives (eq. 6.23):

$$\nabla_\alpha \nabla_\beta \eta_\gamma - \nabla_\beta \nabla_\alpha \eta_\gamma = - \eta_\mu R^\mu{}_{\gamma\alpha\beta} \tag{7.11}$$

We cyclically permote the indices in eq. (7.11), add the three expressions and use Killing's equation to obtain,

$$\nabla_\alpha\nabla_\beta\eta_\gamma + \nabla_\beta\nabla_\gamma\eta_\alpha + \nabla_\gamma\nabla_\alpha\eta_\beta = -(1/2)\eta_\mu(R^\mu{}_{\gamma\alpha\beta} + R^\mu{}_{\alpha\beta\gamma} + R^\mu{}_{\beta\gamma\alpha}). \tag{7.12}$$

The right hand side of eq. (7.12) vanishes because of the symmetry of the Riemann tensor. The left hand side gives,

$$\nabla_\gamma \nabla_\alpha \eta_\beta = \nabla_\beta \nabla_\alpha \eta_\gamma - \nabla_\alpha \nabla_\beta \eta_\gamma = \eta_\mu R^\mu{}_{\gamma\alpha\beta}, \tag{7.13}$$

when eqs. (7.2) and (7.11) are used. ∎

PROOF OF LEMMA 7.2: We expand the left hand side of eq. (7.4)

$$\nabla^\gamma\{\eta_{[\gamma} \nabla_\alpha \eta_{\beta]}\} = \frac{1}{3}\{\eta^\gamma \nabla_\gamma \nabla_\alpha \eta_\beta + \nabla^\gamma \eta_\alpha \nabla_\beta \eta_\gamma + \nabla^\gamma \eta_\beta \nabla_\gamma \eta_\alpha\}$$
$$+ \frac{1}{3}\{- \eta_\alpha \nabla^\gamma \nabla_\gamma \eta_\beta + \eta_\beta \nabla^\gamma \nabla_\gamma \eta_\alpha\} . \tag{7.14}$$

The first three terms in eq. (7.14) are simply the Lie derivative of $\nabla_\alpha \eta_\beta$ along η^γ. Since η^α is a Killing vector field, this Lie derivative vanishes. The last two terms are simplified by applying Lemma 7.1:

$$\nabla^\gamma\{\eta_{[\gamma} \nabla_\alpha \eta_{\beta]}\} = \frac{1}{3} \eta_\alpha R_\beta{}^\mu \eta_\mu - \frac{1}{3} \eta_\beta R_\alpha{}^\mu \eta_\mu .$$ ■

PROOF OF LEMMA 7.3: We recall first two identities (see for example Synge 1966, p. 356):

$$\eta^{\alpha abc} \eta_{\alpha\beta\mu\nu} = - \delta^a_\beta \delta^b_\mu \delta^c_\nu - \delta^a_\mu \delta^b_\nu \delta^c_\beta - \delta^a_\nu \delta^b_\beta \delta^c_\mu + \delta^a_\beta \delta^b_\nu \delta^c_\mu + \delta^a_\mu \delta^b_\beta \delta^c_\nu + \delta^a_\nu \delta^b_\mu \delta^c_\beta , \text{ and} \tag{7.15}$$

$$\eta^{\mu\nu ab} \eta_{\mu\nu\alpha\beta} = - 2\{\delta^a_\alpha \delta^b_\beta - \delta^a_\beta \delta^b_\alpha\}. \tag{7.16}$$

From eq. (7.15) it follows that

$$\eta^{abc\mu} \omega_\mu = 6 \quad \eta^{[a} \nabla^b \eta^{c]} . \tag{7.17}$$

We now consider the left hand side of eq. (7.6). This expression can be rewritten as

$$\nabla_{[\alpha} \omega_{\beta]} = \frac{1}{2} \nabla_\mu\{(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) \omega_\nu\}. \tag{7.18}$$

The combination of Kronecker deltas on the right hand side of eq. (7.18) can be replaced using eq. (7.16):

$$\nabla_{[\alpha} \omega_{\beta]} = - \frac{1}{4} \nabla_\mu\{\eta^{ab\mu\nu} \eta_{ab\alpha\beta} \omega_\nu\}. \tag{7.19}$$

We use eq. (7.17) to find

$$\nabla_{[\alpha} \omega_{\beta]} = \frac{3}{2} \eta_{ab\alpha\beta} \nabla_\mu\{\eta^{[a} \nabla^b \eta^{\mu]}\} . \tag{7.20}$$

Lemma 7.2 is now used on the right hand side of eq. (7.20) to finish the proof. ■

PROOF OF LEMMA 7.4: The gradient of a Killing vector field is antisymmetric (see eq. 7.2), therefore we use eq. (7.16) to write

$$\nabla_\alpha \eta_\beta = -\frac{1}{4} \eta^{\mu\nu ab} \eta_{\mu\nu\alpha\beta} \nabla_a \eta_b . \tag{7.21}$$

We split the contraction on the index ν in eq. (7.21) into the part parallel to η^α and the part normal to η^α ; the result is given by

$$\nabla_\alpha \eta_\beta = -\frac{1}{4}(\eta^\sigma \eta_\sigma)^{-1} \eta_{\alpha\beta\mu\nu} \omega^\mu \eta^\nu$$
$$- \frac{1}{4} \eta^{\mu\sigma ab} \eta_{\mu\gamma\alpha\beta} [\delta^\gamma_\sigma - \eta^\gamma \eta_\sigma/(\eta^\varepsilon \eta_\varepsilon)]\nabla_a\eta_b . \tag{7.22}$$

The second term in eq. (7.22) contains a singly contracted pair of totally antisymmetric tensors. These are replaced by using eq. (7.15). When the resulting expression is simplified by performing all of the contractions involving Kronecker deltes, the result is

$$\nabla_\alpha \eta_\beta = -\frac{1}{4}(\eta^\sigma \eta_\sigma)^{-1} \eta_{\alpha\beta\mu\nu} \omega^\mu \eta^\nu$$
$$+ \frac{1}{2} \nabla_\alpha \eta_\beta - \frac{1}{2}(\eta^\sigma \eta_\sigma)^{-1} \eta_{[\alpha} \nabla_{\beta]} (\eta^\mu \eta_\mu). \tag{7.23}$$

This expression is easily converted to the form of eq. (7.7). ■

PROOF OF LEMMA 7.5 : We use the definition of the twist ω^α, eq. (7.5), and the **Leibniz** property of covarient derivatives to show that

$$\nabla_\alpha \omega^\alpha = \eta^{\alpha\beta\mu\nu}\{\nabla_\alpha \eta_\beta \nabla_\mu \eta_\nu + \eta_\beta \nabla_\alpha \nabla_\mu \eta_\nu\} . \tag{7.24}$$

The second term on the right hand side of eq. (7.24) can be re-expressed in terms of the Riemann tensor by applying Lemma 7.1. The Riemann tensor will be antisymmetrized on its last three indices in eq. (7.24); therefore this term will vanish. The first term on the right hand side of eq. (7.24) can be simplified by using the equation for $\nabla_\alpha \eta_\beta$ from Lemma 7.4. The resulting expression is simplified by applying eqs. (7.15) and (7.16). The result is given by

$$\nabla_\alpha \, \omega^\alpha = 2(\eta^\nu \, \eta_\nu)^{-1} \, \omega^\alpha \, \nabla_\alpha(\eta^\mu \, \eta_\mu). \tag{7.25}$$

Equation (7.25) can be easily converted into eq. (7.8). ■

PROOF OF LEMMA 7.6: If η^α is a Killing vector field then

$$0 = \eta^\alpha \, \nabla_\alpha(\eta^\mu \, \eta_\mu) = \pm \, \eta^\alpha \, \nabla_\alpha \, f^2. \tag{7.26}$$

Therefore the acceleration of u^α may be written

$$u^\alpha \, \nabla_\alpha \, u^\beta = f^2 \, \eta^\alpha \, \nabla_\alpha \, \eta^\beta \, . \tag{7.27}$$

Use Killing's equation to rewrite the right hand side of eq. (7.27) as

$$u^\alpha \, \nabla_\alpha \, u^\beta = \mp \, \frac{1}{2} \, f^{-2} \, \nabla^\beta \, f^2. \tag{7.28}$$

This equation is equivalent to eq. (7.9). We now consider the left hand side of eq. (7.10).

$$P_\alpha{}^\mu \, P_\beta{}^\nu \, (\nabla_\mu \, u_\nu + \nabla_\nu \, u_\mu) = P_\alpha{}^\mu \, P_\beta{}^\nu \{ f^{-1}(\nabla_\mu \, \eta_\nu + \nabla_\nu \, \eta_\mu) + \eta_\nu \, \nabla_\mu \, f^{-1} + \eta_\mu \, \nabla_\nu \, f^{-1} \} \tag{7.29}$$

The first two terms on the right hand side of eq. (7.29) vanish by Killing's equation, while the second two vanish because $P_\alpha{}^\mu \, \eta_\mu = 0$.

We now consider the converse. We assume eqs. (7.9) and (7.10). When the definition of the projection operator is substituted into eq. (7.10) the following non-vanishing terms remain.

$$0 = \nabla_\mu u_\nu + \nabla_\nu u_\mu \mp u_\nu u^\alpha \nabla_\alpha u_\mu \mp u_\mu u^\alpha \nabla_\alpha u_\nu . \qquad (7.30)$$

We use the expression for the acceleration from eq. (7.9), and the definition of η^α to rewrite eq. (7.30) as

$$0 = f^{-1}(\nabla_\mu \eta_\nu + \nabla_\nu \eta_\mu)$$

$$- f^{-2}(\eta_\nu \nabla_\mu f + \eta_\mu \nabla_\nu f) + u_\nu \nabla_\mu \log f + u_\mu \nabla_\nu \log f. \qquad (7.31)$$

The last four terms cancel each other, so that eq. (7.31) becomes Killing's equation. ■

7.3 - The Equivalence of Stationarity and Thermodynamic Equilibrium

The first theorem in this section demonstrates the necessity of thermodynamic equilibrium for a stationary relativistic stellar model. The proofs of the theorems in this section were first given by Lindblom (1976a). Although the proofs of the theorems given here assume that the temperature of the fluid does not vanish, the theorems remain true in that case also. The proofs for the zero temperature case involve fairly straightforeward modifications of the proofs which are given.

THEOREM 7.7 - *A relativistic fluid stellar model which is stationary, globally hyperbolic and asymptotically Minkowskian must be in a state of thermodynamic equilibrium.*

PROOF: A stationary spacetime admits a timelike vector field, η^α, along which all physical fields are Lie transported. In particular the metric tensor, $g_{\alpha\beta}$, and the entropy current vector, $s\,u^\alpha + q^\alpha/T$, have zero Lie derivatives along η^α. Since the spacetime is globally hyperbolic, there exists a spacelike surface τ_o which intersects every integral curve of η^α exactly once. We create a family of surfaces, $\tau(t)$, by Lie transporting the surface τ_o along the integral curves of η^α.

The total entropy of the fluid may be defined by an integral over one of the surfaces $\tau(t)$:

$$S(t) = \int_{\tau(t)} \sqrt{-g}\,(su^\alpha + q^\alpha/T)\,d^3 x_\alpha \,. \tag{7.32}$$

Since the entropy current, and the metric are Lie transported along η^α, the total entropy as defined by eq. (7.32) is independent of the parameter t.

Now consider a region of spacetime, Ω, whose boundary consists of the two surfaces $\tau(t_1)$, $\tau(t_2)$ and a piece at spacelike infinity. The support of the thermodynamic potentials is bounded on any spacelike surface. Therefore the entropy current vanishes on the portion of the boundary of Ω which lies at infinity. The integral of the divergence of the entropy current over the region Ω is therefore zero:

$$\int_{\Omega} \sqrt{-g}\;\; \nabla_\alpha(su^\alpha + q^\alpha/T)\,d^4x = S(t_2) - S(t_1) = 0. \tag{7.33}$$

The entropy production equation (6.30) for a relativistic fluid implies that

$$\nabla_\alpha(su^\alpha + q^\alpha/T) \geq 0. \tag{7.34}$$

Equation (7.33) states that the integral of this non-negative quantity vanishes. We conclude that the divergence of the entropy current itself vanishes, so the fluid must be in a state of thermodynamic equilibrium. ■

In the next theorem we prove the relativistic analogue of Theorem 3.3, that a fluid in thermodynamic equilibrium must be barotropic and the motion of the fluid must be rigid. The concept of rigid motion is not a local one, therefore the extension of this idea to arbitrary curved space-times is not totally straightforeward. We will say that a body is in *rigid motion* if the unit vector field u^α (whose integral curves are the world lines of points in the body) is shear free and expansion free:

$$P^{\alpha\mu} \; P^{\beta\nu} \; (\nabla_\mu \; u_\nu + \nabla_\nu \; u_\mu) = 0. \tag{7.35}$$

This definition of rigid motion for relativistic objects is equivalent to a definition given by Born, Herglotz and Noether: "A body is called rigid if the distance between every neighboring pair of particles, measured orthogonal to the world line of either of them, remains constant along the world line." The concept of rigid motion for relativistic systems is discussed at length by Trautman (1965) §8. In particular Trautman gives the proof of the equivalence of eq. (7.35) and the Born, Herglotz and Noether definition.

THEOREM 7.8 - *If a relativistic fluid, having nonzero coefficients of viscosity and heat conduction, is in a state of thermodynamic equilibrium, then the motion of the fluid is rigid and the fluid is barotropic.*

PROOF: From the definition of thermodynamic equilibrium eq. (7.1) and the equation of entropy production eq. (6.30) it follows that

$$0 = q^\alpha = -\kappa \; P^{\alpha\mu}(\nabla_\mu T + Ta_\mu), \tag{7.36}$$

$$0 = \sigma^{\alpha\beta} \text{ , and} \tag{7.37}$$

$$0 = \theta \ . \tag{7.38}$$

Equations (7.37) and (7.38) are equivalent to eq. (7.35), therefore the fluid motion is rigid.

The entropy and particle number conservation laws eqs. (7.25) and (6.30) reduce to the form

$$u^{\alpha} \nabla_{\alpha} n = 0, \text{ and} \tag{7.39}$$

$$u^{\alpha} \nabla_{\alpha} s = 0 \ . \tag{7.40}$$

Therefore, all of the thermodynamic functions, (e.g. the temperature) have zero derivatives along u^{α}. Equation (7.36) then implies that the acceleration is a gradient

$$a_{\mu} = - \nabla_{\mu} \log T. \tag{7.41}$$

Euler's equation for this system eq. (6.29) simplifies to

$$a_{\mu} = - (\rho+p)^{-1} \nabla_{\mu} p. \tag{7.42}$$

Since the acceleration is a gradient from eq. (7.41) it follows that the right hand side of eq. (7.42) must also be a gradient. It follows that the fluid is barotropic. ■

We next prove a rather trivial corollary of Theorem 7.8, which shows that a fluid in thermodynamic equilibrium is at least locally stationary. It is quite possible that a much stronger result is true for asymptotically Minkowskian relativistic stellar models. I conjecture that thermodynamic equilibrium implies global stationarity for relativistic stellar models. This is a stronger result than the corresponding Newtonian theorem (3.4). Non-stationary examples of thermal equilibrium in the Newtonian case are non-axisymmetric rigid rotators like the Jacobi ellipsoids. In general relativity theory, objects of this sort would radiate gravitational radiation and consequently could not remain in thermodynamic equilibrium. Thus, I suspect that in relativity theory these objects would be eliminated by the assumptions of thermal equilibrium, leaving as possible stellar configurations only the truely stationary models.

THEOREM 7.9 - *If a relativistic fluid, having non-zero coefficients of viscosity and heat conduction, is in a state of thermodynamic equilibrium, then the four velocity of the fluid is proportional to a Killing vector field along which all fluid variables are Lie transported. (Thus, the fluid is "locally" stationary.)*

PROOF: Equations (7.37), (7.38) and (7.41) from the proof of Theorem 7.8 are satisfied by this fluid. According to Lemma 7.6 these equations imply that the four velocity of the fluid is proportional to a Killing vector field, η^α. Equations (7.39) and (7.40) imply that the entropy density, s, and the number density, n, are Lie transported along η^α. That the four velocity is also Lie transported along η^α is easily verified. ∎

§8 ANALYTICITY OF RELATIVISTIC STELLAR MODELS

8.1 Preliminaries

In this chapter we show that the functions which describe the stellar models of general relativity theory must be analytic functions when they are expressed in appropriate coordinates. Therefore the metric, and the other functions of the model can be expressed as power series, and consequently are completely determined by specifying all derivatives of the functions at a single point within each connected analytic region. Thus we can study the global properties of the stellar models by examining their properties in a few local neighborhoods. Hawking (1972) has used these results in his proof that stationary black holes must be axisymmetric; and Lindblom (1976a) uses the results in his proof that stationary stellar models must be axisymmetric (see also §10).

The techniques for proving the analyticity of these spacetimes were developed by Müller zum Hagen (1970a) and (1970b). We reproduce his proof that the vacuum exterior region of a stationary stellar model must be analytic (Theorem 8.7). We also present here an extension of this work which shows that the interior region of a rigidly rotating stellar model must be analytic, if the equation of state is analytic (Theorem 8.8). A slightly weaker version of Theorem 8.8 has been given by Lindblom (1976b). The surface of the star, which forms the boundary between the analytic exterior and the analytic interior of the star, is a place where the functions necessarily fail to be analytic.

The main tool used in the proof of the analyticity of these spacetimes is a theorem by Morrey (1958) which states that every sufficiently differentiable solution to a system of elliptic differential equations must be analytic. We show that when the spacetime is stationary one can choose coordinates in which Einstein's equations become a system of elliptic equations. Morrey's theorem is then used to show that any solutions of the equations must be analytic functions.

The concept of an analytic function on a manifold only makes sense if the manifold has an analytic atlas. In particular, the transformation functions, which relate coordinates on the intersection of two coordinate charts, must be analytic functions. We use *stationary and harmonic coordinates* which are defined by the two conditions:
a) the components of the timelike Killing vector field are given by $\eta^\alpha = \delta^\alpha{}_o$; and b) the Christoffel connection satisfies $g^{\mu\nu}\,\Gamma^\alpha{}_{\mu\nu} = 0$. We show in Lemma 8.3 that stationary and harmonic coordinates exist at each point. We also show (Lemma 8.4) that the transformation function between two stationary harmonic coordinate charts are analytic functions whenever the metric is analytic with respect to these charts. Thus, the stationary and harmonic coordinate charts form an analytic atlas for a stationary spacetime.

In order to use the necessary theorems from the literature on partial differential equations, we must consider a somewhat stronger continuity condition than the usual one. We say that a function $f(x^\alpha)$ is *Hölder continuous of order* μ, $0 < \mu < 1$ (written C^μ) on some domain D, if there exists a constant K so that for all x^α, $y^\alpha \in D$ it follows that

$$|f(x^\alpha) - f(y^\alpha)| < K\,|x^\alpha - y^\alpha|^\mu \,, \tag{8.1}$$

where $|x^\alpha|^2 = (x^o)^2+(x^1)^2+(x^2)^2+(x^3)^2$. Furthermore, we say that a function, f, is of class $C^{m+\mu}$ for integer m and $0 < \mu < 1$ if f has continuous derivatives through order m, and if the m'th derivatives are Hölder continuous of order μ.

We will be considering systems of second order elliptic partial differential equations. The system of equations

$$\Phi^A(x^\alpha, f^B, \partial_\alpha f^B, \partial_\alpha \partial_\beta f^B) = 0 \qquad (8.2)$$

with A,B = 1,2, ...N is said to be *elliptic* in some domain D if for all $x^\gamma \varepsilon$ D and for all vectors $\lambda^\alpha \neq 0$, the determinent on the indices A and B

$$\det \{ \sum_{\alpha,\beta} \lambda^\alpha \lambda^\beta [\frac{\partial}{\partial y^B_{\alpha\beta}} \Phi^A(x^\gamma, y^C, y^C_\gamma, y^C_{\gamma\varepsilon})]\} \neq 0 , \qquad (8.3)$$

does not vanish when evaluated at $y^C = f^C$, $y^C_\alpha = \partial_\alpha f^C$, and $y^C_{\alpha\beta} = \partial_\alpha \partial_\beta f^C$.

The proof of our results will be based on the following two theorems about the solutions of elliptic differential equations. The first theorem is a special case of a theorem by Morrey (1958), which guarantees the analyticity of the solutions of elliptic differential equations

THEOREM 8.1 - *Assumptions:* f^B *is a function which is a solution of a system of elliptic differential equations (eq. 8.2) in a domain* D *, and which is class* $C^{2+\mu}$ *,* $0 < \mu < 1$*. The function* $\Phi^A(x^\alpha, y^B, y^B_\alpha, y^B_{\alpha\beta})$ *(see eq. 8.2) is analytic in the variables* $(x^\alpha, y^B, y^B_\alpha, y^B_{\alpha\beta})$.

Assertion: The function f^B *is analytic in* D.

The second theorem is a local existence theorem for the solutions of elliptic differential equations in one dependent variable.

THEOREM 8.2 - *Let* L *be a second order elliptic operator*

$$L = a^{\alpha\beta} \partial_\alpha \partial_\beta + a^\beta \partial_\beta + a, \tag{8.4}$$

where the coefficients $a^{\alpha\beta}$, a^β, a *and the function* h *are of class* $C^{m+\mu}$, $0 < \mu < 1$. *In a sufficiently small neighborhood of a point, say* x_o, *there exists a solution,* f, *of the equation*

$$Lf = h, \tag{8.5}$$

having the following properties: a) f *is of class* $C^{2+m+\mu}$
b) f *and its first derivatives have prescribed values at the point* x_o.
The proof of Theorem 8.1 may be found in Morrey (1958), while the proof of Theorem 8.2 is given in Bers, John and Schecter (1964) p. 136 and 228.

8.2 Lemmas

In this section we state and prove several lemmas which are needed to prove the analyticity theorems in the next section.

LEMMA 8.3 - *Assumptions: Consider a spacetime* (M,g) *having a* C^{n+2} *differentiable structure for integer* $n \geq 2$. (M,g) *admits a globally timelike Killing vector field,* η^α, *which is* C^{n+1}. *The metric tensor is* C^n.
Assertion: In a neighborhood of each point $x \varepsilon M$ *there exist stationary harmonic coordinates which are* $C^{n+\mu}$, $0 < \mu < 1$, *functions of the* C^{n+2} *coordinates on* M.

PROOF: Since η^α is a C^{n+1} vector field, one can always choose coordinates (in a neighborhood of any point) in which the components of $\eta^\alpha = \delta^\alpha_o$, and which are C^{n+2} functions of the coordinates in the C^{n+2} differentiable structure on M. In these coordinates the metric g^{ij}, i,j = 1,2,3, is positive definite. We note that if one performs a coordinate transformation of the form

$$x^{o'} = x^o + h(x^j), \text{ and } x^{i'} = f^{i'}(x^j), \tag{8.6}$$

then the components of η^α remain unchanged, $\eta^{\alpha'} = \delta^{\alpha'}_{o'}$, and the matrix $g^{i'j'}$ remains positive definite. Killing's equation is given by (see eq. 7.2).

$$\eta^\alpha \, \partial_\alpha \, g_{\mu\nu} + g_{\alpha\nu} \, \partial_\mu \, \eta^\alpha + g_{\mu\alpha} \, \partial_\nu \, \eta^\alpha = 0. \tag{8.7}$$

Therefore in the coordinates in which $\eta^\alpha = \delta^\alpha_o$, Killing's equation reduces to

$$\partial_o \, g_{\alpha\beta} = 0 \, . \tag{8.8}$$

The Christoffel connection transforms according to the formula,

$$\Gamma^\sigma_{\mu\nu} \, \partial_\sigma \, x^{\alpha'} = \Gamma^{\alpha'}_{\beta'\gamma'} \partial_\mu \, x^{\beta'} \partial_\nu \, x^{\gamma'} + \partial_\mu \, \partial_\nu \, x^{\alpha'} \tag{8.9}$$

under a change in coordinates (see e.g. Eisenhart 1926). If we can find new coordinates $x^{\alpha'}$ which satisfy the equations

$$L(x^{\alpha'}) \equiv g^{\mu\nu} \, \partial_\mu \, \partial_\nu \, x^{\alpha'} - g^{\mu\nu} \, \Gamma^\sigma_{\mu\nu} \, \partial_\sigma \, x^{\alpha'} = 0 \, , \tag{8.10}$$

then the resulting coordinates will satisfy the harmonic condition

$$g^{\beta'\gamma'} \, \Gamma^{\alpha'}_{\beta'\gamma'} = 0 \, . \tag{8.11}$$

We make a coordinate transformation in the form of eq. (8.6). The operator L from eq. (8.10) on this transformation reduces to

$$\mathsf{L}(x^{o'}) = g^{ij}\,\partial_i\,\partial_j\,h - g^{\mu\nu}\,\Gamma^{i}_{\mu\nu}\,\partial_i h - g^{\mu\nu}\,\Gamma^{o}_{\mu\nu}\,, \text{ and} \tag{8.12}$$

$$\mathsf{L}(x^{k'}) = g^{ij}\,\partial_i\,\partial_j\,f^{k'} - g^{\mu\nu}\,\Gamma^{i}_{\mu\nu}\,\partial_i\,f^{k'}\,. \tag{8.13}$$

The operators in eqs. (8.12) and (8.13) are elliptic. Consequently the equations $\mathsf{L}(x^{\alpha'}) = 0$ are elliptic equations for h and $f^{k'}$. Theorem 8.2 guarantees the existence of solutions of this equation which are of class $C^{n+\mu}$, $0 < \mu < 1$. Since the values of h and $f^{k'}$ and their first derivitives can be specified at a point, x, we can choose values so that the Jacobian of the coordinate transformation is nonsingular is some neighborhhod of x. Thus, stationary and harmonic coordinates of class $C^{n+\mu}$ exist in a neighborhood of each point. ■

LEMMA 8.4 - *Consider a stationary spacetime* (M, g) *in which the components of the metric tensor are analytic functions of the stationary harmonic coordinate charts. Then, the stationary harmonic coordinate charts form an analytic atlas on* M.

PROOF: To show that the stationary harmonic coordinate charts form an analytic atlas on M, we must show that the transformation functions between thc coordinates in overlapping charts are analytic. Since the coordinates are stationary in both charts, the components of η^{α} in both systems are given by $\eta^{\alpha} = \delta^{\alpha}_{o}$. Therefore

$$\delta^{\alpha'}_{o'} = \eta^{\mu}\,\partial_{\mu}\,x^{\alpha'} = \partial_{o}\,x^{\alpha'}\,. \tag{8.14}$$

By integrating eq. (8.14) it follows that the transformation between two stationary coordinate systems must be in the form of eq. (8.6):

$$x^{o'} = x^{o} + h(x^{j}), \text{ and } x^{i'} = f^{i'}(x^{j}). \tag{8.15}$$

From the transformation law for the Christoffel connection, eq. (8.9), it follows that the transformation functions between two harmonic coordinate systems must satisfy the equation

$$g^{\mu\nu} \partial_{\mu} \partial_{\nu} x^{\alpha'} = 0 . \tag{8.16}$$

From the form of the transformation law given in eq. (8.15) it follows that eq. (8.16) reduces to the following elliptic equation

$$g^{ij} \partial_{i} \partial_{j} x^{\alpha'} = 0. \tag{8.17}$$

Lemma 8.3 guarantees that the transformation functions are at least $C^{2+\mu}$, $0 < \mu < 1$. Theorem 8.1 then shows that the transformation functions must be analytic since eq. (8.17) is an elliptic equation, and since the metric, g^{ij}, was assumed to be an analytic function of the coordinates. ■

LEMMA 8.5 - *The Ricci tensor in a stationary harmonic coordinate system can be written in the form*

$$R_{\alpha\beta} = -\frac{1}{2} g^{ij} \partial_{i} \partial_{j} g_{\alpha\beta} + B_{\alpha\beta}(g,\partial g), \tag{8.18}$$

where the function $B_{\alpha\beta}$ *depends algebraically on the metric* g *and its first derivatives* ∂g, *but not on any higher derivatives.*

PROOF: From the definition of the Ricci tensor, eq. 6.20, it follows that

$$R_{\alpha\beta} = \partial_{\mu} \Gamma^{\mu}_{\alpha\beta} - \partial_{\beta} \Gamma^{\mu}_{\alpha\mu} + C_{\alpha\beta}(g,\partial g). \tag{8.19}$$

The functions $C_{\alpha\beta}$ are the quadratic terms in the Christoffel symbols, therefore $C_{\alpha\beta}$ depends algebraically on g and ∂g, but does not depend on any higher derivatives. We expand the first two terms on the right hand side of eq. (8.19) to make explicit the terms containing second derivatives in g:

$$R_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\{\partial_\mu \partial_\beta g_{\alpha\nu} + \partial_\mu \partial_\alpha g_{\beta\nu} - \partial_\alpha \partial_\beta g_{\mu\nu}\}$$

$$- \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + C'_{\alpha\beta}(g,\partial g). \qquad (8.20)$$

The first three terms on the right hand side of eq. (8.20) can be re-expressed as a term which depends only on the metric and its first derivatives, when one uses the harmonic coordinate condition (see eq. 8.11):

$$g^{\mu\nu}\{2 \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}\} = 0. \qquad (8.21)$$

Therefore eq. (8.20) can be written as

$$R_{\mu\nu} = - \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + B_{\alpha\beta}(g,\partial g). \qquad (8.22)$$

Killing's equation in these stationary harmonic coordinates is given by eq. (8.8). Consequently eq. (8.22) is equivalent to eq. (8.18). ■

LEMMA 8.6 - *Consider a stationary spacetime in which the metric tensor is an analytic function of the stationary harmonic coordinates. If the* C^3 *vector field* ξ^α *commutes with the Killing vector* η^α ,

$$\eta^\mu \nabla_\mu \xi^\alpha - \xi^\mu \nabla_\mu \eta^\alpha = 0 , \qquad (8.23)$$

and if ξ^α *satisfies the equation*

$$\nabla_\mu \nabla^\mu \xi^\alpha = - R^\alpha{}_\mu \xi^\mu , \qquad (8.24)$$

then ξ^α is an analytic vector field

PROOF: In stationary coordinates, eq. (8.23) becomes

$$\partial_o \xi^\alpha = 0 \ . \qquad (8.25)$$

Consequently in these coordinates, eq. (8.24) can be written in the form

$$g^{ij} \partial_i \partial_j \xi^\alpha + B^\alpha(\xi,\partial\xi, g,\partial g, \partial\partial g) = 0 \ . \qquad (8.26)$$

The function B^α depends on each of its arguments algebraicly. Since the metric is assumed to be an analytic function of these coordinates, it follows then from Theorem 8.1 that ξ^α must be analytic. ■

The proof of the first two lemmas (8.3 and 8.4) was first given (in a slightly different form) by Müller zum Hagen (1970a) and (1970b). Lemma 8.5 is a well known result (see e.g. Lichnerowicz [1967] p. 14). The proof of Lemma 8.6 was previously given by Lindblom (1976a).

8.3 The Analyticity Theorems

The boundary of the star is a place where the functions in the stellar model necessarily fail to be analytic. Consequently we need separate theorems to establish the analyticity of the interior and exterior regions respectively. The first theorem, whose proof has been given by Müller zum Hagen (1970b), establishes the analyticity of the vacuum exterior region of a stationary stellar model.

THEOREM 8.7 - *Consider a stationary vacuum spacetime* (M, g). *If the manifold,* M , *admits a* C^5 *differentiable structure with respect to which the timelike Killing vector field,* η^α , *is* C^4 *and the metric is* C^3 , *then* M *admits an analytic atlas in which the metric is an analytic function.*

PROOF: From Lemma 8.3 it follows that there is a stationary harmonic coordinate chart in a neighborhood of each point, and that the metric is a C^3 function of these coordinates. Lemma 8.5 implies that Einstein's equation for a vacuum spacetime can be written in the form

$$\frac{1}{2} g^{ij} \partial_i \partial_j g_{\alpha\beta} = B_{\alpha\beta}(g, \partial g). \qquad (8.27)$$

Equation (8.27) is a system of elliptic equations. Theorem 8.1 proves that the solutions to this equation must be analytic functions since they are assumed to be C^3. Therefore, the metric tensor is an analytic function of the stationary harmonic coordinate system at each point. Lemma 8.4 proves that in this case the stationary harmonic coordinate charts form an analytic atlas for M. ∎

The last theorem of this chapter proves that the interior regions of stellar models which have come to thermodynamic equilibrium (see Theorem 7.8) must be analytic spacetimes. We note that this theorem does not need to assume that the interior of the star is stationary since Theorem 7.9 guarantees the existence of a timelike Killing vector field in the interior region. The proof of a slightly weaker result than Theorem 8.8 has been given by Lindblom (1976b).

THEOREM 8.8 - *Consider a spacetime* (M,g) *which contains a rigidly moving barotropic ideal fluid. If the manifold* M *has a* C^5 *differentiable structure with respect to which the metric,* $g_{\alpha\beta}$ *, is* C^4*, the fluid four velocity* u^α *is* C^4 *and the pressure,* p *, is* C^3*, and if the mass density,* ρ*, is an analytic function of the pressure, then* M *admits an analytic atlas with respect to which* $g_{\alpha\beta}$*,* u^α *and* p *are analytic functions.*

PROOF: Since the fluid moves rigidly and barotropically it follows that the four velocity of the fluid is proportional to a Killing vector field, $\eta^\alpha = f\, u^\alpha$ (see the proof of Theorem 7.9). According to Lemma 7.6 the function f is given by $u^\alpha \nabla_\alpha u^\beta = -\nabla^\beta \log f$. Therefore the function f is C^4 since u^α and $g_{\alpha\beta}$ are also C^4. Consequently the timelike Killing vector field η^α must be C^4 also. It follows therefore from Lemma 8.3 that there exist stationary harmonic coordinate systems in a neighborhood of each point. Since the components of η^α are given by $\eta^\alpha = \delta^\alpha_o$ in these coordinates, it is obvious that η^α is analytic in the stationary harmonic coordinates. The vector field u^α depends, therefore, only on the metric in these coordinates. Since u^α is not null, this dependence on the metric is analytic. To show that the remaining functions $g_{\alpha\beta}$ and p are analytic, we show that they are the solutions of a system of elliptic differential equations.

Einstein's equation for a rigidly moving barotropic fluid are given by

$$R_{\alpha\beta} = 8\pi(\rho+p)u_\alpha u_\beta + 4\pi(\rho-p)g_{\alpha\beta}. \tag{8.28}$$

Consequently, when one uses Lemma 8.5, one can rewrite eq. (8.28) in the form

$$g^{ij}\,\partial_i\,\partial_j\,g_{\alpha\beta} + B_{\alpha\beta}(g,\,\partial g,p). \tag{8.29}$$

The function $B_{\alpha\beta}$ depends on each of its arguments analytically, since ρ depends on p analytically by assumption.

Euler's equation for this type of fluid is given by (see eq. (7.32)

$$(\rho+p)u^u\nabla_u\,u_\alpha = -\nabla_\alpha\,p. \tag{8.30}$$

Taking the divergence of this expression using the Ricci identity (eq. 6.23) and the expansion free nature of the fluid gives

$$\nabla_\alpha\,\nabla^\alpha\,p = -a^\alpha\,\nabla_\alpha(\rho+p) - (\rho+p)\{R_{\alpha\beta}\,u^\alpha\,u^\beta + \nabla_\alpha\,u_\beta\,\nabla^\beta\,u^\alpha\}. \tag{8.31}$$

When eq. (8.28) is substituted into the right hand side of eq. (8.31) for the Ricci tensor, and when the left hand side is written out in stationary harmonic coordinates, equation (8.31) reduces to

$$g^{ij}\,\partial_i\,\partial_j\,p = B(g,\,\partial g,\,p,\,\partial p). \tag{8.32}$$

The function B depends on each of its arguments analytically. Equations (8.29) and (8.32) form a system of elliptic differential equations for $g_{\alpha\beta}$ and p. It follows from Theorem 8.1 that these must be analytic functions of the stationary harmonic coordinate charts. It follows from Lemma 8.4 that the stationary harmonic coordinate charts form an analytic atlas for M. ∎

It is clear from the techniques which we used to prove Theorems 8.7 and 8.8, that it would be possible to prove the analyticity of any stationary spacetime containing matter fields which satisfy elliptic equations in the stationary case. This is true for other common fields of mathematical physics: Yang-Mills fields, scalar fields, Dirac fields, etc. The case of a spacetime containing a source free Maxwell field has been proved by Lindblom (1976b).

§9 STATIC STELLAR MODELS

9-1 Discussion

In this chapter we discuss the properties of a special class of relativistic stellar models, which in addition to being stationary are also static. In our treatment of Newtonian stellar models, we called a star static when the velocity field vanished, $v^i = 0$ (see §4.1). The concept of staticity therefore, is the complete absence of motion; even steady motions such as uniform rotations are to be excluded. There are two different ways of generalizing this notion of staticity to curved spacetimes. The first will be called the *material staticity condition* which requires that the timelike Killing vector η^α (that defines the stationarity of the spacetime) be an eigenvector of the Ricci tensor:

$$\eta_{[\alpha}R_{\beta]}{}^{\mu}\,\eta_\mu = 0 \qquad (9.1)$$

This condition corresponds to the Newtonian definition of staticity in the following way. The flux of momentum, as seen by a stationary observer (one who moves along the integral curves of η^α), is given by

$$P_\alpha = T_\alpha{}^{\mu}\,\eta_\mu \quad . \qquad (9.2)$$

These stationary observers will observe no motion of the mass-energy in their vicinity if the momentum flux, P_α, points purely in their local time direction:

$$\eta_{[\alpha}P_{\beta]} = 0. \qquad (9.3)$$

Equation (9.3) is equivalent to eq. (9.1) whenever Einstein's equations (6.26) are satisfied.

The second notion of staticity, the *metric staticity condition* is said to hold whenever the twist of the timelike Killing vector field, η^α, vanishes:

$$\eta^{\alpha\beta\mu\nu} \eta_\beta \nabla_\mu \eta_\nu = 0 \ . \qquad (9.4)$$

This condition has the following interpretation. Equation (9.4) is equivalent to the requirement that the integral curves of the Killing vector field, η^α, are orthogonal to a family of spacelike surfaces (see Carter 1973, p. 152). These spacelike surfaces can be thought of as the level surfaces of a time function. The requirement that η^α be orthogonal to these surfaces means that the stationary observers will observe themselves to be at rest in these surfaces. Thus, eq. (9.4) is equivalent to the requirement that there exist a time function which makes the stationary observers appear to be at rest. In the usual terminology, a stationary spacetime is called *static* if the metric staticity condition is satisfied.

In section 9.2 we reproduce Lichnerowicz's (1955) proof (see also Carter 1973, p. 151) that the material staticity condition is equivalent to the metric staticity condition in an asymptotically flat singularity free spacetime. For the case of a spacetime containing an ideal fluid, the material staticity condition reduces to the requirement that the fluid four velocity be parallel to the stationary Killing vector field:

$$u_{[\alpha} \eta_{\beta]} = 0. \qquad (9.5)$$

Therefore Lichnerowicz's theorem (9.1) shows that the spacetime of a relativistic stellar model is static if and only if the fluid obeys eq. (9.5). We will need to use this result in the proof of the axi-symmetry theorem in §10.

In the Newtonian theory of gravity we found that a static stellar model was necessarily spherically symmetric (Theorem 4.8). We would expect the analogous result to hold for relativistic stellar models, but no one has found a way to prove this yet. In section 9.3 we discuss the general framework of the problem in general relativity. We discuss the necessary features which any proof that static stars are spherical must posess. In section 9.4 we write out the Einstein equation for a static fluid in a system of coordinates which appear to be useful for the analyses of this problem. Section 9.5 reviews the papers in the literature which address the problem of proving that static stars are spherical. Section 9.6 outlines the methods which this author has used to search for a proof that static stellar models must be spherical.

9-2 Lichnerowicz's Theorem

Lichnerowicz's theorem establishes the equivalence of the spacetime staticity condition and the material staticity condition. The proof which we present here requires that the spacetime be singularity free. This is the case which applies to relativistic stellar models. Generalizations of this result have been given by Hawking and Carter (see Carter 1973 p. 155). Hawking's generalization allows the presence of black holes in the spacetime. Carter generalizes the theorem by including electromagnetic fields. He defines two notions of staticity for the electromagnetic field and its current, in analogy with the conditions on the metric and the stress tensor eqs. (9.1) and (9.4). He then proves the equivalence of the matter (stress energy and electric current) staticity conditions with the field (metric and electromagnetic) staticity conditions. We will not have a need for these more complicated results in this work however.

THEOREM 9.1 - *Consider a spacetime which is stationary, asymptotically flat, asymptotically source free and topologically Minkowskian. Then the material staticity condition,*

$$\eta_{[\alpha}R_{\beta]}{}^{\mu}\,\eta_{\mu} = 0, \tag{9.6}$$

is equivalent to the metric staticity condition,

$$\eta_{[\alpha}\nabla_{\beta}\eta_{\gamma]} = 0. \tag{9.7}$$

PROOF: If we assume the metric staticity condition, eq. (9.7), it follows from Lemma 7.2 that the material staticity condition, eq. (9.6) must hold also. The proof of the converse is somewhat more involved.

When eq. (9.6) is satisfied, Lemma 7.3 states that the twist of η^{α} is curl free

$$\nabla_{[\alpha}\,\omega_{\beta]} = 0. \tag{9.8}$$

Since the topology of the manifold is taken to be simply connected, eq. (9.8) implies the existence of a scalar, ω, which satisfies

$$\nabla_{\alpha}\omega = \omega_{\alpha}. \tag{9.9}$$

Let us consider the following expression

$$2\nabla_{\alpha}\{\omega\omega^{[\alpha}\eta^{\beta]}/(\eta^{\mu}\eta_{\mu})^{2}\}. \tag{9.10}$$

We can expand expression (9.10) in the following way,

$$2\nabla_{\alpha}\left(\frac{\omega\omega^{[\alpha}\eta^{\beta]}}{(\eta^{\mu}\eta_{\mu})^{2}}\right) = \frac{\omega^{\alpha}\omega_{\alpha}}{(\eta^{\mu}\eta_{\mu})^{2}}\,\eta^{\beta} + \nabla_{\alpha}\left(\frac{\omega^{\alpha}}{(\eta^{\mu}\eta_{\mu})^{2}}\right)\omega\,\eta^{\beta}$$

$$+ \frac{\omega}{(\eta^\mu\eta_\mu)^2}\{\omega^\alpha\nabla_\alpha\eta_\beta - \eta^\alpha\nabla_\alpha\omega^\beta\}$$

$$- \frac{\omega\omega^\beta}{(\eta^\mu\eta_\mu)^2}\nabla_\alpha\eta^\alpha - \omega^\beta\eta^\alpha\nabla_\alpha\left(\frac{\omega}{(\eta^\mu\eta_\mu)^2}\right) . \tag{9.11}$$

The second term on the right hand side of eq. (9.11) vanishes by Lemma 7.5. The third is proportional to the commutator of η^α and ω^α which vanishes. The fourth term vanishes because the Killing vector η^α is divergenceless, and the final term vanishes because ω and $\eta^\mu\eta_\mu$ are constant along the integral curves of η^α. Therefore we find that

$$2\nabla_\alpha\left(\frac{\omega\omega^{[\alpha}\eta^{\beta]}}{(\eta^\mu\eta_\mu)^2}\right) = \frac{\omega^\alpha\omega_\alpha}{(\eta^\mu\eta_\mu)^2}\eta^\beta . \tag{9.12}$$

We integrate equation (9.12) over any asymptotically flat spacelike surface, Σ .

$$2\oint_{\partial\Sigma}\sqrt{-g}\;\frac{\omega\omega^{[\alpha}\eta^{\beta]}}{(\eta^\mu\eta_\mu)^2}\,d^2x_{\alpha\beta} = \int_\Sigma \sqrt{-g}\;\frac{\omega^\alpha\omega_\alpha}{(\eta^\mu\eta_\mu)^2}\eta^\beta\,d^3x_\beta . \tag{9.13}$$

The left hand side of eq. (9.13) comes from applying Stokes theorem to the integral of the left hand side of eq. (9.12). When the integrand on the left hand side of eq. (9.13) is evaluated in the asymptotic form of the metric discussed in §6.3, we find that

$$\sqrt{-g}\;\frac{\omega\omega^{[\alpha}\eta^{\beta]}}{(\eta^\mu\eta_\mu)^2}\,d^2x_{\alpha\beta} \simeq O(r^{-1}). \tag{9.14}$$

Therfore, the integral on the left side of eq. (9.13) vanishes. The integrand on the right hand side is negative definite since ω^α is a spacelike vector, $\omega^\alpha\omega_\alpha > 0$, and since the surface is spacelike, $\eta^\alpha d^3x_\alpha < 0$. ∎

9-3 Are Static Stars Spherical?

We would all be quite surprised if someone presented us with a static non-spherical general relativistic stellar model. It has been shown (Theorem 4.8) that there are no Newtonian models of this sort. Thus, it seems plausible to believe that a relativistic example must involve strong fields accompanied by their significant non-linearities. On the other hand, Israel (1967) has shown that static black holes must be spherical. Since the non-linearities of strong gravitational fields cannot conspire to spoil the spherical symmetry of this related problem, it is unlikely that they can prevent spherical symmetry in the case of non-singular stellar models. Yet, no proof has been found which shows that no model exists which lacks spherical symmetry. In the next three sections we attempt to summarize some of the thought which has gone into this problem. Our summary is almost certainly incomplete since this is an unsolved problem, and much of the work has not been published.

Let us first discuss some of the obvious general features which any proof of the "static stars are spherical" theorem must posess. It is clear, for example, that it is necessary to assume that the matter is a fluid which cannot support stresses. If we allowed solid matter we could certainly find solutions which represented a static ice axe (for example) in an otherwise empty and asymptotically flat spacetime. Furthermore, the proof must use the fluid matter assumption in some global way. If the solutions were composed of fluid "almost everywhere" but contained some small amount of non-spherical solid matter, then the entire solution would be non-spherical because of the long range effects of the gravitational field. Therefore, the proof that static stars are spherical must somehow examine the nature of the matter everywhere at once to ensure that it is a fluid. Another important assumption is the boundary condition at spacelike infinity, i.e. asymptotic flatness. If we were to replace this boundary condition by non-spherically symmetric boundary conditions, then the solutions would fail to be spherically symmetric also. For example if we required the asymptotic form of the metric to approach some anisotropic cosmological model, then we would not expect to find that static stars were spherical. Therefore, we see that the proof must somehow check the boundary conditions to see if they are of the appropriate type.

What kind of proof can fulfill these criteria? One type of proof that has been used with great success on problems in general relativity is illustrated by the proof of Lichnerowicz's theorem in the last section. An heuristic outline of the proof goes as follows. We used the assumptions about the matter at each point to derive eq. (9.12). This expression has the form of a divergence equaling a definite signed quantity. Integrating the expression over the star insures that the assumptions are satisfied at all points. The divergence is converted into a surface integral, which vanishes because the appropriate boundary conditions are satisfied. The remaining integral therefore vanishes. Its integrand has definite sign, therefore we conclude that the integrand vanishes, which establishes that which we set out to prove. Other important theorems in relativity theory have been established in qualitatively the same way: e.g. Robinson's (1975) proof of the uniqueness of the stationary black hole solutions, and Israel's (1967) proof that static black holes must be spherically symmetric. This type of proof meets our criteria of being essentially global in nature, and of using the boundary conditions in a crucial way.

Therefore, it seems reasonable to look for a proof of the "static stars are spherical" theorem which is of this form. We will discuss some attempts in §9.6.

9-4 Mathematical Machinery

In this section we will write down Einstein's equations in a system of coordinates which appear to be useful ones for the study of the static star problem. The equations have been written in essentially this way by Avez (1964) and Künzle (1971) in their studies of static stellar models, and by Israel (1967), Müller zum Hagen, Robinson and Seifert (1973) and Robinson (1977) in their studies of static black holes.

We have assumed that the Killing vector field, η^{α}, which defines the stationarity of the stellar model, is also twist free. By Frobenius' theorem (see e.q. Flanders 1973, p. 92), the twist free condition is equivalent to the existence of functions t and V (at least locally) such that

$$\eta_{\alpha} = - V^2 \nabla_{\alpha} t. \tag{9.15}$$

We can choose the function t as a time coordinate. By defining three additional coordinates on one of the t=constant surfaces, and Lie transporting them along the integral curves of η^{α}, we obtain a system of coordinates in which the metric has the form:

$$ds^2 = - V^2 d\, t^2 + g_{ij}\, dx^i\, dx^j \ , \tag{9.16}$$

where i,j = 1,2,3. We note that Killing's equations in this system of coordinates takes the simple form

$$\partial_t V = 0 = \partial_t\, g_{ij} \ . \tag{9.17}$$

The three dimensional metric, g_{ij}, defines a covarient derivative which we denote by semicolon (;) and an associated curvature tensor which we denote by ${}^3R_{abcd}$. For three manifolds we recall that the Riemann tensor can be related to the Ricci and scalar curvatures by the expression

$$
{}^3R_{abcd} = g_{ac}\,{}^3R_{bd} - g_{ad}\,{}^3R_{bc} + g_{bd}\,{}^3R_{ac} - g_{bc}\,{}^3R_{ad} + \frac{1}{2}\,{}^3R\,(g_{ad}g_{bc} - g_{ac}g_{bd}). \qquad (9.18)
$$

Since the Weyl curvature vanishes in three dinensions, the following tensor field is used to measure the conformal structure of the geometry:

$$
R_{ijk} = {}^3R_{ij;k} - {}^3R_{ik;j} + \frac{1}{4}(g_{ik}\,{}^3R_{;j} - g_{ij}\,{}^3R_{;k}). \qquad (9.19)
$$

The tensor R_{ijk} vanishes if and only if the three geometry is conformally flat (see e.g. Eisenhart 1926, p. 89).

Einstein's equations for the static perfect fluid in this special coordinate system are given by

$$
V_{;a}{}^{a} = 4\pi V(\rho+3p), \text{ and} \qquad (9.20)
$$

$$
{}^3R_{ab} = V^{-1}\,V_{;ab} + 4\pi(\rho-p)g_{ab}. \qquad (9.21)
$$

Equations (9.20) and (9.21) can be used to find an expression for the scalar curvature:

$$
{}^3R = 16\pi\rho. \qquad (9.22)
$$

The Bianchi identities for the three dimensional curvature are equivalent to

$$
p_{;a} + (\rho+p)\,V^{-1}\,V_{;a} = 0. \qquad (9.23)
$$

We recognize eq. (9.23) as Euler's equation for this static situation. Note that this equation implies the ρ and p have the same level surfaces as the function V.

We will next pick a system of coordinates within the three dimensional surfaces, t = constant. Let ϕ denote a function with $\partial_t \phi = 0$, and $\phi_{;a} \neq 0$ within some open neighborhood (our coordinate chart). We choose the two additional coordinates, by picking coordinates on some (ϕ,t) = constant two surface, and Lie transporting them along the integral curves of η^α and $\nabla^\alpha \phi$. The resulting form of the metric tensor is given by

$$ds^2 = -V^2 \, dt^2 + W^{-1} \, d\phi^2 + g_{AB} \, dx^A \, dx^B, \tag{9.24}$$

where A,B = 1,2. We denote the intrinsic covarient derivative related to the metric g_{AB} by $D_{A;}$ and the scalar curvature of the (ϕ,t) = constant two-surfaces by ${}^2R$. The extrinsic curvature of these two-surfaces is defined by

$$K_{AB} \equiv \frac{1}{2} \, W^{1/2} \, \partial_\phi \, g_{AB}. \tag{9.25}$$

We also use the notations:

$$\sqrt{g} \equiv \det g_{AB}, \tag{9.26}$$

$$K \equiv g^{AB} \, K_{AB} = W^{1/2} \, \partial_\phi \, \log \sqrt{g} \, , \tag{9.27}$$

$$\psi_{AB} \equiv K_{AB} - \frac{1}{2} \, g_{AB} \, K, \text{ and} \tag{9.28}$$

$$Q \equiv W^{1/2} \, \partial_\phi \, V. \tag{9.29}$$

In terms of these quantities, Einstein's equations can be written as follows:

$$\partial_\phi Q = - D^A\{W^{-1/2} \, D_A V\} - W^{-1/2} \, KQ + 4\pi V \, W^{-1/2}(\rho+3p), \tag{9.30}$$

$$\partial_\phi(KV^{-1}) = - V^{-2} D^A\{V D_A W^{-1/2} - W^{-1/2} D_A V\} - \frac{1}{2} V^{-1} W^{-1/2} K^2$$
$$- V^{-1} W^{-1/2} \psi^{AB} \psi_{AB} - 8\pi V^{-1} W^{-1/2}(\rho+p) \quad , \qquad (9.31)$$

$$D_B(V\psi_A{}^B) = D_A Q + \frac{1}{2} V^2 D_A(KV^{-1}) \quad , \qquad (9.32)$$

$$\partial_\phi(V\psi^A{}_B) = - VW^{-1/2} K \psi^A{}_B - V\{D^A D_B W^{-1/2} - \frac{1}{2} \delta^A{}_B D^C D_C W^{-1/2}\}$$
$$- W^{-1/2}\{D^A D_B V - \frac{1}{2} \delta^A{}_B D^C D_C V\} \quad , \text{ and} \qquad (9.33)$$

$${}^2R = \frac{1}{2} K^2 + 2 V^{-1} KQ - \psi_{AB} \psi^{AB} - 16\pi p + 2V^{-1} D^A D_A V. \qquad (9.34)$$

Einstein's equations as written in eqs. (9.30) - (9.34) represent no loss of generality for the static problem. Coordinates which put the metric is the form of eq. (9.24) can always be found locally.

Why have we gone to the trouble to write out Einstein's equations in this detailed way? We are trying to show that a certain spacetime has spherical symmetry. It is sufficient to show that there is a function ϕ , whose level surfaces have the topology S^2, for which the following geometrical quantities vanish:

$$D_A V = D_A W = D_A K = \psi_{AB} = 0. \qquad (9.35)$$

Let us show why these conditions are sufficient to guarantee spherical symmetry. We first note that it follows from eqs. (9.34) and (9.35) that the two-dimensional curvature is a constant on each $\phi =$ constant surface:

$$D_A {}^2R = 0 . \qquad (9.36)$$

Since each of these surfaces has the topology of a sphere, it follows from, eq. (9.36) that the intrinsic geometry of each surface must be spherical also. Therefore, each ϕ = constant surface admits three independent Killing vector fields which satisfy the intrinsic Killings' equation,

$$D_A \xi_B + D_B \xi_A = 0. \tag{9.37}$$

By writing out the components of the four-dimensional Killing's equation in the coordinates of eq. (9.24), it is not hard to see that these intrinsic Killing vector fields also satisfy the full four-dimensional Killing's equation whenever eq. (9.35) is satisfied. Therefore, the existence of a function ϕ which satisfies the conditions of eq. (9.35) is sufficient to guarantee the spherical symmetry of the spacetime.

How can we find a function ϕ which has the properties of eq. (9.35)? One obvious choice is the function V. This function has no critical points near spacelike infinity, and the topology of its level surfaces is S^2 near infinity (see Künzle 1971). Furthermore, when one chooses $\phi = V$, the condition $D_A V = 0$ is automatically satisfied. Since V has these desirable properties, most authors have made the choice $\phi = V$. This choice is not valid globally however. The function V must have a critical point at the center of the star, and *a priori* may have any number of other critical points also. Israel's (1967) original proof that static black holes are spherical, explicitely assumed that V had no critical points so that the above coordinate choice could be made globally. Müller zum Hagen, Robinson and Seifert (1973) have since shown that this assumption is unnecessary for the black hole case. We shall take Israel's approach and assume that V has no critical points, except for the one at the center of the star. This assumption is probably unnecessary, but we will worry about eliminating it only after proving that static stars must be spherical in this case.

Whenever the function V is free of critical points, Einstein's equations simplify considerably by choosing $\phi = V$ in eqs. (9.24) - (9.34). In particular, we note that eqs. (9.30)-(9.34) simplify to

$$\partial_V W = -2KW^{1/2} + 8\pi V(\rho+3p) \ , \tag{9.38}$$

$$\partial_V K = -\frac{1}{2} W^{-1/2} K^2 + V^{-1}K - D^A D_A (W^{-1/2}) - W^{-1/2} \psi^{AB}\psi_{AB} - 8\pi W^{-1/2}(\rho+p) \ , \tag{9.39}$$

$$D_B \psi^B{}_A = D_A\{W^{1/2} V^{-1} + \frac{1}{2} K\} \ , \text{ and} \tag{9.40}$$

$$\partial_V \psi^A{}_B = - D^A D_B W^{-1/2} + \frac{1}{2} \delta^A{}_B D^C D_C W^{-1/2} - (V^{-1}+W^{-1/2}K)\psi^A{}_B . \tag{9.41}$$

Since $D_A V = 0$ is automatically satisfied for this choice of surface, the Bianchi identities of these equations are given by

$$D_A\rho = D_A p = 0, \text{ and} \tag{9.42}$$

$$\partial_V p = - V^{-1}(\rho+p). \tag{9.43}$$

The conditions for spherical symmetry for this system are given by

$$D_A W = D_A K = \psi_{AB} = 0. \tag{9.44}$$

The problem of proving that static stars are spherical reduces, therefore, to showing that eq. (9.44) is satisfied by any asymptotically flat singularity free solution of eqs. (9.38)-(9.43).

Avez (1964) and Künzle (1971) have shown that the conditions in eq. (9.44) are not independent of each other. They show that the single condition

$$D_A W = 0 \tag{9.45}$$

implies all of eq. (9.44) whenever Einstein's equations are satisfied. The proof of this goes as follows. Take the two dimensional gradient of eq. (9.38). When eq. (9.45) is satisfied this condition implies $D_A K = 0$. Similarly, by taking the gradient of eq. (9.39) we learn that $D_A(\psi^B{}_C\psi^C{}_B) = 0$. Therefore $\psi^A{}_B\psi^B{}_A$ is constant on each level surface of V. This constant must be zero for the following reason. The tensor $\psi^A{}_B$ may always be decomposed in the following way:

$$\psi^A{}_B = -\frac{1}{2}(\chi^c\chi_c)\delta^A{}_B + \chi^A\chi_B\ . \tag{9.46}$$

The vector χ^A is one of the eigenvectors of the tensor $\psi^A{}_B$. Whenever $\psi^A{}_B \neq 0$ the eigenvectors are distinct because $\psi^A{}_B$ is trace free. Since the level surfaces of the function V are topologically S^2, the vector field χ^A must have a zero on each level surface. At the point where χ^A vanishes, $\psi^A{}_B\psi^B{}_A$ vanishes also. Therefore $\psi^A{}_B\psi^B{}_A$ vanishes everywhere, and consequently $\psi^A{}_B$ vanishes everywhere also. Thus we have shown that eq. (9.45) is sufficient to guarantee the spherical symmetry of the stellar model. Avez (1964) and Künzle (1971) actually prove a somewhat more general result. They make no assumption about the critical points of V, but assume that W is a function only of V. They show that this assumption, along with Einstein's equation and the asymptotically flat boundary conditions are sufficient to guarantee the spherical symmetry of the star. Their derivation is somewhat involved, and it will not be discussed here.

9-5 Review of the Literature

This section will present a brief (but complete) review of the published literature related to the problem of proving that static stars are spherical in general relativity theory. The first attempts to prove this result were given by Avez (1963) and (1964). The 1963 paper is a brief anouncement of Avez's result with a summary of the proof, while the 1964 paper presents the details of the arguments. Avez claims to prove that static stars must be spherically symmetric, subject to the assumption that the function V have no degenerate critical points. Unfortunately, Avez made an algebraic error in the proof of his Lemma 2 (see Avez 1964, p. 297). His proof is only correct when one adds the additional assumption that the function W depends only on the function V: $W = W(V)$.

Künzle (1971) generalizes Avez's result somewhat by removing the assumption about the critical points of the function V; however, he was not able to eliminate the unpleasant assumption $W = W(V)$. Künzle (1971) also considers the problem of proving that the staic spherically symmetric stellar models are isolated from other possible static models by proving that they admit no static solutions to the linearized equations. He proves that there are no (non-trivial) static perturbations of a static spherically symmetric stellar model which leave the central pressure and central "gravitational potential" V unchanged. Unfortunately these are not the appropriate physical constraints to put on the perturbations. Rather than holding the central pressure and V fixed for the perturbation, one should demand that the perturbation does not change the total number of particles in the fluid. It is possible that Künzle's restrictions on the perturbations are equivalent to the correct physical restrictions, however, this has not been shown yet.

The most recent publication concerning the spherical symmetry of static stars has been given by Marks (1977). Marks claims to prove the general result, static stellar models must be spherical. His proof is fallacious however, and his arguments do not appear to add any new insight into the problem.

Two other interesting papers which discuss static asymptotically flat spacetimes in general relativity are given by Müller zum Hagen (1970) and (1974). He considers the question of whether gravity is always attractive in the case of static models. In the course of these studies, he proves some results about the topology of the level surfaces of the function V. He also considers the possibility of having two separated bodies in a static spacetime.

9-6 Lasciate Ogni Speranza, Voi Ch' Entrate!

The problem of proving that static general relativistic stellar models must be spherical has intrigued a large number of scholars. A great deal of effort has been expended on this problem. With an unsolved problem of this sort, the largest portion of the ideas and work on the problem never appear in print, even though a description of the blind alleys and false starts would be extremely helpful to anyone who cared to follow. The purpose of this section therefore is to present some of the main thoughts and approaches which this author has made in his studies of static stellar models. Hopefully this material will be useful to anyone who studies this problem in the future. We feel obligated, however, to issue Dante's (1310) warning to anyone who would follow the approach given here too closely "Lasciate ogni speranza, voi ch' entrate!" Abandon all hope, ye who enter here.

The results presented here will be given without detailed proofs since none of the results is of great inherent interest, and the proofs require only a rather straightforward application of the appropriate Einstein equations to verify there validity.

In terms of the 3+1 dimensional decomposition of the spacetime introduced in eq. (9.16) we can write out explicit formulas for some geometrical objects of interest. The three-dimensional curvature tensor is given by

$$^3R_{eabc} = V^{-1}\{g_{eb}V_{;ac} - g_{ec}V_{;ab} + g_{ac}V_{;eb} - g_{ab}V_{;ec}\} + 8\pi p(g_{ec}g_{ab} - g_{eb}g_{ac}). \qquad (9.47)$$

The three dimensional conformal tensor (see eq. 9.19) is given by

$$R_{abc} = -V^{-2}\{4V_{;a[b}V_{;c]} + g_{a[b}W_{;c]} - 8\pi V(\rho+3p)g_{a[b}V_{;c]}\}. \qquad (9.48)$$

The square of the conformal tensor is also an interesting object.

$$R_{abc}R^{abc} = 4V^{-4}WW_{;a}{}^{a} - 4V^{-5}WV^{;a}W_{;a} - 3V^{-4}W_{;a}W^{;a} - 32\pi V^{-3}W\,V^{;a}(\rho+3p)_{;a} - 64\pi V^{-4}W^2(\rho+p) + 16\pi V^{-3}(\rho+3p)V^{;a}W_{;a} - 64\pi^2V^{-2}W(\rho+3p)^2. \qquad (9.49)$$

Another object which has potential interest for this problem is the following combination of four-dimensional curvature tensors:

$$I = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \qquad (9.50)$$

The integral of this object is a topological invarient of a four manifold, just as the scalar curvature ^2R in two dimensions is related to the Euler characteristic of a two manifold via the Gauss-Bonnett theorem. If we assume that the topology of our four-manifold is R^4 and if the space-time is static and asymptotically flat, then the integral of I vanishes. In terms of the 3+1 dimensional decomposition of the geometry, I is given by

$$I = 8V^{-2}\, V_{;ab}\, V^{;ab} - 128\pi^2(\rho+p)(\rho+3p) \tag{9.51}$$

Since our stellar model is static, the integral of I over any t = constant surface must also vanish. Perhaps knowing that the integral of eq. (9.51) is zero will help prove that static stars are spherical.

We can also write these objects in terms of the 2+1+1 decomposition of the geometry introduced by eq. (9.24) with $\phi = V$. The square of the conformal tensor (eq. 9.49) reduces to a particularly elegant form in this notation

$$R_{abc}\, R^{abc} = 8V^{-4}\, W^2\{\psi_{AB}\psi^{AB} + \frac{1}{8}\, W^{-2}\, D_A W D^A W\} \ . \tag{9.52}$$

Note that eq. (9.52) does not depend on the density and pressure at all. This is exactly the same expression as the one for a vacuum spacetime. Also note that if the left hand side of eq. (9.52) vanishes, the stellar model is necessarily spherical because of eq. (9.45). Therefore, spatially conformally flat stellar models are necessarily spherical.

The proof that static black holes must be spherical is accomplished by showing that the scalar $R_{abc}\, R^{abc}$ is proportional to a divergence (see Robinson 1977). For the vacuum case, eq. (9.49) can be written as a divergence in the following two ways

$$\frac{1}{4}\, V^3\, W^{-7/4}\, R_{abc}R^{abc} = (W^{-3/4}V^{-1}W^{;a})_{;a}, \quad \text{and} \tag{9.53}$$

$$\frac{1}{16}\, V^5\, W^{-7/4}\, R_{abc}R^{abc} = (\frac{1}{4}\, W^{-3/4}\, VW^{;a} - 2W^{1/4}V^{;a})_{;a} \ . \tag{9.54}$$

One argues from the boundary conditions that the integrals of these quantities must vanish. It follows that the black hole must be spatially conformally flat, and then from eq. (9.52) that it must be spherical. This author, along with Steven Detweiler, attempted to carry out an analogous program for the case of fluid stellar models. We attempted to find a divergence proprotional to the right hand side of eq. (9.49). We were unsuccessful in these attempts however.

Continuing now with our program of writing out the interesting geometrical quantities in the coordinates of eq. (9.24), we find that the scalar I from eq. (9.51) can be written as

$$\sqrt{g}\ VW^{-1/2}\ I = \partial_V\{-8\ \sqrt{g}\ WV^{-1}K + 64\pi\sqrt{g}\ W^{1/2}\ p\} + 4\ \sqrt{g}\ V^{-1}\ D^A(W^{-1/2}\ D_A W). \tag{9.55}$$

The factor $\sqrt{g}\ VW^{-1/2}$ is just the four dimensional volume element. We see that the right hand side of eq. (9.55) is a divergence. But we already knew that the integral of I vanished, therefore eq. (9.55) is not really a useful one in our search for a proof that static stars are spherical.

To conclude this chapter, let us describe a systematic approach which this author has taken in the search for an appropriate divergence identity. Each of the functions which appear in Einstein's equations (9.38)-(9.41) can be classified by how many derivatives of the metric tensor it represents. Thus the functions $\sqrt{g}$, W, and g_{AB} are assigned the order zero. The functions K, ψ_{AB}, $D_A W$, etc. are assigned order one. Similarly the fuctions ${}^2R$, p, $D^A D_A W$, $\partial_V \psi^A{}_B$, etc. have order two. And finally, to make everything consistent, we assign to the function V the order minus one. In terms of this classification scheme, we can observe that Einstein's equations each have a well defined order: eq. (9.38) has order one, while the eqs. (9.39)-(9.41) each have order two.

We can now consider writing down divergences of the form

$$\partial_v(\sqrt{g}\ \lambda) \tag{9.56}$$

where λ is a scalar function composed of the various functions of the problem: W, K, ψ_{AB}, etc. We will attempt to write down all divergences of this form, where λ is a polynomial function of W, K, ψ_{AB} and their derivatives, and we will classify these divergences by the order of λ. We use Einstein's equations to evaluate the derivative ∂_v in these expressions. Since Einstein equations have definite order, the divergence equations which are produced in this way will all have definite order. The object of our approach is to produce all possible divergences of each order. Then by taking linear combinations of the various divergences within each order, we will attempt to find an equality between a divergence and a definite signed quantity.

The possible expressions for the scalar λ will now be given for orders zero through three:

Order zero: W^n

Order one: $W^n K$

Order two: $W^n\ D^A W\ D_A W$

$W^n K^2$

$W^n\ \psi^A{}_B\ \psi^B{}_A$

$W^n\ p$

Order three: $W^n K\, D^A W\, D_A W$

$W^n\, \psi^{AB}\, D_A W\, D_B W$

$W^n\, K^3$

$W^n K\, \psi^A{}_B\, \psi^B{}_A$

$W^n\, D_A W\, D^A K$

$W^n\, K\, p$

In these expressions n is an arbitrary constant. We note that there are other expressions in some of these orders which when introduced into expression (9.56), differ from those given here only by a divergence, e.g. in second order the expression $W^n\, D^A\, D_A W$. These terms have not been included in our list here. We use Einstein's equations to evaluate these functions in expression (9.56):

<u>Order zero:</u>

$$\partial_v(\sqrt{g}\, W^n) = (1-2n)\sqrt{g}\; W^{n-1/2}\, K + 8\pi n\, \sqrt{g}\, V\, W^{n-1}(\rho+3p) \tag{9.57}$$

<u>Order one:</u>

$$\partial_v(\sqrt{g}\, W^n K) = (\tfrac{1}{2} - 2n)\sqrt{g}\, W^{n-1/2}\, K^2 + \sqrt{g}\, V^{-1}\, W^n\, K - \sqrt{g}\, W^n\, D^A\, D_A(W^{-1/2})$$
$$+ 8\pi n\sqrt{g}\; VW^{n-1}K(\rho+3p) - \sqrt{g}\; W^{n-1/2}\{\psi_{AB}\, \psi^{AB} + 8\pi(\rho+p)\} \tag{9.58}$$

<u>Order two:</u>

$$\partial_v(\sqrt{g}\, W^n\, D^A W\, D_A W) = -2(n+1)\sqrt{g}W^{n-1/2}\, K\, D^A W\, D_A W - 2\sqrt{g}\, W^{n-1/2}\, \psi^{AB}\, D_A W\, D_B W$$
$$+ 8\pi n\sqrt{g}\; VW^{n-1}(\rho+3p) D^A W\, D_A W - 4\sqrt{g}\, W^{n+1/2}\, D^A W\, D_A K \tag{9.59}$$

$$\partial_v(\sqrt{g}\ W^n K^2) = 2\sqrt{g}\ W^n\ V^{-1}\ K^2 - 2n\sqrt{g}\ W^{n-1/2}\ K^3 + 8\pi n\sqrt{g}\ VW^{n-1}K^2(\rho+3p)$$

$$-2\sqrt{g}\ W^n K\ D^A D_A(W^{-1/2}) - 2\sqrt{g}\ W^{n-1/2}K\{\psi^{AB}\ \psi_{AB} + 8\pi(\rho+p)\} \qquad (9.60)$$

$$\partial_v(\sqrt{g}\ W^n\ \psi_{AB}\ \psi^{AB}) = -2\sqrt{g}\ V^{-1}\ W^n\ \psi_{AB}\ \psi^{AB} - (1+2n)\sqrt{g}\ W^{n-1/2}\ K\ \psi_{AB}\ \psi^{AB}$$

$$+ 8\pi n\sqrt{g}\ W^{n-1}V\ \psi_{AB}\psi^{AB}(\rho+3p) - 2\sqrt{g}\ W^n\ \psi^{AB}\ D_A\ D_B(W^{-1/2}) \qquad (9.61)$$

$$\partial_v(\sqrt{g}\ W^n p) = (1-2n)\sqrt{g}\ W^{n-1/2}\ K\ p + 8\pi n\sqrt{g}\ W^{n-1}p\ (\rho+3p)$$

$$- \sqrt{g}\ V^{-1}\ W^n(\rho+p) \qquad (9.62)$$

<u>Order three:</u>

$$\partial_v(\sqrt{g}\ W^n K\ D_A W\ D^A W) = \sqrt{g}\ V^{-1}\ W^n K\ D^A\ WD_A W - 4\sqrt{g}\ W^{n+1/2}K\ D^A W\ D_A H$$

$$-\sqrt{g}\ W^{n-1/2}\{\psi_{AB}\psi^{AB} + 8\pi(\rho+p) + (\tfrac{5}{2} + 2n)\ K^2\}D^A W\ D_A W$$

$$+ 8\pi n\sqrt{g}\ V\ W^{n-1}K(\rho+3p)D^A W\ D_A W - \sqrt{g}\ W^n\ D^A D_A(W^{-1/2})D^B W\ D_B W$$

$$- 2\sqrt{g}\ W^{n-1/2}\ K\ \psi^{AB}\ D_A W\ D_B W \qquad (9.63)$$

$$\partial_v(\sqrt{g}\ W^n\ \psi^{AB}\ D_A W\ D_B W) = -\sqrt{g}\ V^{-1}\ W^n\ \psi^{AB}\ D_A W\ D_B W$$

$$-2\sqrt{g}\ W^{n-1/2}\ \psi^{AB}\ D_A W\ \psi_A{}^C\ D_C W + 8\pi n\sqrt{g}\ VW^{n-1}(\rho+3p)\psi^{AB} D_A W\ D_B W$$

$$-\sqrt{g}\ W^n\ D^A D^B(W^{-1/2})D_A W\ D_B W + \tfrac{1}{2}\sqrt{g}\ W^n\ D^A\ D_A(W^{-1/2})D^B W\ D_B W$$

$$-(3+2n)\sqrt{g}\ W^{n-1/2}K\ \psi^{AB}\ D_A W\ D_B W - 4\sqrt{g}\ W^{n+1/2}\psi^{AB}\ D_A W\ D_B K \qquad (9.64)$$

$$\partial_v(\sqrt{g}\, W^n K^3) = 3\sqrt{g}\, V^{-1} W^n K^3 - (\tfrac{1}{2} + 2n)\sqrt{g}\, W^{n-1/2} K^4$$

$$+ 8\pi n\sqrt{g}\, VW^{n-1} K^3(\rho+3p) - 3\sqrt{g}\, W^n K^2 D^A D_A(W^{-1/2})$$

$$- 3\sqrt{g}\, W^{n-1/2} K^2\{ \psi_{AB}\psi^{AB} + 8\pi(\rho+p) \} \tag{9.65}$$

$$\partial_v(\sqrt{g}\, W^n K\psi^{AB}\psi_{AB}) = - \sqrt{g}\, W^n V^{-1} K \psi^{AB}\psi_{AB} - 2\sqrt{g}\, W^n K\psi^{AB} D_A D_B(W^{-1/2})$$

$$-\sqrt{g}\, W^{n-1/2} \psi_{AB}\psi^{AB}\{\psi_{CE}\psi^{CE} + (\tfrac{3}{2} + 2n)K^2 + 8\pi(\rho+p) \}$$

$$+ 8\pi n\sqrt{g}\, V W^{n-1} K \psi_{AB}\psi^{AB}(\rho+3p) - \sqrt{g}\, W^n\psi_{AB}\psi^{AB} D^C D_C(W^{-1/2}) \tag{9.66}$$

$$\partial_v(\sqrt{g}\, W^n D_A W D^A K) = \sqrt{g}\, V^{-1} W^n D_A W D^A K - 2\sqrt{g}\, W^{n+1/2} D^A K D_A K$$

$$+ \tfrac{1}{2}\sqrt{g}\, W^{n-3/2}\{ \tfrac{1}{2} K^2 + \psi_{AB}\psi^{AB} + 8\pi(\rho+p)\} D^A W D_A W$$

$$-2(1+n)\sqrt{g}\, W^{n-1/2} K D_A W D^A K + 8\pi n\sqrt{g}\, VW^{n-1}(\rho+3p) D^A W D_A K$$

$$-2\sqrt{g}\, W^{n-1/2} \psi^{AB} D_A W D_B K - \sqrt{g}\, W^n D^A W D_A(D^B D_B[W^{-1/2}])$$

$$-\sqrt{g}\, W^{n-1/2} D^A W D_A(\psi_{BE}\psi^{BE}) \tag{9.67}$$

$$\partial_v(\sqrt{g}\, W^n K p) = - \sqrt{g}\, V^{-1} W^n K \rho + (\tfrac{1}{2} - 2n)\sqrt{g}\, W^{n-1/2} K^2 p$$

$$-\sqrt{g}\, W^{n-1/2} p \{\psi_{AB}\psi^{AB} + 8\pi(\rho+p)\} - \sqrt{g}\, W^n p D^A D_A(W^{-1/2})$$

$$+ 8\pi n\sqrt{g}\, W^{n-1} V Kp (\rho+3p) \tag{9.68}$$

This approach was not pursued to fourth order for reasons that must be obvious at this point.

What good are all of these equations? Let us try to make a useful expression from these by adjusting the value of the parameter n, and also by taking suitable linear combinations of the various equations. We note that the right hand sides of equations within the same order tend to have the same sorts of terms. Thus by taking linear combinations of expressions within each order we can eliminate some of the terms having indefinite signs. The hope is to find an expression which is an equality between a divergence and a definite signed quantity, which vanishes if and only if the star is spherical.

Let us begin with order zero. Here we have only one equation, (9.57) so we can only attempt to adjust the parameter n to obtain interesting expressions. The obvious choice for n is n = 1/2.

$$\partial_V(\sqrt{g}\ W^{1/2}) = 4\pi\sqrt{g}\ VW^{-1/2}(\rho+3p) \ . \tag{9.69}$$

This equation is interesting. In the vacuum case this equation, (9.69), is one of the divergences used in the proof that static black holes must be spherical (see Israel 1967). The integral of eq. (9.69) gives an expression for the total mass of the star as seen from infinity in terms of the local energy densities:

$$m = \int \sqrt{g}\ VW^{-1/2}(\rho+3p)\ dVd^3x \tag{9.70}$$

In order one (eq. 9.58) the only obvious simplification is made by choosing n = 1/4.

$$\begin{aligned}\partial_V(\sqrt{g}\ W^{1/4}\ V^{-1}K) = &- 2\sqrt{g}\ D^A\ D_A(W^{-1/4}) - 2\pi\sqrt{g}\ W^{-3/4}\ K(\rho+3p)\\ &-\sqrt{g}\ V^{-1}\ W^{-1/4}\{\psi_{AB}\psi^{AB} + \frac{1}{8}W^{-2}\ D^AW\ D_AW\}\\ &- 8\pi\sqrt{g}\ V^{-1}\ W^{-1/4}(\rho+p).\end{aligned} \tag{9.71}$$

Although this equation does not appear to be useful in the fluid case, the vacuum limit of this equation has precisely the correct form. In fact, this is another of the divergences which are used to prove that static black holes are spherical. Thus, it appears that our technique is working somewhat, since we are able to recover the black hole results so easily.

Moving on to second order, we have a larger number of equations to work with. We take a general linear combination of eqs. (9.59)-(9.62), and adjust the coefficients in this sum to eliminate terms which do not have a definite sign such as $D^A K\, D_A W$, $\psi^{AB}\, D_A W\, D_B W$ and $\psi^{AB}\, D_A D_B (W^{-1/2})$. The resulting expression is given by,

$$\begin{aligned}
&\partial_v(\sqrt{g}\, W^n\, D_A W\, D^A W) - 8\, \partial_v(\sqrt{g}\, W^{n+2}\, \psi_{AB}\psi^{AB}) \\
&\quad + 8\sqrt{g}\, D_A\{W^{n+1/2}\, \psi^{AB}\, D_B W\} = \\
&\quad = 16\sqrt{g}\, W^{n+2}\{V^{-1} + (n + \tfrac{5}{2})W^{-1/2}K - 4(n+2)\pi V W^{-1}(\rho+3p)\}\psi_{AB}\psi^{AB} \\
&\quad + 4\sqrt{g}\, W^n\{V^{-1} - \tfrac{1}{2}(n+1)W^{-1/2}\, K + 2n\pi\, V W^{-1}(\rho+3p)\}D_A W\, D^A W.
\end{aligned} \tag{9.72}$$

This expression appears quite encouraging. It almost has the correct form. However, no choice of n gives all of the terms on the right hand side the same sign so this equation is probably not useful in our search for a proof that static stars are spherical.

The case of the third order divergences seems somewhat worse than the case of the second order expressions. It is not possible to take linear combinations of eqs. (9.63)-(9.67) and solve for the coefficients in the sum by setting to zero those terms which have unknown sign. There are too many terms with unknown sign, so it is not possible to set them all to zero.

The situation does not appear hopeful for going to higher order in this way. At each level, the terms on the right hand side of the equation will be of one higher order than those on the left. There appears to be an ever increasing diversity of terms as one goes to higher order, thus one never has enough equations to eliminate completely the terms of unknown sign, and the situation appears to get worse and worse as the order of the equations is increased. Perhaps the appropriate next step would be to consider somewhat more complicated forms for the function λ in eq. (9.56). Instead of polynomials, perhaps one should try rational fractions. Or perhaps what is needed is a really fresh new outlook on the problem, a new method of performing these global proofs which does not involve the divergence equals definite signed quantity construction.

§10 STATIONARY STARS ARE AXISYMMETRIC

10-1 Preliminaries

In this chapter we prove that stationary stellar models, which are made of a viscous heat-conducting fluid, must be axisymmetric. This theorem is an example of the "Multum Non Ex Parvo" theorems in general relativity (see Wyler 1974): one assumes that the solutions of Einstein's equations have a certain symmetry and then one finds that those solutions must posess additional symmetries. An example of this kind of result is Birkhoff's (1923) theorem: any spherically symmetric solution of the vacuum Einstein equations must also be static. Other examples are Israel's (1967) theorem that static black holes must be spherically symmetric, and also Hawking's (1972) theorem that stationary black holes must be axisymmetric. I have always found this kind of result to be quite fascinating.

The proof of the "stationary stars are axisymmetric" theorem was first given by Lindblom (1976a). The precise statement of the theorem is as follows.

THEOREM 10.1 - *A stationary (non-static) general relativistic stellar model composed of a viscous, heat-conducting fluid (having non-zero coefficients of heat conduction and viscosity) which satisfies conditions i) and ii) must be axisymmetric.*

In addition to the assumptions included in our definition of a stellar model in §6.3, we must make the following two assumptions:

i) The Killing vector field which defines the stationarity of the spacetime is C^4.

ii) There exists an open subset of the surface of the star which is a level surface of an analytic function, f, of the stationary harmonic coordinates in the exterior of the star with $df \neq 0$ on the surface of the star.

This second condition is somewhat stronger than we should have liked, however, in physics we generally describe nature with piecewise analytic functions. This condition is much weaker than that.

The proof of this theorem is lengthy and somewhat complicated. We present a summary of the proof here, and give the details in the following sections. We begin by recalling Theorem 7.7 and Theorem 7.9. These results show that the interior region of a stationary imperfect fluid must have a Killing vector field which is proportional to the fluid velocity. We have assumed the stellar model is stationary but not static. Theorem 9.1 shows that in this case the fluid velocity cannot be proportional to the Killing vector field which defines the stationarity of the spacetime. Therefore the interior of the fluid has two linearly independent Killing vector fields.

The remaining portion of the proof extends the second Killing vector field to the exterior of the stellar model, and shows that the symmetry which the second Killing vector represents is a rotation. To extend the Killing vector, ξ^α (which is proportional to the fluid velocity), into the exterior of the star we apply several theorems from the literature of partial differential equations to the Cauchy problem for the differential equation $\nabla^\alpha \nabla_\alpha \xi^\beta = 0$, on the surface of the star. This equation is necessarily satisfied by any Killing vector field in the exterior region of the star, thus it is a natural one to use for the extension of ξ^α. A portion of the surface of the star is used as the initial surface, on which the Cauchy data (consisting of the values of the field ξ^α and their first derivatives) are defined by continuity from the interior of the star. The existence of this extension is guaranteed by the Cauchy-Kowalewsky theorem. It is shown that an extension obtained in this way is a Killing vector which commutes with the globally timelike Killing vector. Once extended a short way past the boundary of the star, the field ξ^α can be analytically continued to cover the remainder of the exterior. In this way, the additional symmetry found in the interior of the star is extended to include the entire spacetime.

The final problem is to show that the additional symmetry, which the second Killing vector field represents, is a rotation. An heuristic version of this argument is as follows. The spacetime near spacelike infinity behaves asymptotically as flat Minkowski spacetime, whose symmetries are elements of the Poincare group. A star is not invariant under spacelike translations or velocity boosts. Thus, asymptotically the star admits at most time translations and space rotations. The additional symmetry, being linearly independent of the time translation symmetry (defined by the stationarity of the space), must be some linear combination of a rotation and a time translation. Therefore the star is rotationally or axially symmetric. The rigerous proof of this point has just recently been given by Ashtekar and Xanthopoulos (1978). We state this result as Theorem 10.4, but the details of their proof are lengthy and will not be included here.

To present the rigorous proof of Theorem 10.1 we make use of several theorems from the literature, whose proofs we will not give here. The first is the Cauchy-Kowalewsky theorem, which guarantees the existence of solutions to certain partial differential equations (see Courant and Hilbert, 1962, p. 39).

THEOREM 10.2 - *Consider a system of m partial differential equations, each of order* k, *for the* m *functions* u^1, u^2, ... u^m *of the* n+1 *independent variables* t , y_1,... y_n. *If this system can be written in "normal form",*

$$\partial^k_t\, u^i = f^i(t, y^j, \partial_t\, u^j, \; \dots \; \partial^k\, u^j/\partial y^k_n\,). \tag{10.1}$$

where f^i *depends analytically on each of its arguments, then there exists* *one and only one solution (in some neighborhood of the point* $t = y^j = 0$) *to eq. (10.1) which has prescribed analytic values of the functions* $u^i, \partial_t u^i, \ldots, \partial_t^{k-1} u^i$ *on the surface* $t = 0$.

We will use this theorem to prove the existence of an extension of the Killing vector field ξ^α into the exterior of the star.

The second theorem from the literature which we need to prove Theorem 10.1 is a uniqueness theorem for the initial value problem. The Cauchy-Kowalewski theorem (10.2) proves the uniqueness of analytic solutions of this problem, but leaves open the question of non-analytic solutions. The following theorem due to Holmgren (see Courant and Hilbert 1962 p. 23f) eliminates the possibility of multiple non-analytic solutions also.

THEOREM 10.3 - *If* $L(u)$ *is a linear differential operator with analytic coefficients and if the Cauchy initial data vanish on a smooth noncharacteristic surface* S_o *, then any solution* u *of* $L(u) = 0$ *with these initial data vanishes identically in a suitably small neighborhood of any closed subset of* S_o .

The third result which we need to complete the proof of the axisymmetry theorem classifies the possible symmetries of asymptotically flat spacetimes. Ashtekar and Xanthopoulos (1978) prove that the isometry group of any asymptotically flat spacetime must be a subgroup of the Poincare group. Furthermore for stationary spacetimes they prove the following:

THEOREM 10.4 - *If a stationary asymptotically flat spacetime (having not identically zero Bondi mass) admits a second independent Killing vector field, then the spacetime is axisymmetric.*

The definition of asymptotic flatness which this theorem uses is new and somewhat stronger than the requirement of weak asymptotic simplicity. The reader is advised, therefore, to check the definition given in Ashtekar and Xanthopoulos (1978); (see also Geroch and Horowitz 1978 and §6.3 of this work).

10.2 Extending the Killing Vector

PROOF OF THEOREM 10.1: It follows from theorem 7.7, 7.9 and 9.1 that the interior of the stellar model admits a second Killing vector field, ξ^α, which is linearly independent of the timelike Killing vector, η^α (which defines the stationarity of the spacetime). We will now prove that ξ^α can be extended into the exterior of the star.

We extend ξ^α into the exterior of the star via an initial value problem on the surface of the star. The propagation of ξ^α will be determined by the differential equation

$$\nabla_\alpha \nabla^\alpha \xi^\beta = 0 \tag{10.2}$$

This equation is chosen to define the extension of ξ^α since it is satisfied by any Killing vector field in the exterior of the star. The initial values of the field, ξ^α and $\partial_\alpha \xi^\beta$, will be specified on the surface of the star by taking the limits of the corresponding quantities from the star's interior. These initial values plus the differential equation (10.2) form a Cauchy initial value problem for ξ^α on the surface of the star. The mathematical tool which is used to show the existence of this extension of ξ^α is the Cauchy-Kowalewsky theorem (10.2). This theorem guarantees the existence of a solution of the Cauchy problem in a small neighborhood of the initial surface if a) the differential equation depends analytically on the unknown functions, their derivatives and on the coordinates; and if b) the Cauchy data are analytic functions of the coordinates on the initial surface. The theorem of Müller zum Hagen (1970b) (Theorem 8.7) proves that the components of the metric tensor, $g_{\alpha\beta}$, are analytic functions in the exterior of the star; therefore, condition a) of the Cauchy-Kowalewsky theorem is satisfied by eq. (10.2). The next task is to show that the condition b), the analyticity of ξ^α and $\partial_\mu \xi^\beta$, is satisfied on the surface of the star.

In order to show that the initial values of functions ξ^α and $\partial_\alpha \xi^\beta$ are analytic on the surface of the star, we have had to make a slightly stronger assumption about the surface of the star than would normally have been desirable. We have assumed that the surface of the star must be the level surface of a C^2 function of the external coordinates, and that in a neighborhood of some point x on the surface of the star, the surface is the level surface of an analytic function, f, of the external stationary harmonic coordinates with $df(x) \neq 0$. Given this restriction on the surface of the star, we can show that ξ^α and $\partial_\alpha \xi^\beta$ are analytic functions on the surface in some neighborhood of the point x. Begin by noting that the vector field ξ^α, within the star, is a Killing vector not only of the four-geometry, but also of the three-geometry intrinsic to each surface of constant pressure. To see this, let n^α represent the unit normal to the surfaces of constant pressure. (Note that ξ^α and n^α commute.) The metric tensor intrinsic to these surfaces is given by $\gamma_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta$. Its Lie derivative along ξ^α vanishes. Equivalently, ξ^α satisfies Killing's equation within the surface: $D_i\xi_j + D_j\xi_i = 0$, $i,j = 0,2,3$. D_i represents the covariant derivative related to the intrinsic geometry. Furthermore, since ξ_i is a Killing vector field, it must satisfy (see Lemma 7.1),

$$D_i D^i \xi^j = - {}^3R^j{}_i \xi^i. \qquad (10.3)$$

Equation (10.3) must hold on each surface of constant pressure within the star. In particular then, it must hold on the surface of the star.

We have assumed that a portion of the surface of the star is an analytic function, f, of the stationary harmonic coordinates of the exterior of the star. Since we have assumed that $df(x) \neq 0$ we can choose f as one of the coordinates in a neighborhood of x. The transformation to these adapted coordinates will be analytic. Therefore, the metric tensor will be an analytic function of these adapted coordinates since Theorem 8.7 showed that the metric was an analytic function of the stationary harmonic coordinates. Furthermore, the intrinsic metric on each surface of constant f must be analytic. In particular, then the intrinsic metric of the surface of the star must be analytic in a neighborhood of the point x. Therefore, in this region, eq. (10.3) is an analytic equation for ξ^i. Lemma 8.6 proves that the solutions of eq. (10.3) must be analytic functions in this case. Thus the functions ξ^α are analytic on the surface of the star in some neighborhood of the point x.

All that remains to establish condition b) of the Cauchy-Kowalewsky theorem, is to show that the first derivatives $\partial_\alpha \xi^\beta$ are also analytic functions on the surface of the star. Let n^α be the components of the unit normal vector, and let e^α be the components of an arbitrary analytic vector field which is tangent to the surface of the star. Since ξ^α are analytic functions, it follows that $e^\alpha \partial_\alpha \xi^\beta$ will also be analytic. To learn about the derivatives of ξ^α in the direction normal to the surface, the four dimensional Killing equation is used:

$$0 = \xi^\mu \partial_\mu g_{\alpha\beta} + g_{\alpha\mu} \partial_\beta \xi^\mu + g_{\beta\mu} \partial_\alpha \xi^\mu \ . \tag{10.4}$$

The inner products of this equation with the vectors n^α and e^α give expressions for the normal derivatives:

$$n_\alpha n^\beta \partial_\beta \xi^\alpha = - \frac{1}{2} n^\alpha n^\beta \xi^\mu \partial_\mu g_{\alpha\beta} \ , \tag{10.5}$$

$$e_\alpha n^\beta \partial_\beta \xi^\alpha = - n^\alpha e^\beta \xi^\mu \partial_\mu g_{\alpha\beta} - n_\alpha e^\beta \partial_\beta \xi^\alpha \ . \tag{10.6}$$

The left hand side of eqs. (10.6) and (10.7) give all possible components of the normal derivatives of ξ^α. The right hand sides are composed entirely of functions which are known to be analytic. Thus, we conclude that the Cauchy data ξ^α, $\partial_\alpha \xi^\beta$ are analytic functions on the surface of the star. The Cauchy-Kowalewsky theorem therefore guarantees the existence of a solution of eq. (10.2) with the initial data given above. The vector field ξ^α is thereby extended, at least a small distance into the exterior of the star.

10.3 Properties of the Extension

The vector field ξ^α has been extended a small way into the exterior of the star in the previous section. It is now shown that this extension is a Killing vector field which commutes with the globally timelike Killing vector field η^α. The following identity is satisfied by any vector field in a vacuum spacetime:

$$\nabla_\alpha \nabla^\alpha (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) + 2\,(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) R^{\alpha\ \beta}_{\ \mu\ \nu} = \nabla_\mu \nabla_\alpha \nabla^\alpha \xi_\nu + \nabla_\nu \nabla_\alpha \nabla^\alpha \xi_\mu \ . \tag{10.7}$$

When ξ^α is extended using equation (10.2) the right hand side of equation (10.7) vanishes. The left hand side then becomes an equation for $t_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$, the Killing tester (which vanishes if and only if ξ^α is a Killing vector):

$$\nabla_\alpha \nabla^\alpha t_{\mu\nu} + 2R_{\mu\ \nu}^{\ \alpha\ \beta}\ t_{\alpha\beta} = 0. \tag{10.8}$$

To prove that ξ^α is a Killing vector field, it must be shown that the only solution to eq. (10.8) which is consistent with the boundary conditions is $t_{\alpha\beta} = 0$. The boundary conditions, on the surface of the star, must therefore be examined so that the values of $t_{\alpha\beta}$ and $\partial_\alpha t_{\beta\gamma}$ may be evaluated.

It will now be shown that the tensor $t_{\alpha\beta}$, and its first derivatives, $\partial_\alpha t_{\beta\gamma}$, vanish on the surface. This fact follows from the continuity required by the juction conditions at the surface of the star. The tensor $t_{\alpha\beta}$ is a function of the vector ξ^α, the metric $g_{\alpha\beta}$ and their first derivatives:

$$t_{\alpha\beta} = \xi^\mu \partial_\mu g_{\alpha\beta} + g_{\alpha\mu} \partial_\beta \xi^\mu + g_{\beta\mu} \partial_\alpha \xi^\mu \ . \tag{10.9}$$

The metric and its first derivatives are required to be continuous by Synge's juction conditions (see §6.3). The components of the vector field ξ^α and its first derivatives $\partial_\alpha \xi^\beta$ were required to be continuous in the extensnion of ξ^α. Therefore $t_{\alpha\beta}$ must be continuous across the surface.

Since $t_{\alpha\beta}$ vanishes within the star, it must therefore vanish also on the surface of the star.

The derivatives $\partial_\alpha t_{\beta\gamma}$ must also be continuous across the surface of the star, but this is not as easily seen. These derivatives depend on the metric, the vector field, plus their first and second derivatives:

$$\partial_\alpha t_{\beta\gamma} = \partial_\alpha(\xi^\mu \partial_\mu g_{\beta\gamma} + g_{\beta\mu}\partial_\gamma \xi^\mu + g_{\mu\gamma}\partial_\beta \xi^\mu). \tag{10.10}$$

The only term in eq. (10.10) involving second derivatives of the metric is $\xi^\mu \partial_\alpha \partial_\mu g_{\beta\gamma}$. The vector field ξ^α is tangent to the surface. (This follows from the fact that ξ^α is a Killing vector field within the star, which implies that the gradient of the pressure is orthogonal to ξ^α: $\xi^\mu \nabla_\mu p = 0$; therefore ξ^α must be tangent to the surface of the star.) The Synge juction conditions require that only the second derivatives of the form $n^\alpha n^\beta \partial_\alpha \partial_\beta g_{\mu\nu}$ have discontinuities. None of these terms are present in eq. (10.10) since ξ^μ is tangent to the surface.

It must also be shown that the second derivatives of the form $\partial_\alpha \partial_\beta \xi^\gamma$ are continuous. The second derivatives of the form, $e^\mu \partial_\mu \partial_\alpha \xi^\beta$ will be continuous whenever e^μ is tangent to the surface of the star. Only the second derivatives of the form $n^\alpha n^\beta \partial_\alpha \partial_\beta \xi^\mu$ need to be considered. These derivatives are not determined by the juction conditions, but by the differential equations governing ξ^α. In the exterior of the star, ξ^α satisfied eq. (10.2) by construction. Within the star ξ^α must satisfy the equation

$$\nabla_\alpha \nabla^\alpha \xi^\beta = - R^\beta{}_\alpha \xi^\alpha , \tag{10.11}$$

since it is a Killing vector field (see Lemma 7.1). Since the exterior of the star is a vacuum region, eq. (10.11) is equivalent to eq. (10.2); and therefore eq. (10.11) is satisfied everywhere. The left hand side of eq. (10.11) can be written out in the following way

$$\nabla_\alpha \nabla^\alpha \xi^\beta = - R^\beta{}_\alpha \xi^\alpha + g^{\mu\nu}\partial_\mu \partial_\nu \xi^\beta + \xi^\alpha \partial_\alpha (g^{\mu\nu}\Gamma^\beta{}_{\mu\nu})$$
$$+ B^\beta(\xi,\partial\xi,g,\partial g). \tag{10.12}$$

The term B^β is a function only if the quantities ξ^α, $g_{\alpha\beta}$ and their first derivatives. Therefore B^β is continuous across the surface of the star. We use eqs. (10.11) and (10.12) to find the following expression

$$g^{\mu\nu}\partial_\mu \partial_\nu \xi^\beta = - \xi^\alpha \partial_\alpha (g^{\mu\nu}\Gamma^\beta{}_{\mu\nu}) - B^\beta(\xi,\partial\xi,g,\partial g). \tag{10.13}$$

The right hand side of eq. (10.13) is continuous across the surface of star since the only second derivatives which occur are of the form $\xi^\alpha \partial_\alpha \partial_\beta g_{\mu\nu}$. These second derivatives are continuous since ξ^α is tangent to the surface of the star. Therefore $g^{\mu\nu}\partial_\mu \partial_\nu \xi^\beta$ must be continuous across the surface. All of the second derivatives of ξ^α are continuous except $n^\mu n^\nu \partial_\mu \partial_\nu \xi^\beta$. Since $g^{\mu\nu}\partial_\mu \partial_\nu \xi^\beta$ is continuous, it follows that $n^\mu n^\nu \partial_\mu \partial_\nu \xi^\beta$ must be continuous also. This implies that all of the second derivatives $\partial_\alpha \partial_\beta \xi^\mu$ must be continuous. This completes our argument that the functions $t_{\alpha\beta}$ and $\partial_\alpha t_{\beta\gamma}$ are continuous across the surface of the star. Since ξ^α is a Killing vector field within the star, it follows that $t_{\alpha\beta}$ and $\partial_\alpha t_{\beta\gamma}$ both must vanish on the surface of the star.

The functions $t_{\alpha\beta}$ and $\partial_\alpha t_{\beta\gamma}$ form Cauchy data for the linear differential eq. (10.8). In stationary vacuum spacetimes such as the exterior of the star, the components of the metric tensor are analytic functions when expressed in suitable coordinates. Therefore, eq. (10.8) forms a linear system of partial differential equations with analytic coefficients, for the quantities $t_{\alpha\beta}$. The theorem of Holmgren (Theorem 10.3) guarantees the uniqueness of the solutions of this Cauchy problem. Since $t_{\alpha\beta} = 0$ is a solution, it must be the unique solution. Thus, we finally conclude that when ξ^α is extended according to eq. (10.2), the extension must be a Killing vector field.

It is also useful to show that the extension of ξ^α via eq. (10.2) commutes with the timelike Killing vector η^α. Let us define $\ell^\alpha = \eta^\mu \nabla_\mu \xi^\alpha - \xi^\mu \nabla_\mu \eta^\alpha$, the commutator of ξ^α and η^α. Since η^α is a Killing vector field, it is straightforward to show that,

$$\nabla_\alpha \nabla^\alpha \ell^\beta = \eta^\mu \nabla_\mu \; (\nabla^\alpha \nabla_\alpha \xi^\beta) - (\nabla^\alpha \nabla_\alpha \xi^\mu) \nabla_\mu \eta^\beta \; . \tag{10.14}$$

The right hand side vanishes whenever eq. (10.2) is satisfied. It is therefore possible to use eq. (10.14) and the initial values of ℓ^α and its derivatives as a Cauchy problem on the surface of the star. The initial value data ℓ^α and $\partial_\alpha \ell^\beta$ are functions of ξ^α, η^α and their first and second derivatives. It has already been shown that ξ^α and its first two derivatives are continuous at the surface. The vector field η^α is assumed to be C^4. Thus both ℓ^α and $\partial_\alpha \ell^\beta$ must be continuous functions. Each vanishes within the star; therefore each must vanish on the surface of the star. As before, Theorem 10.3 guarantees that $\ell^\alpha = 0$ is the unique solution of this problem. Thus, the extension of ξ^α commutes with η^α.

The Killing vector ξ^α has been shown to exist inside the star, and in at least a small open neighborhood in the exterior. Since the exterior geometry of the star is analytic, the components ξ^α must also be analytic functions (see Lemma 8.6). These can be extended to cover the entire exterior spacetime by analytic continuation. Ashtekar and Xanthopulos' theorem (10.4) completes the proof of Theorem 10.1 by showing that the stellar model must be axisymmetric since it admits a second Killing vector field. ∎

§11 STATIONARY AXISYMMETRIC STELLAR MODELS

11-1 Introduction

In this section we discuss the stationary and axisymmetric ideal fluid stellar models in general relativity theory. In §10 we showed that stationary viscous stellar models are necessarily axisymmetric. Therefore, the stationary axisymmetric models are an important special case of the general class of stationary ideal fluid models, which may include non-axisymmetric objects analogous to the Dedekind ellipsoids (see Chandrasekhar and Elbert 1974 and Friedman and Schutz 1975).

The majority of the work which is done on rotating stellar models within general relativity theory is concerned with the properties of stationary and axisymmetric models. This work takes place in three main areas. The first area is the study of approximate solutions of Einsteins equations. A great deal of work has been done, for example, on the study of slowly rotating nearly spherical models (which may have strong gravitational fields) by Hartle and Thorne (1968) (for a review see Thorne 1971). Another type of approximate solution has weak fields and slow motions, but need not be almost spherical: the post-Newtonian theory of gravitation (see Shapiro and Lightman 1976). The second major area is the search for exact solutions to the full set of general relativistic field equations. A comprehensive review of the known exact interior fluid solutions is given by Krasinski (1975). Stationary axisymmetric vacuum solutions, which could represent the exterior of a stellar model; are reviewed by Reina and Treves (1976). None of the exact interior solutions has been successfully matched onto an asymptotically flat exterior solution, however. A few numerical solutions, which include both interior and exterior solutions, have also been given, see for example Wilson (1973) and Butterworth (1976).

The third major area is the study of the fundamental properties of stationary axisymmetric stellar models, without reference to any particular equation of state for the fluid, and without using any approximations. In keeping with the main theme of this dissertation, we will review in detail the fundamental properties of stationary axisymmetric models, while neglecting completely the study of any particular models.

We begin by defining some important terms. A spacetime will be called *axisymmetric* if there exists a global spacelike Killing vector field, ξ^α, whose integral curves are diffeomorphic to circles (generically). Carter (1970) has shown that an asymptotically flat axisymmetric spacetime must contain a rotation axis. A *rotation axis* is a timelike two-surface on which the rotational Killing vector field vanishes. Furthermore, Carter (1970) (also Ashtekar and Xanthopoulos 1978) has shown that if a spacetime is stationary and axisymmetric, then the rotational Killing vector field, ξ^α, necessarily commutes with the timelike Killing vector field, η^α:

$$\mathcal{L}_\eta \xi^\beta = \eta^\alpha \nabla_\alpha \xi^\beta - \xi^\alpha \nabla_\alpha \eta^\beta = 0. \qquad (11.1)$$

Whenever the two Killing vector fields, ξ^α and η^α, commute it follows from Frobenius' theorem (see for example Misner 1963) that there exists a family of two-surfaces to which the vector fields, ξ^α and η^α, are everywhere tangent. These two surfaces are called the *surfaces of transitivity* of the symmetry group of the spacetime. Another interesting property which a stationary axisymmetric spacetime may posess is a second family of two-surfaces which are everywhere orthogonal to the surfaces of transitivity. Such a spacetime is called *orthogonally transitive*. The conditions on the Killing vector fields η^α and ξ^α which guarantee orthogonal transitivity are given by

$$\eta^{\alpha\beta\mu\nu}\ \xi_\alpha \eta_\beta\ \nabla_\mu\ \eta_\nu = 0, \text{ and} \tag{11.2}$$

$$\eta^{\alpha\beta\mu\nu}\ \eta_\alpha\ \xi_\beta\ \nabla_\mu\ \xi_\nu = 0. \tag{11.3}$$

These conditions follow from another application of Frobenius' theorem (see for example Flanders 1963, p. 97).

A stationary axisymmetric spacetime, which is orthogonally transsitive, admits a coordinate system (locally) which greatly simplifies the form of the metric tensor. This special coordinate system can be constructed in the following way. Consider one of the two-surfaces which is orthogonal to the Killing vector fields η^α and ξ^α. On this two-surface pick two tangent vector fields, X_1 and X_2, which commute on the two-surface. These two vector fields can be extended to a four-dimensional neighborhood of the original surface by Lie transport along the normal vector fields η^α and ξ^α:

$$\mathcal{L}_\eta\ X_1^\alpha\ = \mathcal{L}_\xi\ X_1^\alpha\ = \mathcal{L}_\eta\ X_2^\alpha\ = \mathcal{L}_\xi\ X_2^\alpha\ = 0. \tag{11.4}$$

It is easy to check that the vector fields X_1^α and X_1^α when defined according to eq. (11.4) have the following properties:

$$\mathcal{L}_{X_1}\ X_2\ = 0, \text{ and} \tag{11.5}$$

$$X_1^\alpha\ \eta_\alpha = X_2^\alpha\ \eta_\alpha\ = X_1^\alpha\ \xi_\alpha\ = X_2^\alpha\ \xi_\alpha\ = 0. \tag{11.6}$$

Equations (11.1), (11.4) and (11.5) show that the four linearly independent vector fields η^α, ξ^α , X_1^α and X_2^α commute with each other. It follows (see Spivak 1970, p. 5-36) that one can find coordiantes $(t,\phi,\ x^1,\ x^2)$ locally such that $\eta = \partial/\partial t$, $\xi = \partial/\partial\phi$, $X_i = \partial/\partial x^i$, $i = 1,2$. Because of eq. (11.6) the metric tensor has an extremely simple form when expressed in these coordinates:

$$ds^2 = Adt^2 + B\,dt\,d\phi + C\,d\phi^2 + g_{ij}\,dx^i\,dx^j;\ i,j = 1,2 \qquad (11.7)$$

Furthermore, since the vectors η^α and ξ^α are Killing vector fields, it follows that the functions A,B,C and g_{ij} depend only on the two coordinates x^i. (We note that this form of the metric is preserved under coordinate transformations of the form $x^{j'} = x^{j'}(x^i)$.)

Therefore, in a stationary axisymmetric spacetime, which is orthogonally transitive, we can choose coordinates in which at least four of the off diagonal components of the metric are zero, and the remaining components depend on only two of the coordinates. Obviously, this choice of coordinates vastly simplifies the resulting form of Einstein's equations (see for example Bardeen 1973).

In a stationary axisymmetric spacetime we can consider the concept of convection. We will call a spacetime *convection free* if

$$\xi_{[\alpha}\eta_\beta R_{\gamma]}{}^\mu\,\eta_\mu = 0, \text{ and} \qquad (11.8)$$

$$\xi_{[\alpha}\,\eta_\beta R_{\gamma]}{}^\mu\,\xi_\mu = 0. \qquad (11.9)$$

This definition is motivated by the notion of convection free motion of a fluid, i.e. purely aximuthal motion,

$$u^\alpha = (-\psi)^{-1/2}\{\eta^\alpha + \Omega\,\xi^\alpha\} \quad . \qquad (11.10)$$

Conditions (11.8) and (11.9) for an ideal fluid spacetime are equivalent to the condition (11.10) on the velocity field of the fluid. We will show in §11.2 that the assumption of orthogonal transitivity is equivalent to the assumption that a spacetime is convection free.

Our discussion for the remainder of this chapter will proceed as follows. Section 11.2 will derive a number of useful and interesting identities for geometries which admit two linearly independent Killing vector fields. Section 11.3 will write out Einsteins equations in a useful and elegant form. We also show how coordinates may be chosen to minimize the number of dependent variables in the convection free case. The final section, 11.4, derives a number of properties of stationary axisymmetric fluids: integral formulas for the total mass and angular momentum of a stellar model; the generalization of Poincare's limit on the angular velcity of a rotating star; the generalization of the theorem that barotropic fluids must rotate on cylinders; and other constraints and relationships among the functions which describe the stellar model.

11-2 More Killingvectorology

In this section we present a number of identities involving two Killing vector fields. We present the statement of the results and discussion at the beginning of the section, and leave the proofs to the end. The results presented here are not new. They can be found scattered through the literature. Most of the results are at least stated in the works of Papapetrou (1966) and Carter (1973). We collect the results together here, along with their proofs, because these Killing vector identities form an invaluable toolbox for the future study of stationary axisymmetric stellar models.

LEMMA 11.1 - *Let* η^{α} *and* ξ^{α} *be commuting Killing vector fields, then:*

$$\nabla^{\gamma}\{\xi_{[\gamma}\nabla_{\alpha}\eta_{\beta]}\} = \frac{2}{3}\,\xi_{[\alpha}R_{\beta]}{}^{\mu}\eta_{\mu}\ . \tag{11.11}$$

LEMMA 11.2 - *Let η^α and ξ^α be commuting Killing vector fields, then:*

$$\nabla_\alpha\{\eta^{\beta\gamma\mu\nu}\ \eta_\beta\ \xi_\gamma\ \nabla_\mu\ \eta_\nu\} = 2\ \eta_{\alpha\beta\mu\nu}\ \eta^\beta\ \xi^\mu\ R^\nu{}_\gamma\ \eta^\gamma\ . \tag{11.12}$$

The next theorem is an important one in the study of stationary axisymmetric spacetimes. The vacuum case of this theorem was proved by Papapetrou (1966). The non-vacuum proof was given first by Kundt and Trumper (1966) and independently by Carter (1969).

THEOREM 11.3 - *A stationary axisymmetric asymptotically flat spacetime is convection free (i.e. eqs.11.8 and 11.9) if and only if it is orthogonally transitive (i.e. eqs. 11.2 and 11.3).*

A vacuum spacetime is automatically convection free according to eqs. (11.2) and (11.3). Therefore, Theorem 11.3 proves that all stationary axisymmetric vacuum spacetimes are orthogonally transitive.

The next five lemmas derive useful expressions for certain derivatives of the Killing vector fields in an orthogonally transitive geometry. We will use the notation $\gamma_{\alpha\beta}$ for the metric intrinsic to the surfaces of transitivity. This intrinsic metric is given by the following formula:

$$\gamma^{\alpha\beta} = \sigma^{-2}\{(\eta^\mu\xi_\mu)[\xi^\alpha\eta^\beta + \xi^\beta\eta^\alpha] - (\xi^\mu\xi_\mu)\eta^\alpha\eta^\beta - (\eta^\mu\eta_\mu)\xi^\alpha\xi^\beta\}\ , \tag{11.13}$$

where σ^2 is the determinent of $\gamma_{\alpha\beta}$ in the coordinate system in which ξ^α and η^α are coordinate vectors, i.e.

$$\sigma^2 = (\xi^\mu\eta_\mu)^2 - (\eta^\mu\eta_\mu)(\xi^\nu\xi_\nu). \tag{11.14}$$

Also, we will use the notation $\varepsilon^{\alpha\beta}$ to denote the antisymmetric tensor on on the two-surfaces orthogonal to the surfaces of transitivity:

$$\varepsilon^{\alpha\beta} = \sigma^{-1} \eta^{\alpha\beta\mu\nu} \eta_\mu \xi_\nu . \tag{11.15}$$

The first lemma gives a useful expression for the gradient of one of the Killing vector fields:

LEMMA 11.4 - *If* η^α *and* ξ^α *are commuting, orthogonally transitive Killing vector fields, then*

$$\nabla_\alpha \eta_\beta = \sigma^{-2} \xi_{[\alpha} \{(\eta^\mu\eta_\mu)\nabla_{\beta]}(\eta^\nu\xi_\nu)-(\eta^\nu\xi_\nu)\nabla_{\beta]}(\eta^\mu\eta_\mu)\}$$

$$+ \sigma^{-2} \eta_{[\alpha} \{(\xi^\mu\xi_\mu)\nabla_{\beta]}(\eta^\nu\eta_\nu)-(\xi^\mu\eta_\mu)\nabla_{\beta]}(\xi^\mu\eta_\mu)\} . \tag{11.16}$$

The next four lemmas compute expressions for certain components of the four dimensional curvature tensor in terms of invariant scalars formed from the Killing vector fields: $\eta^\alpha\eta_\alpha$, $\xi^\alpha\xi_\alpha$, $\eta^\alpha\xi_\alpha$, etc.

LEMMA 11.5 - *If* η^α *and* ξ^α *are commuting, orthogonally transitive Killing vector fields, then*

$$\eta^{\alpha\beta\mu\nu} \xi_\beta \nabla_\mu \eta_\nu = - \sigma \varepsilon^{\alpha\beta} V_\beta, \text{ and} \tag{11.17}$$

$$\nabla^\alpha V_\alpha = - 4 \sigma^{-2} \xi^\alpha \eta^\beta \xi_{[\alpha} R_{\beta]}{}^\mu \eta_\mu, \tag{11.18}$$

where

$$V_\alpha = \sigma^{-2} \{(\xi^\mu\xi_\mu)\nabla_\alpha(\eta^\nu\eta_\nu)-(\xi^\mu\eta_\mu)\nabla_\alpha(\xi^\nu\eta_\nu)\}.$$

LEMMA 11.6 - *If* η^α *and* ξ^α *are commuting, orthogonally transitive Killing vector fields, then*

$$\eta^{\alpha\beta\mu\nu}\ \eta_\beta\ \nabla_\mu\ \eta_\nu = \sigma\ \varepsilon^{\alpha\beta}\ W_\beta \quad \textit{and} \tag{11.19}$$

$$\nabla^\alpha\ W_\alpha = 4\ \sigma^{-2}\ \xi^\alpha\ \eta^\beta\ \eta_{[\alpha}\ R_{\beta]}{}^\mu\ \eta_\mu \quad , \tag{11.20}$$

where

$$W_\alpha = \sigma^{-2}\{(\eta^\mu\eta_\mu)\nabla_\alpha(\eta^\nu\xi_\nu)-(\eta^\nu\xi_\nu)\nabla_\alpha(\eta^\mu\eta_\mu)\}\ .$$

LEMMA 11.7 - *If* η^α *and* ξ^α *are commuting, orthogonally transitive Killing vector fields, then*

$$\eta^{\alpha\beta\mu\nu}[\xi_\beta\nabla_\mu\eta_\nu - \eta_\beta\nabla_\mu\xi_\nu] = 2\ \varepsilon^{\alpha\beta}\ \nabla_\beta\sigma\ , \text{ and} \tag{11.21}$$

$$\nabla_\alpha\nabla^\alpha\ \log\sigma = -\ \gamma^{\alpha\beta}\ R_{\alpha\beta}\ . \tag{11.22}$$

LEMMA 11.8 - *If* η^α *and* ξ^α *are commuting, orthogonally transitive Killing vector fields, then*

$$\begin{aligned}\nabla^\alpha\{\sigma^{-2}\ \nabla_\alpha(\eta^\mu\eta_\mu)\} = &-\ 2\sigma^{-2}\ R_{\alpha\beta}\ \eta^\alpha\ \eta^\beta \\ &-\sigma^{-4}(\eta^\mu\eta_\mu)\{\nabla^\alpha(\eta^\nu\xi_\nu)\nabla_\alpha(\eta^\beta\xi_\beta)-\nabla^\alpha(\eta^\nu\eta_\nu)\nabla_\alpha(\xi^\beta\xi_\beta)\}\ , \textit{ and}\end{aligned} \tag{11.23}$$

$$\begin{aligned}\nabla^\alpha\{\sigma^{-2}\ \nabla_\alpha(\xi^\mu\eta_\mu)\} = &-\ 2\sigma^{-2}\ R_{\alpha\beta}\ \eta^\alpha\ \xi^\beta \\ &-\sigma^{-4}(\eta^\mu\xi_\mu)\{\nabla^\alpha(\eta^\nu\xi_\nu)\nabla_\alpha(\eta^\beta\xi_\beta)-\nabla^\alpha(\eta^\nu\eta_\nu)\nabla_\alpha(\xi_\beta\xi^\beta)\}\ .\end{aligned} \tag{11.24}$$

The metric tensor intrinsic to the surfaces of transitivity, $\gamma_{\alpha\beta}$, is independent of the coordinates t and ϕ defined by $\eta = \partial/\partial t$ and $\xi = \partial/\phi$. Therefore the intrinsic curvature of these surfaces is zero. The intrinsic geometry of the two-surfaces orthogonal to the surfaces of transitivity is not flat, however. We denote by

$$h_{\alpha\beta} = g_{\alpha\beta} - \gamma_{\alpha\beta}, \tag{11.25}$$

the intrinsic metric of the orthogonal two-surfaces and by ${}^2R_{\alpha\beta\mu\nu}$ the intrinsic curvature. The following lemma relates the components of the two-dimensional curvature to the components of the four-curvature.

LEMMA 11.9 - *In a stationary axisymmetric orthogonally transitive space-time, the intrinsic curvatures of the two-surfaces with intrinsic metric* $h_{\alpha\beta}$ *(see eq. 11.25) are given by*

$${}^2R_{\alpha\beta\mu\nu} = h_\alpha{}^\gamma\, h_\beta{}^\varepsilon\, h_\mu{}^\tau\, h_\nu{}^\theta\, R_{\gamma\varepsilon\tau\theta}, \tag{11.26}$$

$${}^2R_{\alpha\beta} = h_\alpha{}^\mu\, h_\beta{}^\nu\, R_{\mu\nu} + \sigma^{-1}\, h_\alpha{}^\mu\, h_\beta{}^\nu\, \nabla_\mu \nabla_\nu \sigma$$
$$+ \frac{1}{2}\sigma^{-2}\{\nabla_{(\alpha}(\eta^\nu\eta_\mu)\nabla_{\beta)}(\xi^\nu\xi_\nu) - \nabla_\alpha(\eta^\mu\xi_\mu)\nabla_\beta(\eta^\nu\xi_\nu)\}, \tag{11.27}$$

$${}^2R = (h^{\mu\nu} - \gamma^{\mu\nu})R_{\mu\nu}$$
$$+ \frac{1}{2}\sigma^{-2}\{\nabla_\mu(\eta^\alpha\eta_\alpha)\nabla^\mu(\xi^\beta\xi_\beta) - \nabla_\mu(\eta^\alpha\xi_\alpha)\nabla^\mu(\eta^\beta\xi_\beta)\} \quad . \tag{11.28}$$

These last five lemmas will be useful to us in our discussion of Einstein's equations in the next section. Each of these lemmas has related the derivatives of invariant scalar quantities such as σ, $\eta^\mu\eta_\mu$, $\xi^\mu\eta_\mu$, etc to certain components of the four-dimensional Ricci tensor.

We now present the proofs of these results:

PROOF OF LEMMA 11.1: Explicitly write out the terms of the anti-symmetrization indicated on the left hand side of eq. (11.11) and perform the differentation to find:

$$\nabla^{\gamma}\{\xi_{[\gamma}\nabla_{\alpha}\eta_{\beta]}\} = \frac{1}{3}\{\xi^{\gamma}\nabla_{\gamma}\nabla_{\alpha}\eta_{\beta} + \nabla_{\alpha}\xi^{\gamma}\nabla_{\gamma}\eta_{\beta} + \nabla_{\beta}\xi^{\gamma}\nabla_{\alpha}\eta_{\gamma}\}$$

$$+ \frac{1}{3}\{\nabla^{\gamma}\xi_{\gamma}\nabla_{\alpha}\eta_{\beta} - \xi_{\alpha}\nabla^{\gamma}\nabla_{\gamma}\eta_{\beta} + \xi_{\beta}\nabla^{\gamma}\nabla_{\gamma}\eta_{\alpha}\} . \qquad (11.29)$$

The first three terms equal the Lie derivative of $\nabla_{\alpha}\eta_{\beta}$ along the Killing vector field ξ^{α}. This Lie derivative vanishes because η^{α} and ξ^{α} commute, and because ξ^{α} is a Killing vector. The fourth term vanishes because ξ^{α} is divergenceless. The last two terms are transformed using Lemma 7.1. The result is given by

$$\nabla^{\gamma}\{\xi_{[\gamma}\nabla_{\alpha}\eta_{\beta]}\} = \frac{2}{3}\xi_{[\alpha}R_{\beta]}{}^{\mu}\eta_{\mu} .$$ ∎

PROOF OF LEMMA 11.2: Consider the result of Lemma 7.3:

$$\nabla_{[\alpha}\{\eta_{\beta]\gamma\mu\nu}\eta^{\gamma}\nabla^{\mu}\eta^{\nu}\} = \eta_{\alpha\beta\mu\nu}\eta^{\mu}R^{\nu}{}_{\gamma}\eta^{\gamma} \qquad (11.30)$$

Contracting both sides of eq. (11.30) with the vector ξ^{α} , we find

$$\nabla_{\alpha}\{\eta_{\beta\gamma\mu\nu}\xi^{\beta}\eta^{\gamma}\nabla^{\mu}\eta^{\nu}\} = 2\,\eta_{\alpha\beta\mu\nu}\xi^{\beta}\eta^{\mu}R^{\nu}{}_{\gamma}\eta^{\gamma}$$

$$+ \xi^{\beta}\nabla_{\beta}\{\eta_{\alpha\gamma\mu\nu}\eta^{\gamma}\nabla^{\mu}\eta^{\nu}\} - \eta^{\beta}{}_{\gamma\mu\nu}\eta^{\gamma}\nabla^{\mu}\eta^{\nu}\nabla_{\beta}\xi_{\alpha} . \qquad (11.31)$$

The last two terms on the right hand side of eq. (11.31) are the commutator of the vectors ξ^{α} and $\eta^{\alpha}{}_{\beta\mu\nu}\eta^{\beta}\nabla^{\mu}\eta^{\nu}$. This commutator vanishes because ξ^{α}

is a Killing vector field which commutes with η^{α}. The remaining terms are equivalent to eq. (11.12). ∎

PROOF OF THEOREM 11.3: In the proof of Lemma 11.2, no distinction was made between the Killing vector fields η^{α} and ξ^{α}. Consequently it follows that both of the following expressions are implied by Lemma 11.2:

$$\nabla_{\alpha}\{\eta^{\beta\gamma\mu\nu}\ \eta_{\beta}\xi_{\gamma}\nabla_{\mu}\eta_{\nu}\} = 2\ \eta_{\alpha\beta\mu\nu}\ \eta^{\beta}\xi^{\mu}\ R^{\nu}{}_{\gamma}\ \eta^{\gamma} \quad , \tag{11.32}$$

$$\nabla_{\alpha}\{\eta^{\beta\gamma\mu\nu}\ \xi_{\beta}\eta_{\gamma}\nabla_{\mu}\xi_{\nu}\} = 2\ \eta_{\alpha\beta\mu\nu}\ \xi^{\beta}\eta^{\mu}R^{\nu}{}_{\gamma}\xi^{\alpha} \quad . \tag{11.33}$$

If the spacetime is orthogonally transitive, the left hand sides of eqs. (11.32) and (11.33) vanish. This implies that the right hand sides must vanish also. These conditions are equivalent to the convection free conditions, eqs. (11.8) and (11.9). If the spacetime is convection free, eqs. (11.32) and (11.33) imply that the two scalars

$$c_{\eta} = \eta^{\alpha}\xi^{\beta}\nabla^{\mu}\eta^{\nu}\ \eta_{\alpha\beta\mu\nu} \quad , \text{ and} \tag{11.34}$$

$$c_{\xi} = \xi^{\alpha}\eta^{\beta}\nabla^{\mu}\xi^{\nu}\eta_{\alpha\beta\mu\nu} \quad , \tag{11.35}$$

are constants. Since an asymptotically flat, stationary, axisymmetric spacetime must have a rotation axis (see Carter 1970) the rotation Killing vector field ξ^{ν} must vanish somewhere. Consequently, the constants c_{η} and c_{ξ} must have the value zero. The vanishing of these constants is equivalent to the orthogonal transitivity conditions eqs. (11.2) and (11.3).

∎

PROOF OF LEMMA 11.4: Since the vectors η^α and ξ^α are orthogonally transitive, it follows that

$$\nabla_\alpha \eta_\beta = \eta_\alpha V_\beta + \xi_\alpha W_\beta + X \eta_\alpha \xi_\beta \tag{11.36}$$

where V^α and W^α are orthogonal to η^α and ξ^α. Since η^α and ξ^α are commuting Killing vector fields, it follows that $X = 0$ by contracting the tensor $\xi^\alpha \eta^\beta$ into both sides of eq. (11.36). Contracting η^β and ξ^β successively into eq. (11.36) yields the two expressions:

$$-\nabla_\alpha(\eta^\beta \eta_\beta) = (\eta^\beta \eta_\beta) V_\alpha + (\eta^\beta \xi_\beta) W_\alpha \text{ , and} \tag{11.37}$$

$$-\nabla_\alpha(\eta^\beta \xi_\beta) = (\eta^\beta \xi_\beta) V_\alpha + (\xi^\beta \xi_\beta) W_\alpha \ . \tag{11.38}$$

Equations (11.37) and (11.38) can be inverted to find expressions for V_α and W_α is terms of $\nabla_\alpha(\eta^\beta \eta_\beta)$ and $\nabla_\alpha(\eta^\beta \xi_\beta)$. The resulting expressions are given by

$$V_\alpha = \sigma^{-2} \ (\xi^\mu \xi_\mu) \nabla_\alpha (\eta^\nu \eta_\nu) - (\xi^\mu \eta_\mu) \nabla_\alpha (\xi^\nu \eta_\nu)\}, \text{ and} \tag{11.39}$$

$$W_\alpha = \sigma^{-2}\{(\eta^\mu \eta_\mu) \nabla_\alpha (\eta^\nu \xi_\nu) - (\eta^\nu \xi_\nu) \nabla_\alpha (\eta^\mu \eta_\mu)\} \ . \tag{11.40}$$

Substituting eqs. (11.39) and (11.40) into eq. (11.36) gives the desired result, eq. (11.16). ■

PROOF OF LEMMA 11.5: Use eq. (11.16) from Lemma 11.4 to evaluate the left hand side eq. (11.17). The resulting expression is equivalent to the right hand side of eq. (11.17) when the definition of $\varepsilon^{\alpha\beta}$ (eq. 11.15) and V_α are used. Equation (11.18) is derived from eq. (11.11) of Lemma 11.1 Contract both sides of eq. (11.11) with the tensor $\xi^\alpha \eta^\beta$ and rearrange the terms somewhat:

$$\nabla^\gamma\{\xi^\alpha\eta^\beta\ \xi_{[\gamma}\nabla_\alpha\eta_{\beta]}\} = \frac{2}{3}\ \xi^\alpha\eta^\beta\xi_{[\alpha}R^\mu{}_{\beta]}\eta_\mu$$

$$+ \{(\nabla^\gamma\xi^\alpha)\eta^\beta\ + \xi^\alpha\nabla^\gamma\eta^\beta\}\xi_{[\gamma}\nabla_\alpha\eta_{\beta]}\ . \qquad (11.41)$$

Use Lemma 11.4 to evaluate the gradients of the Killing vector fields, $\nabla_\alpha\eta_\beta$ and $\nabla_\alpha\xi_\beta$ on both sides of eq. (11.41). The resulting expression is given by

$$-\frac{1}{6}\ \nabla^\gamma(\sigma^2 V_\gamma) = \frac{2}{3}\ \xi^\alpha\eta\ \xi_{[\alpha}\ \ R_{\beta]}{}^\mu\ \eta_\mu - \frac{1}{6}\ V^\alpha\nabla_\alpha\ \sigma^2. \qquad (11.42)$$

This equation (11.42) is easily transformed into eq. (11.18). ■

The proof of Lemma 11.6 is exactly analogous to the proof just given for Lemma 11.5, except that one uses Lemma 7.2 instead of Lemma 11.1. Therefore, we will not outline the proof further here.

PROOF OF LEMMA 11.7: We begin by interchanging the vectors η^α and ξ^α in Lemma 11.5 to obtain the following expressions.

$$\eta^{\alpha\beta\mu\nu}\ \eta_\beta\nabla_\mu\xi_\nu = \sigma\ \varepsilon^{\alpha\beta}\ U_\beta, \text{ and} \qquad (11.43)$$

$$\nabla^\alpha\ U_\alpha = -\ 4\ \sigma^{-2}\ \eta^\alpha\xi^\beta\eta_{[\alpha}\ R_{\beta]}{}^\mu\ \xi_\mu\ , \qquad (11.44)$$

where $U_\alpha = \sigma^{-2}\{(\eta^\mu\eta_\mu)\nabla_\alpha(\xi^\nu\xi_\nu)-(\eta^\mu\xi_\mu)\nabla_\alpha(\eta^\nu\xi_\nu)\}$.
We can easily verify that

$$V_\alpha + U_\alpha = -\ \nabla_\alpha\ \log\sigma^2\ . \qquad (11.45)$$

We add eqs. (11.17) and (11.43) to obtain the expression:

$$\eta^{\alpha\beta\mu\nu}\{\xi_\beta\nabla_\mu\eta_\nu\ -\ \eta_\beta\nabla_\mu\xi_\nu\}\ =\ -\sigma\ \varepsilon^{\alpha\beta}\{\ V_\beta + U_\beta\}. \qquad (11.46)$$

This expression (11.46) is equivalent to eq. (11.21) when eq. (11.45) is used. Next, we add eqs. (11.18) and (11.44) to find

$$\nabla_\alpha\nabla^\alpha\log\sigma^2\ = 4\sigma^{-2}\xi^\alpha\eta^\beta\{\xi_{[\alpha}R_{\beta]}{}^\mu\ \eta_\mu\ -\ \eta_{[\alpha}R_{\beta]}{}^\mu\xi_\mu\} \qquad (11.47)$$

When the antisymmetrizations indicated on the right hand side of eq. (11.47) are written out, and the definition of $\gamma^{\alpha\beta}$ from eq. (11.13) is recalled, we can see that eq. (11.47 is equivalent to eq. (11.22). ■

PROOF OF LEMMA 11.8: We will use the vectors V_α and W_α defined in Lemmas 11.5 and 11.6 respectively. From the definitions of these quantities it is easily seen that

$$\nabla_\alpha(\eta^\mu\eta_\mu) = -(\eta^\mu\eta_\mu)V_\alpha - (\eta^\mu\xi_\mu)W_\alpha \text{ , and} \tag{11.48}$$

$$\nabla_\alpha(\eta^\mu\xi_\mu) = -(\eta^\mu\xi_\mu)V_\alpha - (\xi^\mu\xi_\mu)W_\alpha \text{ .} \tag{11.49}$$

We compute the divergences of these equations:

$$\begin{aligned}\nabla^\alpha\{\sigma^{-2}\nabla_\alpha(\eta^\mu\eta_\mu)\} = &- \sigma^{-2}\{(\eta^\mu\eta_\mu)\nabla^\alpha V_\alpha + (\xi^\mu\eta_\mu)\nabla^\alpha W_\alpha\} \\ &- \sigma^{-2}\{V^\alpha\nabla_\alpha(\eta^\mu\eta_\mu) + W^\alpha\nabla_\alpha(\xi^\mu\eta_\mu)\} \\ &+ \nabla^\alpha(\eta^\mu\eta_\mu)\nabla_\alpha\sigma^{-2}\end{aligned} \tag{11.50}$$

An analagous equation can be written down for $\nabla^\alpha\{\sigma^{-2}\nabla_\alpha(\xi^\mu\eta_\mu)\}$. The right hand side of eq. (11.50) is simplified by application of eqs. (11.18), (11.20) and the definitions of V_α and W_α. After a bit of straightforeward algebra, these divergence expressions can be put in the form of eqs. (11.23) and (11.24). ■

PROOF OF LEMMA 11.9: Let D_α denote the covariant derivative intrinsic to the surface whose metric is $h_{\alpha\beta}$. Let X^α denote an arbitrary vector field tangent to these surfaces. The intrinsic curvature to these surfaces can be computed from Ricci's identity:

$$D_\alpha D_\beta X_\gamma - D_\beta D_\alpha X_\gamma = - {}^2R^\mu{}_{\gamma\alpha\beta} X_\mu \ . \qquad (11.51)$$

We now express the intrinsic covariant derivative D_α in terms of the full four-dimensional derivative ∇_α:

$$D_\alpha D_\beta X_\gamma - D_\beta D_\alpha X_\gamma = h^a_\alpha h^b_\beta h^c_\gamma \{\nabla_a [h^\mu_b h^\nu_c \nabla_\mu X_\nu] - \nabla_b [h^\mu_a h^\nu_c \nabla_\mu X_\nu]\}. \qquad (11.52)$$

The right hand side of eq. (11.) can be expanded in the following way:

$$h^a_\alpha h^b_\beta h^c_\gamma \{\nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c\} + h^a_\alpha h^b_\beta h^c_\gamma \{\nabla_a [h^\mu_b h^\nu_c] - \nabla_b [h^\mu_a h^\nu]\} \nabla_\mu X_\nu \ . \qquad (11.53)$$

The first term in expression (11.53) is a projection of the four-dimensional curvature tensor:

$$h^a_\alpha h^b_\beta h^c_\gamma \{\nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c\} = - R^e{}_{cab} X_e h^a_\alpha h^b_\beta h^c_\gamma \qquad (11.54)$$

The second term in (11.53) has several pieces, each proportional to an expression of the form $h^a_\alpha h^b_\beta \nabla_a h^\mu_b$. We can show that terms of this form vanish. Use the definition of h^μ_b from eq. (11.25) to note that

$$h^a_\alpha h^b_\beta \nabla_a h^\mu_b = - h^a_\alpha h^b_\beta \nabla_a \gamma^\mu_b \ . \qquad (11.55)$$

Now use the expression for γ^μ_b given in eq. (11.13). Perform the differentiation, and remember that $h^a_\alpha \eta_a = h^a_\alpha \xi_a = 0$. The resulting expression is given by

$$h^a_\alpha \, h^b_\beta \, \nabla_a \, h^\mu_b = - \sigma^{-2}[(\eta^\nu \xi_\nu)\eta^\mu - (\eta^\nu \eta_\nu)\xi^\mu] h^a_\alpha \, h^b_\beta \, \nabla_a \, \xi_b$$

$$- \sigma^{-2}[(\eta^\nu \xi_\nu)\xi^\mu - (\xi^\nu \xi_\nu)\eta^\mu] h^a_\alpha \, h^b_\beta \, \nabla_a \, \eta_b \, . \qquad (11.56)$$

When the equation for the gradients of η^α and ξ^α from Lemma 11.4 are used, the right hand side of eq. (11.56) vanishes. Therefore, the right hand side of eq. (11.52) reduces to the right hand side of eq. (11.54). This establishes the validity of eq. (11.26).

To prove the validity of eqs. (11.27) and (11.28) we must perform the contractions of eq. (11.26) with the tensor $h_{\alpha\beta}$. It follows that

$$^2R_{\alpha\beta} = h^a_\alpha \, h^b_\beta \, h^{\mu\nu} \, R_{a\mu b\nu}$$

$$= h^\mu_\alpha \, h^\nu_\beta \, R_{\mu\nu} - h^a_\alpha \, h^b_\beta \, \gamma^{\mu\nu} \, R_{a\mu b\nu} \qquad (11.57)$$

We recognize the first term on the right hand side of eq. (11.5) as one of the terms in eq. (11.27). The remaining task, therefore is to convert the second term into the appropriate derivatives of the Killing vector fields. The computation is rather lengthy, we will omit the details here. The appropriate strategy is as follows. Use Lemma 7.1 to convert the components of $\gamma^{\mu\nu} \, R_{\alpha\mu\beta\nu}$ into derivatives of the Killing vector fields. Then repeatedly use Lemma 11.4 to evaluate the gradients $\nabla_\alpha \, \eta_\beta$ and $\nabla_\alpha \, \xi_\beta$. The resulting expression can be simplified to the form of eq. (11.27). Deriving eq. (11.28) by contracting eq. (11.27) with the tensor $h^{\alpha\beta}$ is a fairly straightforeward process, when one remembers to use Lemma 11.7. ∎

11-3 Einstein's Equations

In this section we discuss two approaches which have been taken in the study of Einstein's equations in stationary and axisymmetric spacetimes. These equations have been studied for many years because it was recognized that their solutions would represent rotating bodies in general relativity theory. The field equations for a stationary axisymmetric orthogonally transitive spacetime (eq. 11.7) were given by Andress (1930), and have been studied by a large number of authors since then. The two approaches which we present here are i) a covariant decomposition of the equations in a way that utilizes the symmetries of the Killing vector fields to simplify the equations and ii) an attempt to reduce the number of functions which describe the metric by a clever choice of coordinates.

The first approach decomposes the geometry of the spacetime using $\gamma_{\alpha\beta}$, the intrinsic metric on the surfaces of transitivity, and the coprojection tensor $h_{\alpha\beta}$. Since $\gamma_{\alpha\beta}$ describes the intrinsic geometry of the surface of transitivity of the Killing vector fields, all geometric quantities will be constant on these surfaces. Consequently, all derivatives in Einstein's equations can be expressed in terms of the intrinsic covarient derivative defined by the projection tensor $h_{\alpha\beta}$. This approach simplifies the equations as much as possible by utilizing the available symmetries, yet it does this in a completely coordinate independent fashion. This technique was developed by Geroch (1972) for the study of vacuum spacetimes, and was extended for the study of ideal fluid spacetimes by Hansen and Winicour (1975).

The second approach, which we discuss here, attempts to minimize the number of unknown functions, and the number of independent variables which describe the gravitational field, by a suitable choice of coordinates. Harrison (1970) constructs a coordinate system for rigidly rotating fluids which reduces the number of functions describing the gravitational field to three. For differentially rotating fluids the number of functions has also been reduced to three, by Abramowicz and Muchotrzeb (1976), when the form of the rotation law of the fluid is specified.

We will now summarize how Einstein's equations are decomposed using the projection tensors $\gamma_{\alpha\beta}$ and $h_{\alpha\beta}$. For simplicity we discuss only the orthogonally transitive case here. (The general equations have been worked out in this way by Hansen and Winicour (1975).) We introduce the covarient derivative, D_α, which is intrinsic to the surfaces whose metric is $h_{\alpha\beta}$. This derivative acts only on vectors which satisfy $X^\alpha = h^\alpha_\beta\, X^\beta$; for these vector fields we define

$$D_\alpha X_\beta = h^\mu_\alpha\, h^\nu_\beta\, \nabla_\mu X_\nu. \tag{11.58}$$

Einstein's equations in a stationary axisymmetric orthogonally transitive spacetime are given as follows:

$$\sigma\, D_\alpha[\sigma^{-1} D^\alpha(\eta^\mu\eta_\mu)] = -\,2\, R_{\alpha\beta}\, \eta^\alpha\eta^\beta + \eta^\mu\eta_\mu\, T\ , \tag{11.59}$$

$$\sigma\, D_\alpha[\sigma^{-1} D^\alpha(\eta^\mu\xi_\mu)] = -2\, R_{\alpha\beta}\xi^\alpha\xi^\beta + \eta^\mu\xi_\mu\, T\ , \tag{11.60}$$

$$\sigma\, D_\alpha[\sigma^{-1} D^\alpha(\xi^\mu\xi_\mu)] = -2\, R_{\alpha\beta}\xi^\alpha\xi^\beta + \xi^\mu\xi_\mu\, T\ , \text{ and} \tag{11.61}$$

$${}^2R = (h^{\mu\nu} - \gamma^{\mu\nu})R_{\mu\nu} + \frac{1}{2}\,T\ ; \text{ where} \tag{11.62}$$

$$T = \sigma^{-2}\{D^\alpha(\eta^\mu\eta_\mu)D_\alpha(\xi^\nu\xi_\nu) - D^\alpha(\xi^\mu\eta_\mu)D_\alpha(\xi^\nu\eta_\nu)\}. \tag{11.63}$$

Equations (11.59)-(11.62) along with the Bianchi identities

$$\nabla_\alpha(R^{\alpha\beta} - \frac{1}{2}\ g^{\alpha\beta}\, R) = 0 \tag{11.64}$$

are sufficient to determine the gravitational field. Equations (11.59)-(11.61) determine the Killing scalers $\eta^\mu\eta_\mu$, $\xi^\mu\eta_\mu$ and $\xi^\mu\xi_\mu$. Equation (11.62) determines the Gaussian curvature of the two-surfaces whose metric is $h_{\alpha\beta}$.

Consequently eq. (11.62) determines the metric $h_{\alpha\beta}$. Equations (11.59)-(11.61) represent the $\gamma_\alpha{}^\mu \gamma_\beta{}^\nu R_{\mu\nu}$ components of the Ricci tensor, while eq. (11.62) represents the $h^{\mu\nu}R_{\mu\nu}$ component. The components of the form $h_\alpha{}^\mu \gamma_\beta{}^\nu R_{\mu\nu}$ vanish identically for an orthogonally transitive spacetime, as shown by Lemma 11.2. The other possible components $h_\alpha{}^\mu h_\beta{}^\nu R_{\mu\nu} - \frac{1}{2} h_{\alpha\beta} h^{\mu\nu} R_{\mu\nu}$ vanish whenever the Bianchi identity is satisfied. A more complete discussion of these equations are given by Hansen and Winicour (1975).

We can verify that eqs. (11.59)-(11-62) are correct by converting the four-dimensional covariant derivatives, ∇_α, into intrinsic covariant derivatives, D_α, in Lemmas 11.8 and 11.9. The following lemma establishes the needed relationship:

LEMMA 11.10 - $$D_\alpha X^\alpha = \sigma \nabla_\alpha \{\sigma^{-1} X^\alpha\} \ . \tag{11.65}$$

PROOF: We take the trace of eq. (11.58) and use eq. (11.25) for the tensor $h_{\alpha\beta}$. The resulting equation can be put into the following form:

$$D_\alpha X^\alpha = \nabla_\alpha X^\alpha + X_\beta \nabla_\alpha \gamma^{\alpha\beta} \quad . \tag{11.66}$$

Now compute the divergence of $\gamma^{\alpha\beta}$ with the aid of eq. (11.13). The result is given by

$$\nabla_\alpha \gamma^{\alpha\beta} = - \nabla^\beta \log\sigma \ . \tag{11.67}$$

Equations (11.66) and (11.67) together imply eq. (11.65). ∎

For the special case of a convection free ideal fluid spacetime, the Ricci tensor is related to the fluid variables by

$$R_{\mu\nu} = 4\pi(\rho-p)g_{\mu\nu} - 8\pi(\rho+p)\psi^{-1}[\eta_\mu+\Omega\xi_\mu][\eta_\nu+\Omega\xi_\nu], \tag{11.68}$$

$$\psi = \eta^\mu\eta_\mu + 2\Omega\, \xi^\mu\eta_\mu + \Omega^2\, \xi^\mu\xi_\mu\ . \tag{11.69}$$

By substituting eq. (11.68) into eqs. (11.59)-(11.64) we see that the Einstein equations reduce to equations on a two-dimensional surface, Σ , for the scalar functions $\eta^\mu\eta_\mu$, $\eta^\mu\xi_\mu$, $\xi^\mu\xi_\mu$, p, ρ and Ω as well as the intrinsic metric of the surface $h_{\alpha\beta}$. The equations as we have written them are invariant under changes of coordinates within the surface Σ. Consequently, we are free to choose these coordinates in any way which will simplify the equations. We will discuss some simplifying choices of coordinates later in this section.

Before considering simplifications of the coordinate, however, it is possible to make certain simplification in the system of equations (11.59)-(11.64). For example, by taking linear combinations of eqs. (11.59)-(11.61) the following simpler expression can be obtained (see Lemma 11.7):

$$D_\alpha D^\alpha \sigma = - \sigma\gamma^{\alpha\beta}R_{\alpha\beta} = 16\pi\sigma p. \tag{11.70}$$

The second equality applies only to the ideal fluid spacetimes, eq. (11.68). The other equations, which can be cast in a simple form for convection free fluids, are the Bianchi identities. Part of this simplification has been done in §6.2 where the Bianchi identities were related to Euler's equation (6.32) and the conservation of energy equation (6.28). The conservation of energy equation is identically satisfied for stationary axisymmetric convection free fluids, while Euler's equation has the simple form:

$$\frac{1}{2} D_\alpha \log\psi - \psi^{-1}(\eta^\mu \xi_\mu + \Omega\, \xi^\mu \xi_\mu) D_\alpha \Omega = -\ (\rho+p)^{-1}\ D_\alpha p. \tag{11.71}$$

This concludes our discussion of the covariant decomposition of Einstein's equations for general stationary axisymmetric convection free fluids. Next, we focus our attention on the special case of rigidly rotating fluids. We show how the covariant equations can be considerably simplified in this case, and how coordinates can be chosen to reduce the number of unknown functions to three.

In a rigidly rotating stellar model, the fluid velocity is proportional to a Killing vector field (this follows from Lemma 7.6). For the stationary axisymmetric convection free fluids considered here, the fluid velocity is given by eq. (11.10), and the rigidity condition is equivalent to requiring that the function Ω be a constant. Thus, the vector field

$$k^\alpha = \eta^\alpha + \Omega\, \xi^\alpha \tag{11.72}$$

is a Killing vector field which is linearly independent of ξ^α. Furthermore, k^α and ξ^α commute. Therefore, we could use the pair of Killing vector fields k^α and ξ^α rather than the pair η^α and ξ^α in all of our discussions in this chapter. When we choose the pair k^α and ξ^α, the field equations can be simplified considerably.

We note that the vector field k^α is an eigenvector of the Ricci tensor since it is proportional to the fluid velocity:

$$k_{[\alpha}\, R_{\beta]}{}^\mu k_\mu = 0. \tag{11.73}$$

This fact and Lemma 7.3 imply that the twist of k^μ is curl free. Therefore, there exists a scalar function, ω, such that

$$\nabla_\alpha \omega = \eta_{\alpha\beta\mu\nu} \, k^\beta \nabla^\mu k^\nu \; . \tag{11.74}$$

Furthermore, Lemmas 7.5 and 11.6 imply that ω satisfies the differential equation:

$$D_\alpha \{\sigma\psi^{-2} \, D^\alpha \omega\} \;=\; 0 \; . \tag{11.75}$$

The scalers σ and ψ are related to the Killing scalars by

$$\psi = k^\mu k_\mu, \text{ and} \tag{11.76}$$

$$\sigma^2 = (k^\mu \xi_\mu)^2 - \psi \, \xi^\mu \xi_\mu \quad . \tag{11.77}$$

The scalar ω is related implicitly to the Killing scalers by the following equation (see Lemma 11.6):

$$D_\alpha \omega = \sigma^{-1} \, \psi^2 \, \varepsilon^{\alpha\beta} \, D_\alpha (k^\mu \xi_\mu / \psi). \tag{11.78}$$

Consequently we may use eq. (11.75) to replace one of Einstein's equations (11.59)-(11.61). Another of the equations, (11.59), takes a very simple form when the Killing vector k^α replaces η^α. The equivalent expression is given by

$$D_\alpha \{\sigma\psi^{-2} \, D^\alpha (\psi^2 + \omega^2)\} = 16\pi\sigma(\rho + 3p). \tag{11.79}$$

Therefore in terms of the three functions, ψ, σ and ω defined in eqs. (11.76)-(11.78), Einstein's equations for a rigidly rotating stationary axisymmetric ideal fluid spacetime reduce to the three simple equations.

$$D^{\alpha}\{\sigma\psi^{-2} D_{\alpha}\omega\} = 0 , \tag{11.80}$$

$$D^{\alpha}D_{\alpha}\sigma = 16\pi\sigma p, \tag{11.81}$$

$$D^{\alpha}\{\sigma\psi^{-2} D_{\alpha}(\psi^2+\omega^2)\} = 16\pi\sigma(\rho+3p) , \tag{11.82}$$

and the one not so simple equation

$${}^{2}R = 8\pi(\rho+p) + \frac{1}{2}\psi^{-2} D^{\alpha}\psi D_{\alpha}\psi - \frac{1}{2}\psi^{-2} D^{\alpha}\omega D_{\alpha}\omega - \sigma^{-1}\psi^{-1} D^{\alpha}\psi D_{\alpha}\sigma. \tag{11.83}$$

The Bianchi identity (or Euler's equation) for this system is given by

$$\frac{1}{2}\psi^{-1} D_{\alpha}\psi = - (\rho+p)^{-1} D_{\alpha}p. \tag{11.84}$$

The choice of potentials used here, σ, ψ and ω, are essentially equivalent to those introduced by Ernst (1968) in his study of the stationary axi-symmetric vacuum equations.

We now turn to the problem of choosing coordinates on the two-surfaces whose intrinsic metric tensor is $h_{\alpha\beta}$. Since these surfaces are two-dimensional, one obvious choice of coordinates are the two-dimensional harmonic coordinates:

$$h_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} = e^{2\mu}(dx^2 + dy^2) \tag{11.85}$$

The coordinates x and y can be any functions which satisfy the equations

$$D^{\alpha}D_{\alpha}x = 0 \text{ and } D_{\alpha}y = \varepsilon_{\alpha}{}^{\beta} D_{\beta}x. \tag{11.86}$$

In this choice of coordinates, the left hand sides of eqs. (11.80)-(11.82) each have the reasonably simple form

$$D_{\alpha}(AD^{\alpha}B) = e^{-2\mu}\{\partial_x(A\partial_x B) + \partial_y(A\partial_y B)\}. \tag{11.87}$$

Furthermore, the two dimensional Gaussian curvature is quite simple in this choice of coordinates,

$$^{2}R = -2\ D^{\alpha}D_{\alpha}\mu\ . \tag{11.88}$$

Therefore, two-dimensional harmonic coordinates give us a system of equations for the four functions σ, μ, ψ and ω which describe the gravitational field.

The vacuum limit of Einstein's equations for stationary axisymmetric spacetimes are unexpectedly simple in these coordiantes. The first simplification comes from eq. (11.81) which implies that σ is a harmonic function. Consequently, σ, can be used as a coordinate, which reduces the number of dependent functions needed to specify the geometry to ω, ψ and μ. The second major simplification is that eqs. (11.80) and (11.82) do not depend on the function μ in the vacuum case. Consequently the system of equations separates. The functions ω and ψ are determined by solving eqs. (11.80)-(11.82) and then μ is determined later by solving eq. (11.83).

No one has found a way to simplify the fluid equations to the extent that was possible for the vacuum equation. Harrison (1970) found that the number of dependent functions can be reduced to three by a suitable choice of coordinates however. The idea is to use some combination of the functions σ, ω, ψ and μ as a coordinate, and therefore be able to eliminate one dependent function. This can be accomplished by choosing ω as one coordinate. From eq. (11.78) it is clear that an appropriate choice for the second coordinate is $\ell = \psi^{-1}\ k^{\mu}\xi_{\mu}$. We also define

$$D^{\alpha}\omega D_{\alpha}\omega = e^{-2\tau}\ . \tag{11.89}$$

Consequently the two-dimensional metric has the form:

$$h_{\alpha\beta}\, dx^\alpha dx^\beta = e^{2\tau}\, \{d\omega^2 + \sigma^{-2}\, \psi^4\, d\ell^2\}. \tag{11.90}$$

In these coordinates, the divergences on the left hand side of eqs. (11.80)-(11.82) have the form:

$$D_\alpha (AD^\alpha B) = \sigma\, \psi^{-2} e^{-2\tau} \{\partial_\omega (\sigma^{-1} \psi^2 A \partial_\omega B) + \partial_\ell (\sigma\psi^{-2}\, A\partial_\ell B)\}. \tag{11.91}$$

The Gaussian curvature in these coordinates has the form:

$$\begin{aligned} {}^2R = & -2\sigma\psi^{-2} e^{-2\tau} \{\partial_\omega (\sigma^{-1}\psi^2 \partial_\omega \tau) + \partial_\ell (\sigma\psi^{-2} \partial_\ell \tau)\} \\ & - 2\sigma\psi^{-2} e^{-2\tau} \partial_\omega \partial_\omega\, (\sigma^{-1}\psi^2). \end{aligned} \tag{11.92}$$

We note that one of Einstein's equations (11.80) is automatically satisfied. Therefore the three equations (11.81)-(11.83) determine the three functions which describe the gravitational field: σ, ψ and τ.

The gravitational field of a differentially rotating fluid can also be represented by three functions through a suitable choice of coordinates. The analysis is qualitatively similar to that given here for the case of rigid rotation, but the resulting field equations are not as simple. The reader is referred to the work of Abramowicz and Muchotrzeb (1976) for the details of the analysis.

11-4 Properties of Rotating Stellar Models

In this section we present the properties of stationary axisymmetric general relativistic stellar models which have been derived to this date. The picture of a general relativistic star which emerges from the results presented here is far less complete than our view of Newtonian stars obtined from the results in Part I of this dissertation. Hopefully the formal results which have been presented in this work will be useful in future efforts to make this picture more complete.

The first property which we present is the general relativistic analogue of Poincare's limit on the angular velocity of a rotating star (Theorem 5.1). The result as presented here was derived by Abramowicz (1973).

THEOREM 11.11 - *The velocity field,* u^α *, of a stationary ideal fluid stellar model which has expansion-free flow* $\nabla^\alpha u_\alpha = 0$, *must satisfy the following inequality*

$$\int \sqrt{-g}\ \omega^{\alpha\beta}\omega_{\alpha\beta}\ d^3x \leq 16\pi \int \sqrt{-g}\ (\rho+3p)d^3x + \int \sqrt{-g}\ \sigma_{\alpha\beta}\sigma^{\alpha\beta}\ d^3x \ . \tag{11.93}$$

Here, $\omega^{\alpha\beta}$ *and* $\sigma^{\alpha\beta}$ *are the rotation and shear of the velocity field (see eqs. 6.14 and 6.15) and the integrals are performed on the intersection of a spacelike surface with the world tube of the stellar model.*

PROOF: We compute the divergence of the acceleration for an expansion free congruence using eqs. (6.12)-(6.17) and (6.23). The result, often called Raychandhuri's equation, is given by

$$\nabla^\alpha a_\alpha = R_{\alpha\beta}u^\alpha u^\beta + \frac{1}{4}\sigma_{\alpha\beta}\sigma^{\alpha\beta} - \frac{1}{4}\omega_{\alpha\beta}\ \omega^{\alpha\beta}. \tag{11.94}$$

We use Euler's equation (6.32) to re-express the left hand side of eq. (11.94) in terms of the fluid variables:

$$\nabla^\alpha a_\alpha = -\nabla^\alpha\{(\rho+p)^{-1}\nabla_\alpha p\}. \tag{11.95}$$

We integrate eq. (11.95) over a finite length world tube formed by Lie transporting the support of the fluid on a spacelike surface along the integral curves of the timelike Killing vector field. Since the pressure must be decreasing at the surface of the star it follows that

$$\int \sqrt{-g}\, \nabla^{\alpha} a_{\alpha}\, d^3x dt \;=\; -\oint \sqrt{-g}\, (\, \rho + p\,)^{-1}\, \nabla^{\alpha} p\, d^3x_{\alpha} \;\geq\; 0. \qquad (11.96)$$

Since the solution is stationary, we may perform the integral over t to find that

$$\int \sqrt{-g}\, \nabla^{\alpha} a_{\alpha}\, d^3x \;\geq 0. \qquad (11.97)$$

Now integrate eq. (11.94) over one spacelike surface. Use eq. (11.97) and the form of the Ricci tensor for a fluid, eq. (6.31), to obtain eq. (11.93).■

In the special case of a rigidly rotating stationary axisymmetric stellar model, this theorem has a simpler and more useful form. In the coordinate system intruduced in eq. (11.91), the rotation scalar is given by:

$$\omega^{\alpha\beta}\omega_{\alpha\beta} \;=\; 2\psi^{-2}e^{-2\tau} \;. \qquad (11.98)$$

The shear vanishes for rigid rotation. Consequently, Theorem 11.11 for this situation implies the following inequality:

$$\int d\ell d\omega \;\;\geq\;\; 8\pi \int (\, \rho + 3p\,)\, \psi^2 e^{2\tau}\, d\ell d\omega \;. \qquad (11.99)$$

The integrals here are performed over the intersection of one of the two-surfaces orthogonal to the surfaces of transitivity, with the world tube of the stellar model.

The next result gives equations for the total mass and angular momentum of stationary and axisymmetric asymptotically flat spacetimes as three dimensional integrals over spacelike surfaces. These formulae were derived originally by Komar (1959).

THEOREM 11.12- *The total mass, M, and angular momentum, J, of a stationary and axisymmetric spacetime, which is asymptotically flat and singularity free, are given by the integrals*

$$M = \frac{1}{4\pi} \int \sqrt{-g}\, R^{\alpha}{}_{\beta} \eta^{\beta}\, d^3x_{\alpha} \quad \text{, and} \tag{11.100}$$

$$J = -\frac{1}{8\pi} \int \sqrt{-g}\, R^{\alpha}{}_{\beta} \xi^{\beta} d^3x_{\alpha} \; . \tag{11.101}$$

The integrals are performed over any asymptotically flat spacelike surface; and η^{α} *and* ξ^{α} *represent the stationary and axisymmetric Killing vector fields respectively.*

PROOF: We integrate the gradients of the Killing vector fields over the boundary of the asymptotically flat spacelike surface. Since the spacetime is singularity free this boundary is at spacelike infinity, so we can use the asymptotic form of the metric given in §6.3 to show that

$$M = -\frac{1}{4\pi} \oint \sqrt{-g}\, \nabla^{\alpha} \eta^{\beta}\, d^2x_{\alpha\beta} \quad \text{, and} \tag{11.102}$$

$$J = \frac{1}{8\pi} \oint \sqrt{-g}\, \nabla^{\alpha} \xi^{\beta}\, d^2x_{\alpha\beta} \; . \tag{11.103}$$

Now we use Stokes theorem and Lemma 7.1 to convert these surface integrals into the desired form. For example the integral for the stationary Killing vector field can be transformed as follows:

$$\oint \sqrt{-g}\, \nabla^{\alpha} \eta^{\beta}\, d^2x_{\alpha\beta} = \int \sqrt{-g}\, \nabla^{\alpha} \nabla_{\alpha} \eta^{\beta}\, d^3x_{\beta} \tag{11.104}$$

$$= -\int \sqrt{-g}\, R^{\beta}{}_{\alpha} \eta^{\alpha}\, d^3x_{\beta} \; . \tag{11.105}$$

A similar transformation on the axisymmetric Killing vector field gives eq. (11.101) from eq. (11.103). ■

For the case of a spacetime which contains only convection-free ideal fluid, the expressions for the total mass and angular momentum have the following forms:

$$M = \int \frac{\sqrt{-g}\,\sigma}{(\xi^\mu \xi_\mu)^{1/2}} \{\rho + 3p - \frac{2\Omega}{\psi} (\rho + p)[\eta^\alpha + \Omega\xi^\alpha]\xi_\alpha\} \, d^3x \,, \qquad (11.106)$$

$$J = -\int \frac{\sqrt{-g}\,\sigma}{(\xi^\mu \xi_\mu)^{1/2}} (\rho + p)\, \psi^{-1} [\eta^\alpha + \Omega\xi^\alpha]\xi_\alpha \, d^3x \,. \qquad (11.107)$$

These expressions are obtained by using the form of the Ricci tensor given by eq. (11.68) with Theorem 11.12 and integrating over a t = constant surface.

The next two properties of rotating stellar models which we present are derived by applying the maximum principle which describes the solutions to certain elliptic differential equations. We recall the needed form of the maximum principle here without proof. A complete discussion of this result, along with its proof may be found in Bers, John and Schecter (1964). We need to consider the uniformly elliptic differential operator

$$L = a^{ij}\partial_i\partial_j + a^i\partial_i \,. \qquad (11.108)$$

The continuous functions a^{ij} and a^i must satisfy the following conditions. In some bounded open set, Σ , there exist constants $K > 0$ and $m > 0$ such that

$$a^{ij} X_i X_j \geq m\, \delta^{ij} X_i X_j, \quad \text{and} \qquad (11.109)$$

$$|a^{ii}|^2 + |a^i|^2 \leq K, \qquad (11.110)$$

for all vectors X_i.

THEOREM 11.13- *If* L *is a uniformly elliptic differential operator on the bounded open set* Σ, *and if* $\mathsf{L}(u) \geq 0$ *in* Σ, *then* u *does not have a maximum in* Σ *or* u *is a constant.*

The next two theorems set limits on the possible values which may be taken by the functions describing stationary axisymmetric convection-free stellar models. The first theorem, proved by the author and independently by Hansen and Winicour (1977), shows that the "frame dragging" angular velocity, $-\xi^{\mu}\eta_{\mu}/\xi^{\nu}\xi_{\nu}$, (see for example Cohen and Brill 1968 or Lindblom and Brill 1974) is always positive whenever the angular velocity of the fluid is positive. Note that this result does not require the fluid to rotate rigidly.

THEOREM 11.14- *A stationary axisymmetric convection-free ideal fluid stellar model in which the fluid rotates with positive angular velocity,* $\Omega \geq 0$ *(see eq. 11.10), must have negative Killing scalar,* $\xi^{\alpha}\eta_{\alpha} \leq 0$, *with equality only on the rotation axis.*

PROOF: Rewrite eq. (11.20) of Lemma 11.6 using the Ricci tensor for an ideal fluid eq. (11.68) and Lemma 11.10. The result is given by

$$D^{\alpha}\{\sigma^{-1}(\eta_{\mu}\eta^{\mu})^{2}D_{\alpha}[\eta^{\nu}\xi_{\nu}/\eta^{\beta}\eta_{\beta}]\} = -16\pi\sigma(\rho + p)\Omega\psi^{-1}(\eta^{\alpha} + \Omega\xi^{\alpha})\eta_{\alpha} . \quad (11.111)$$

The vector field $\eta^{\alpha} + \Omega\xi^{\alpha}$ is timelike and future directed since it is proportional to the fluid velocity. Consequently we have the following inequalities:

$$\psi = (\eta^{\alpha} + \Omega\xi^{\alpha})(\eta_{\alpha} + \Omega\xi_{\alpha}) < 0, \text{ and} \quad (11.112)$$

$$(\eta^{\alpha} + \Omega\xi^{\alpha})\eta_{\alpha} < 0. \quad (11.113)$$

Since the fluid angular velocity, Ω, is assumed positive it follows that the right hand side of eq. (11.111) is negative, therefore

$$D^{\alpha}\{\sigma^{-1}(\eta^{\mu}\eta_{\mu})^{2}\, D_{\alpha}[\, \eta^{\nu}\xi_{\nu}/\eta^{\beta}\eta_{\beta}\,]\} \;\leq\; 0. \tag{11.114}$$

Equation (11.114) is defined on the two-surfaces orthogonal to the surfaces of transitivity of the Killing vector fields. The boundaries of this two-surface are the rotation axis and spacelike infinity (since the stellar model is singularity free by assumption). The function $\eta^{\nu}\xi_{\nu}/\eta^{\mu}\eta_{\mu}$ vanishes both on the rotation axis and at spacelike infinity. We now apply Theorem 11.13 to eq. (11.114) for the function $\eta^{\nu}\xi_{\nu}/\eta^{\mu}\eta_{\mu}$. It follows that this function has no minimum except on the boundary. Since the function vanishes on the boundary, it follows that

$$\eta^{\nu}\xi_{\nu}/\eta^{\mu}\eta_{\mu} \;\geq\; 0. \tag{11.115}$$

From the stationarity of the spacetime it follows that $\eta^{\nu}\eta_{\nu} < 0$ everywhere, consequently $\eta^{\nu}\xi_{\nu} \leq 0$. ■

The next theorem also gives inequalities on the Killing scalars, but only for the case of rigidly rotating stellar models. One of the scalars, $(\eta^{\alpha} + \Omega\xi^{\alpha})\xi_{\alpha}$, determines the sign of the angular momentum density of Theorem 11.12 (see eq. 11.107). Thus, this theorem demonstrates that the angular momentum density of a uniformly rotating stellar model has the same sign (everywhere) as the angular velocity of the fluid. The proof of this result is given by Hansen and Winicour (1975). Attempts to prove generalizations of this theorem for differentially rotating fluids have been made by Hansen and Winicour (1977); their preliminary results contain unphysical assumptions, however.

THEOREM 11.15- *A stationary axisymmetric convection-free rigidly rotating fluid stellar model, having positive fluid angular velocity* $\Omega > 0$, *must satisfy the following inequalities everywhere in the spacetime:*

$$(\eta^\alpha + \Omega\xi^\alpha)\xi_\alpha \geq 0, \qquad \textit{and} \tag{11.116}$$

$$(\eta^\alpha + \Omega\xi^\alpha)\eta_\alpha < 0, \tag{11.117}$$

with equality holding only on the rotation axis.

PROOF: We will introduce, as in §11.3, the Killing vector field $k^\alpha = \eta^\alpha + \Omega\xi^\alpha$. We take the curl of eq. (11.78) to show that

$$D^\alpha\{\sigma^{-1}\psi^2 D_\alpha(k^\mu\xi_\mu/\psi)\} = 0. \tag{11.118}$$

Theorem 11.13 shows that the function $k^\mu\xi_\mu/\psi$ cannot have any maxima or minima in the regions where eq. (11.118) is a uniformly elliptic equation. This equation fails to be uniformly elliptic whenever $\sigma = 0$, the rotation axis, or where $\psi = 0$, the "speed of light cylinder" where the vector k^α is null. Consider the region "inside" of the speed of light cylinder, where $\psi < 0$. This region is bounded by the rotation axis, pieces at spacelike infinity, and the speed of light cylinder. On the rotation axis $k^\mu\xi_\mu$ vanishes since ξ^α is zero there. At spacelike infinity $k^\mu\xi_\mu$ is positive: $k^\mu\xi_\mu = \Omega\xi^\mu\xi_\mu \geq 0$. On the speed of light cylinder $\psi = 0$, consequently $k^\mu\xi_\mu = \frac{1}{2}\,\Omega^{-1}(\Omega^2\xi^\mu\xi_\mu - \eta^\mu\eta_\mu) > 0$. Therefore, the function $k^\mu\xi_\mu/\psi$ is negative everywhere on the boundary of the interior of the speed of light cylinder. Since it can have no maxima or minima in this region because of eq. (11.118) it follows that $k^\mu\xi_\mu \geq 0$ in this region, with equality only on the rotation axis. A similar argument for the "exterior" of the speed of light cylinder,

$\psi > 0$ also gives $k^{\mu}\xi_{\mu} > 0$ in this region. This establishes eq. (11.116).

Equation (11.117) is automatically satisfied within the speed of light cylinder, since k^{α} is a future directed vector field there. Outside of the light cylinder is necessarily a vacuum region of the spacetime since the orbits of k^{α} are spacelike there. In a vacuum region the following identity follows from an applecation of Lemma 11.6

$$D^{\alpha}\{\sigma^{-1}\psi^{2} D_{\alpha}(k^{\mu}\eta_{\mu}/\psi)\} = 0. \qquad (11.119)$$

Thus, the function $k^{\mu}\eta_{\mu}/\psi$ can have no maxima or minima in this region. An analysis similar to that given above for $k^{\mu}\xi_{\mu}$, implies that $k^{\mu}\eta_{\mu} < 0$ everywhere. ∎

The next theorem is the general relativistic generalization of the Newtonian "rotation on cylinders" theorem (3.7). The proof of the relativistic theorem has been given in various forms by a number of people: Boyer (1965)(1966), Thorne (1971), Abramowicz (1971) (1973).

THEOREM 11.16– *Consider a stationary axisymmetric convection-free ideal fluid spacetime. If the fluid rotates rigidly, then the fluid is barotropic, and the level surfaces of the density function coincide with the level surfaces of the function* ψ *(see eq. 11.69). If the fluid is barotropic, then the level surfaces of the angular velcoity of the fluid coincide with the level surfaces of* $\psi^{-1}(\eta^{\alpha} + \Omega\xi^{\alpha})\xi_{\alpha}$.

PROOF: The results follow trivially by taking the curl of eq. (11.71) and applying the definition of a barotropic fluid, eq. (6.34). ∎

Abramowicz attempts to extend this theorem by proving that the level surfaces of Ω must have the topology of cylinders. It appears to me, however, that his analysis is faulty. He uses results derived by Carter (1973) for vacuum spacetimes, and incorrectly applies them to fluid spacetimes.

The final result which we present here classifies the convection-free stationary axisymmetric spacetimes according to their Petrov type. The proof, given by Glass and Wilkinson (1978), involves the analysis of the curvature tensors, of a convection-free spacetime, in the null tetrad formalism of Newman and Penrose (1962). Since we have not introduced the null tetrad formalism here, we will omit the proof.

THEOREM 11.17- *A stationary axisymmetric convection-free spacetime is either Petrov type I or D.*

This concludes our review of the properties of rotating stellar models in general relativity theory. The picture which we present is incomplete. One would expect to find that relativistic stellar models must have some sort of mirror symmetry in analogy with the Newtonian models, Theorem 4.6. But no proof of this has yet been found. As we discussed in §9, one would expect to find that static general relativistic stellar models must be spherical, but no proof has yet been found. We hope that this review will help us to comprehend the present state of knowlege of relativistic stars, and that it will inspire us to further insights.

APPENDIX I.

ON THE EVOLUTION OF THE HOMOGENEOUS ELLIPSOIDAL FIGURES†

By Steven L. Detweiler and Lee Lindblom

I. INTRODUCTION

In this paper we examine some of the effects of viscosity and of gravitational radiation on the homogeneous ellipsoidal figures. When an astrophysical system undergoes gravitational collapse (e.g., to form a white dwarf or a neutron star), the resulting compact object may be rapidly rotating so that the secular instabilities caused by the dissipative forces could cause evolution away from an axisymmetric state. The purpose of this paper is to examine how that evolution actually occurs, within the approximation of the homogeneous ellipsoidal figures.

The analysis of the secular instabilities of the Maclaurin spheroids illustrates how important the combined effects of viscosity and gravitational radiation can be on the homogeneous ellipsoidal figures. Chandrasekhar (1969 [hereafter referred to as E.F.E.], 1970*a*) demonstrates that the presence of either viscosity or gravitational radiation reaction induces a secular instability in the Maclaurin spheroids beyond the point of bifurcation of the Jacobi and Dedekind sequences. More recently, Lindblom and Detweiler (1977 [Paper I]) show that the presence of both viscosity and gravitational radiation reaction moves the point of the onset of secular instability beyond the point of bifurcation to a point determined by the ratio of the strengths of the dissipative forces. And, for a spheroid of given mass and density, one specific value of the viscosity of the fluid will cause the Maclaurin sequence to be stable all the way to the point of the onset of dynamical instability. Thus, the presence of both gravitational radiation reaction and viscosity drastically changes the discussion of the stability of the Maclaurin spheroids from the case where only one or the other of the dissipative forces is acting.

In the present work we find similar qualitative changes in the evolution of the slowly varying ellipsoids when both dissipative effects are included. In § II we review briefly the general equations of motion which govern the evolution of the ellipsoidal figures, including the effects of viscosity and gravitational radiation reaction.

Miller (1974) and, and Press and Teukolsky (1973) have studied numerically the evolution of the ellipsoidal figures including the effects of gravitational radiation and viscosity, respectively. Their analyses show that a perturbed Maclaurin spheroid, which lies in the region of secular instability, will evolve through a sequence of ellipsoids which lie near the Riemann *S* family. We use this qualitative feature of their results to derive an approximation scheme which allows the economical large-scale integration of the equations of motion. When the effects of viscosity and gravitational radiation are weak, the general motion of an ellipsoid will consist of (i) a large-scale motion from one place to another along the Riemann *S* surface, and (ii) small-scale hydrodynamical oscillations about the quasi-equilibrium Riemann *S* configurations. The second of these effects, although of secondary interest, causes the numerical integration of the equations of motion to be very inefficient. The time step size must be kept small with respect to the oscillation period in order to maintain numerical accuracy. We develop in § III an approximation scheme which suppresses the oscillations, and therefore allows for a quick and efficient integration of the large-scale effects of dissipation on the evolution.

In § IV the results of the numerical integration of the equations of motion for the slowly varying ellipsoidal figures are presented. We illustrate the qualitative features of the evolution (over most of the Riemann *S* surface) for ellipsoids having varying amounts of viscosity and gravitational radiation reaction. In particular, we illustrate the evolution in the limiting cases of purely viscous or purely radiative evolution. We also examine the critical case where the Maclaurin sequence is stabilized all the way to the point of the onset of a dynamical instability. Several intermediate cases are also examined.

† Reprinted from "On the Evolution of the Homogeneous Ellipsoidal Figures" in the Astrophysical Journal 213, 193-199 (1977).

II. THE EQUATIONS OF MOTION

The uniform density ellipsoidal figures are described by the 10 time-dependent parameters a_i, Ω_i, Λ_i $(i = 1, 2, 3)$, and p_c (see E.F.E.). The functions a_i are the lengths of the principal axes of the ellipsoid, Ω_i represents the angular velocity of the principal axes with respect to a nonrotating inertial reference frame, Λ_i measures the internal motion of the fluid as seen by an observer in the principal axis frame, p_c represents the central pressure of the fluid, and ρ is the uniform and time-independent mass density. The Riemann-Lebovitz equations describe the hydrodynamical evolution of these functions. They are conveniently written as

$$H_{ij} = 0 \,. \tag{1}$$

The elements H_{ij} are defined by cyclically permuting the subscripts 1, 2, 3 in equations (2a, b, c):

$$H_{11} = \frac{d^2a_1}{dt^2} - a_1(\Omega_2^2 + \Omega_3^2 + \Lambda_2^2 + \Lambda_3^2) + 2(a_2\Lambda_3\Omega_3 + a_3\Lambda_2\Omega_2) + 2\pi G\rho A_1 a_1 - \frac{2p_c}{\rho a_1}\,, \tag{2a}$$

$$H_{12} = a_1\frac{d\Lambda_3}{dt} - a_2\frac{d\Omega_3}{dt} + 2\left(\Lambda_3\frac{da_1}{dt} - \Omega_3\frac{da_2}{dt}\right) + a_1\Lambda_1\Lambda_2 + a_2\Omega_1\Omega_2 - 2a_3\Lambda_1\Omega_2\,, \tag{2b}$$

$$H_{13} = a_3\frac{d\Omega_2}{dt} - a_1\frac{d\Lambda_2}{dt} + 2\left(\Omega_2\frac{da_3}{dt} - \Lambda_2\frac{da_1}{dt}\right) + a_3\Omega_3\Omega_1 + a_1\Lambda_3\Lambda_1 - 2a_2\Omega_3\Lambda_1\,. \tag{2c}$$

In equations (2) the Newtonian gravitational potentials A_i are given by the integral expression,

$$A_i = \int_0^\infty \frac{a_1a_2a_3\,du}{(a_i^2 + u)[(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)]^{1/2}} \tag{3}$$

(note that $A_1 + A_2 + A_3 = 2$). To the nine equations of motion represented by equation (1) an additional constraint must be added, corresponding to the conservation of mass. Since the mass density is assumed to be a constant, this additional constraint reduces to

$$a_1a_2a_3 = \bar{a}^3 = \text{constant}\,. \tag{4}$$

The unit of length for this paper will be scaled so that $\bar{a} = 1$.

Equation (1) describes the hydrodynamical and the Newtonian gravitational effects on the evolution of the ellipsoid. In this discussion, the effects of viscosity and of gravitational radiation reaction will be of interest; therefore terms which describe those effects must be added to equation (1). The terms which describe the viscous interaction were first written for these ellipsoidal objects by Rosenkilde (1967) and later in a notation more closely related to the one used here by Press and Teukolsky (1973). We describe the effects of viscosity with the average viscosity of the ellipsoid ν and the matrix V_{ij}:

$$V_{11} = \frac{10}{a_1^2}\frac{da_1}{dt}\,, \tag{5a}$$

$$V_{12} = \frac{5}{a_2}\left(\frac{a_1}{a_2} - \frac{a_2}{a_1}\right)\Lambda_3\,, \tag{5b}$$

$$V_{13} = \frac{5}{a_3}\left(\frac{a_3}{a_1} - \frac{a_1}{a_3}\right)\Lambda_2\,. \tag{5c}$$

The other elements of V_{ij} can be obtained from equations (5) by cyclically permuting the subscripts 1, 2, 3.

The terms describing gravitational radiation reaction for the ellipsoidal figures were derived by Chandrasekhar (1970*a*, *b*) and modified to the form employed here by Miller (1974). This interaction employs the coupling constant $\mathscr{G} = 2GM/25c^5$ and the matrix G_{ij} defined by

$$G_{ij} = \sum_{\alpha=0}^{5}\sum_{\beta=0}^{\alpha} C^5{}_\alpha C^\alpha{}_\beta R^\beta{}_{in}\frac{d^{5-\alpha}Q_{nk}}{dt^{5-\alpha}}R^{\alpha-\beta}{}_{lk}A_{lj}\,, \tag{6}$$

where

$$C^\alpha{}_\beta = \alpha!/[(\alpha-\beta)!\beta!] \qquad \text{and} \qquad A_{ij} = \text{diag}\,(a_1, a_2, a_3)\,.$$

The quantity Q_{ij} is defined by

$$Q_{ij} = A_{ik}A_{kj} - \tfrac{1}{3}\delta_{ij}A_{kl}A_{kl}\,,$$

and is proportional to the quadrupole tensor for the ellipsoids. Lowercase Latin indices take the values 1, 2, 3, and summation is implied for repeated indices. The rotation matrices are defined by

$$R^0{}_{ij} = \delta_{ij} \qquad \text{and} \qquad R^{\alpha+1}{}_{ij} = \left(\delta_{il}\frac{d}{dt} - \epsilon_{ilm}\Omega_m\right)R^\alpha{}_{lj} .$$

In terms of the quantities defined above, the complete description of the evolution of the ellipsoidal figures is given by

$$H_{ij} + \nu V_{ij} + \mathscr{G}G_{ij} = 0 . \tag{7}$$

III. AN APPROXIMATION TO THE EQUATIONS OF MOTION

Equation (7) forms a set of ordinary differential equations which, in principle, may be integrated in a straightforward manner. However, in practice there is a major drawback: these equations govern not only the long-time-scale evolution of an ellipsoid due to the dissipative forces, but also the short-time-scale hydrodynamical oscillations of an ellipsoid which is not in perfect equilibrium. Thus, while we are primarily interested in the long-time-scale evolution near the stationary Riemann S ellipsoids, a numerical integration of equation (7) necessitates an extremely short time step size to refrain from losing information about the hydrodynamical oscillations.

In the derivation of equation (7) it is necessarily assumed that the effects of radiation reaction and of viscosity are small, but cumulative. We are led, therefore, to seek solutions to the equations of motion which are slowly evolving from one quasi-equilibrium configuration to another. Thus, in the equations of motion we assume that the velocities $d(a_i, \Omega_i, \Lambda_i)/dt$ and the dissipative coefficients ν and $\mathscr{G}$ are small, and subsequently drop terms which are higher than first order in these small quantities. These assumptions are equivalent to assuming as an initial configuration an ellipsoid which lies very near the Riemann S surface, and whose subsequent evolution consists of a slow cumulative motion in addition to small amplitude oscillations.

We now expand the terms in equation (7) to first order in the small quantities: ν, $\mathscr{G}$, and $d(a_i, \Omega_i, \Lambda_i)/dt$. The initial configuration is taken to be near the Riemann S surface; in particular we take $\Lambda_1 = \Lambda_2 = \Omega_1 = \Omega_2 = 0$ initially (we set $\Omega_3 = \Omega$, $\Lambda_3 = \Lambda$). The nonzero components of the viscous and radiative matrices to this order are given by

$$V_{12} = a_1V_{21}/a_2 = 5(a_1{}^2 - a_2{}^2)\Lambda/a_1a_2{}^2 , \tag{8}$$

and

$$G_{12} = a_2G_{21}/a_1 = 16\Omega^5a_2(a_1{}^2 - a_2{}^2) . \tag{9}$$

The resulting equations of motion are the following:

$$H_{11} = -a_1(\Lambda^2 + \Omega^2) + 2a_2\Lambda\Omega + 2\pi G\rho a_1A_1 - 2p_c/\rho a_1 = 0 , \tag{10a}$$

$$H_{22} = -a_2(\Lambda^2 + \Omega^2) + 2a_1\Lambda\Omega + 2\pi G\rho a_2A_2 - 2p_c/\rho a_2 = 0 , \tag{10b}$$

$$H_{33} = 2\pi G\rho a_3A_3 - 2p_c/\rho a_3 = 0 , \tag{10c}$$

$$H_{12} + \nu V_{12} + \mathscr{G}G_{12} = a_1\frac{d\Lambda}{dt} - a_2\frac{d\Omega}{dt} + 2\Lambda\frac{da_1}{dt} - 2\Omega\frac{da_2}{dt} + 5\nu\Lambda\frac{a_1{}^2 - a_2{}^2}{a_1a_2{}^2} + 16\mathscr{G}\Omega^5a_2(a_1{}^2 - a_2{}^2) = 0 , \tag{11a}$$

and

$$H_{21} + \nu V_{21} + \mathscr{G}G_{21} = a_1\frac{d\Omega}{dt} - a_2\frac{d\Lambda}{dt} + 2\Omega\frac{da_1}{dt} - 2\Lambda\frac{da_2}{dt} + 5\nu\Lambda\frac{a_1{}^2 - a_2{}^2}{a_1{}^2a_2} + 16\mathscr{G}\Omega^5a_1(a_1{}^2 - a_2{}^2) = 0 . \tag{11b}$$

The off diagonal equations,

$$H_{13} = H_{31} = H_{23} = H_{32} = 0 , \tag{12}$$

guarantee that the vorticity and the rotation axes maintain their orientation ($\Lambda_1 = \Lambda_2 = \Omega_1 = \Omega_2 = 0$) as the ellipsoid evolves.

Equations (10) may be solved for Λ, Ω, and p_c in terms of the values of a_i:

$$(\Lambda - \Omega)^2 = 2\pi G\rho\left[\frac{a_1A_1 + a_2A_2}{a_1 + a_2} - \frac{a_3{}^2A_3}{a_1a_2}\right] , \tag{13a}$$

$$(\Lambda + \Omega)^2 = 2\pi G\rho\left[\frac{a_1A_1 - a_2A_2}{a_1 - a_2} + \frac{a_3{}^2A_3}{a_1a_2}\right] , \tag{13b}$$

and

$$2p_c/\rho = 2\pi G\rho a_3{}^2A_3 . \tag{13c}$$

The relationships illustrated in equations (13) are identical to those of a Riemann S-type ellipsoid. Thus to the zeroth order in the small quantities the ellipsoid is instantaneously of Riemann S type; and to the first order in the small quantities the evolution of the ellipsoid is governed by equations (11).

The equations of evolution may be cast in a more useful form. To accomplish this, the equations relating Λ and Ω to a_1 and a_2 (eqs. [13a, b]) may be differentiated with respect to time to obtain expressions for $d\Lambda/dt$ and $d\Omega/dt$ as linear combinations of da_1/dt and da_2/dt. These expressions can then be used to eliminate $d\Lambda/dt$ and $d\Omega/dt$ from the equations of motion (11) and the resulting relationships solved to obtain equations for da_1/dt and da_2/dt. The final form of these equations are given by

$$\frac{da_i}{dt} = \nu b_i + \mathscr{G} c_i\,, \quad i = 1, 2, 3\,. \tag{14}$$

The quantities b_i and c_i are functions of a_1, a_2, a_3, Λ, and Ω; thus, these equations can be easily integrated numerically to explore the orbits of the slowly varying ellipsoidal figures. The precise expressions for the coefficients b_i and c_i are given by

$$b_1 = -5\Lambda \frac{a_1^2 - a_2^2}{a_1^2 a_2^2} \frac{Q(2, 1)(a_1 + a_2) + Q(2, -1)(a_1 - a_2)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{15a}$$

$$b_2 = -5\Lambda \frac{a_1^2 - a_2^2}{a_1^2 a_2^2} \frac{Q(1, -1)(a_1 - a_2) - Q(1, 1)(a_1 + a_2)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{15b}$$

$$b_3 = -a_3\left(\frac{b_1}{a_1} + \frac{b_2}{a_2}\right), \tag{15c}$$

$$c_1 = -16\Omega^5(a_1^2 - a_2^2) \frac{Q(2, 1)(a_1 + a_2) + Q(2, -1)(a_2 - a_1)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{16a}$$

$$c_2 = -16\Omega^5(a_1^2 - a_2^2) \frac{Q(1, -1)(a_2 - a_1) - Q(1, 1)(a_1 + a_2)}{Q(1, 1)Q(2, -1) + Q(1, -1)Q(2, 1)}\,, \tag{16b}$$

$$c_3 = -a_3\left(\frac{c_1}{a_1} + \frac{c_2}{a_2}\right). \tag{16c}$$

In equations (15) and (16) we have made use of the symbols $Q(\alpha, \epsilon)$ which are defined by

$$Q(\alpha, \epsilon) = \frac{\pi G\rho}{\Lambda - \epsilon\Omega}\Bigg[A_\alpha + a_\alpha A_{\alpha,\alpha} + \epsilon a_\beta A_{\beta,\alpha} - a_3 A_{\alpha,3} - \frac{a_3 a_\beta}{a_\alpha} A_{\beta,3} - \frac{a_1 A_1 + \epsilon a_2 A_2}{a_1 + \epsilon a_2} + (\epsilon)^\alpha \frac{a_3^2}{a_\alpha^2 a_\beta}(a_1 + \epsilon a_2)(3A_3 + a_3 A_{3,3} - a_\alpha A_{3,\alpha}) + \frac{2}{\pi G\rho}(\Lambda - \epsilon\Omega)^2\Bigg]\,, \tag{17}$$

with $\alpha, \beta = 1, 2$; $\alpha \neq \beta$; $\epsilon = \pm 1$; and $A_{i,j} \equiv \partial A_i/\partial a_j$.

The Riemann S algebraic constraints, equations (13), along with the evolution equations (14), form the complete description of the slowly varying ellipsoidal figures. The evolution equations are rather complicated; however, the form in which they are presented allows them to be integrated numerically in a straightforward manner. We have performed these integrations and discuss the results in § IV.

IV. NUMERICAL RESULTS

The evolution of the slowly varying ellipsoidal figures is described by equations (14). From these equations it is clear that the evolutionary trajectory of a given initial ellipsoid is determined solely by the ratio of the viscous time scale to the radiation-reaction time scale. (The rate at which the ellipsoid evolves along the trajectory is proportional to the magnitudes of both dissipative time scales, however.) Thus, as in Paper I, it is convenient to introduce a dimensionless constant

$$X = \frac{125(1 - e_0^2)^{2/3}}{2\Omega_0^4} \frac{\nu}{GM\bar{a}^4/c^5}\,, \tag{18}$$

where Ω_0 and e_0 are the angular velocity and the eccentricity of the Maclaurin spheroid at the point of the onset of the dynamical instability; so that

$$\Omega_0^2/\pi G\rho = 0.44022 \quad \text{and} \quad e_0 = 0.95289\,.$$

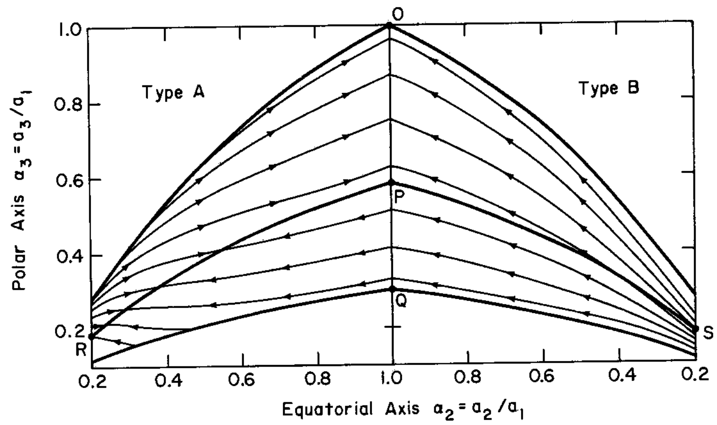

FIG. 1.—The evolutionary paths of the slowly varying ellipsoids for $X = 0$ (only radiation reaction) or for $X = \infty$ (only viscosity). The Jacobi and Dedekind sequences are RP and PS; the stable Maclaurin sequence is OP, and the secularly unstable Maclaurin sequence is PQ.

The constants appearing in equation (18) are chosen such that if $X = 1$, then (as discussed in Paper I) the entire Maclaurin sequence is stable up to the point of the onset of a dynamical instability, point Q in Figure 1. X is a certain ratio of the viscous to the gravitational radiation time scale; thus, the trajectories of the ellipsoids are determined simply by specifying X. The evolution of the ellipsoid will tend to be viscosity dominated if $X > 1$ and radiation reaction dominated if $X < 1$.

We have examined the evolution of the slowly varying ellipsoidal figures by numerically integrating equations (14). As a test of the approximation discussed in § III we compared some of our trajectories with those tabulated by Miller (1974) and found, as expected, that her ellipsoids performed small oscillations about a sequence of Riemann S-type ellipsoids which satisfied equations (14). As a second check we evolved ellipsoids with either the viscosity or the radiation reaction forces absent, and found that either the circulation or the angular momentum, respectively, were conserved as required for these interactions (see E.F.E. and Miller 1974).

The results of our numerical analysis are presented in Figures 1–5. For different choices of the parameter X the evolutionary tracks effectively cover the Riemann S surface. Figure 1 serves the dual roles of illustrating the evolution of an ellipsoid with X either zero or infinite. For $X = 0$ the evolution is caused solely by the radiation reaction; in this case Type A corresponds to the Dedekind-like ($|\Omega| < |\Lambda|$) ellipsoids, Type B to the Jacobi-like ($|\Omega| > |\Lambda|$); the trajectories are contours of constant circulation; all of the evolution is directed either toward the stable portion of the Maclaurin sequence, OP, or toward the Dedekind sequence, RP. Similarly for $X = \infty$, the evolution is caused solely by the viscosity; Type A corresponds to the Jacobi-like ellipsoids and Type B to the Dedekind-like; the trajectories are contours of constant angular momentum; all of the evolution is directed either toward OP or toward the Jacobi sequence, RP.

Figures 2–4 illustrate the evolutionary tracks for values of X ranging from 10^{-4} to 50. For $X = 50$ (Fig. 5) the typical evolution may proceed as follows: starting below the Dedekind sequence on the left half of the diagram, the

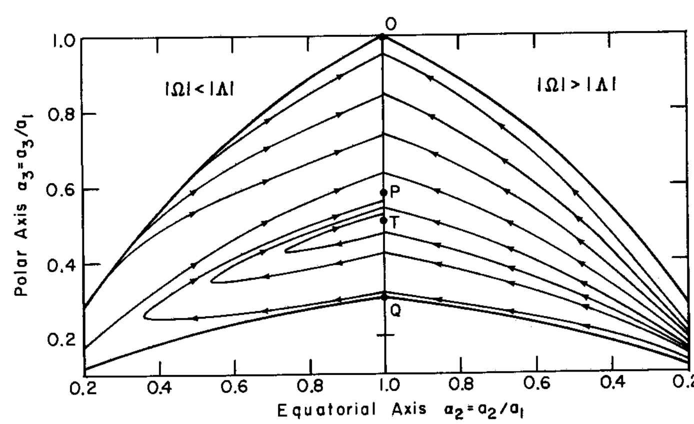

FIG. 2.—The evolutionary paths for $X = 10^{-4}$. The point P is the point of bifurcation of the Jacobi and Dedekind sequences. The stable Maclaurin spheroids are OT; the secularly unstable Maclaurin spheroids are TQ.

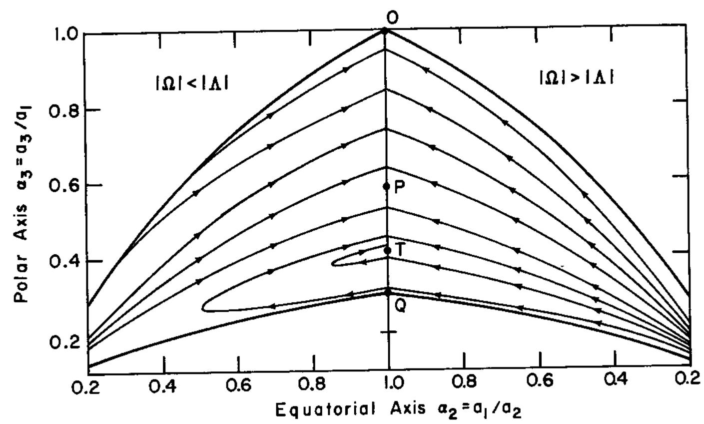

FIG. 3.—The evolutionary paths for $X = 10^{-2}$

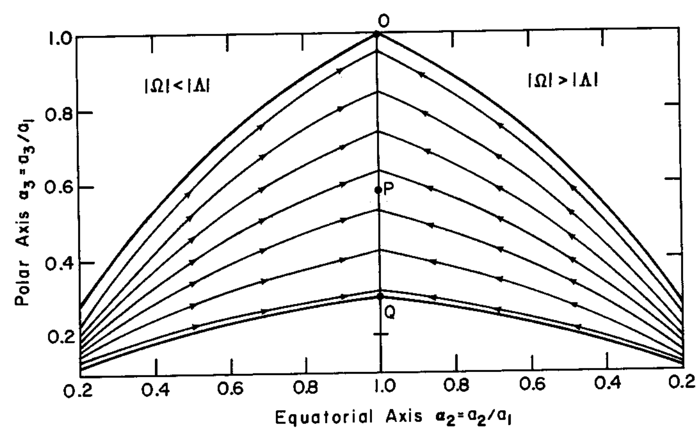

FIG. 4.—The evolutionary paths for $X = 1$. The point Q is the point of the onset of dynamical instability. All of the Maclaurin spheroids along OQ are stable.

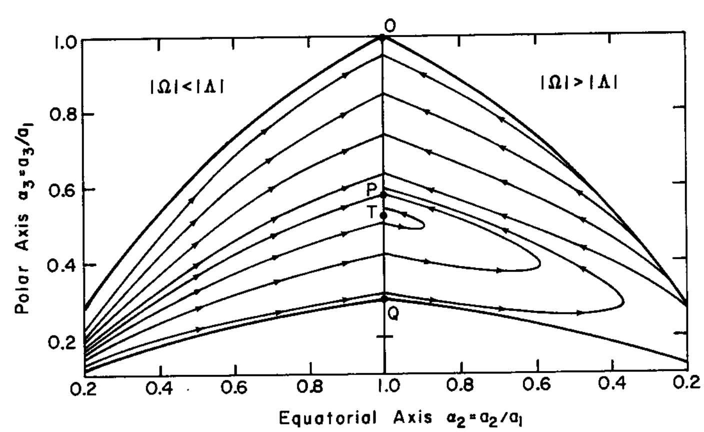

FIG. 5.—The evolutionary paths for $X = 50$

ellipsoid is driven up and toward the Maclaurin sequence nearly along a contour of constant angular momentum. If it approaches the Maclaurin sequence below the point T, the point of the onset of secular instability, it will be evolving toward an unstable Maclaurin spheroid. In a physically realistic situation a small perturbation is now needed to move the ellipsoid across the Maclaurin sequence into the Jacobi-like region. The evolution continues then toward the Jacobi sequence (PS in Fig. 1); the circulation decreases because of viscous dissipation, and the quadrupole moment increases as the ellipsoid tends toward a "rotating cigar" configuration. Eventually viscosity ceases to play the dominant role in the evolution—not because of the lack of viscosity but rather because the ellipsoid is nearly rigidly rotating. With a large rotating quadrupole moment, the ellipsoid now loses angular momentum in the form of gravitational radiation and the evolution proceeds back toward a stable member of the Maclaurin sequence. Analogous evolutionary scenarios describe each of the other figures.

For a choice of viscosity such that $X = 1$ (Fig. 4) all of the evolutionary tracks lead directly toward the Maclaurin sequence implying that the entire line OQ is in stable equilibrium. If a Maclaurin spheroid anywhere along the line OQ is displaced slightly, it simply evolves back toward the Maclaurin sequence.

Figures 1–5 clearly illustrate the results of Paper I. A Maclaurin spheroid which lies on the sequence OT is stable; if perturbed slightly, it simply evolves back toward the Maclaurin sequence. On the other hand, a spheroid on the sequence TQ is unstable; if it is perturbed, it evolves away from the Maclaurin sequence. For the limiting cases of purely viscous or purely radiative dissipation, Figure 1 shows that the point T, the onset of secular instability, coincides with the bifurcation point P. Figure 3 illustrates the case $X = 1$, showing that the point T coincides with Q, the point of dynamical instability.

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APPENDIX II.

ON THE SECULAR INSTABILITIES OF THE MACLAURIN SPHEROIDS†

BY Lee Lindblom and Steven L. Detweiler

The secular instability of the Maclaurin spheroids due to viscosity or gravitational radiation reaction (Chandrasekhar 1969, 1970*a*) is by now well understood; and similar instabilities of more general classes of stellar models are known to exist (Ostriker and Bodenheimer 1973; Miller 1973). The overall effect of such a secular instability is to place an upper limit on the ratio of the rotational kinetic energy, T, to the gravitational potential energy, W. This upper limit occurs at the point of bifurcation where the value of $|T/W|$ is given approximately by 0.14. After exhausting its nuclear fuel, a star may collapse to a state having $|T/W| > 0.14$, if it is rotating rapidly enough. In such a case it was thought that gravitational radiation reaction and viscosity would force the star to evolve through nonaxisymmetric configurations wherein T would be dissipated until $|T/W|$ approached 0.14.

We find that for Maclaurin spheroids this picture is not correct in the presence of *both* viscous and radiation reaction forces. Rather, we find that the secular instabilities caused by viscosity and by gravitational radiation tend to cancel each other. The particular point at which the secular instability actually sets in depends on X (see eq. [6]), the ratio of the strengths of the viscous and the gravitational forces. For a particular choice of X the stable portion of the Maclaurin sequence can be extended all the way to the point of the onset of dynamical instability, corresponding to $|T/W| = 0.274$.

The cancellation of the secular instabilities occurs because viscous dissipation and radiation reaction cause different modes to become unstable. In particular, the mode which is not unstable to a particular dissipative force is in fact stabilized by that force. Thus, for example, the mode which is unstable to radiation reaction is stabilized by viscosity. For a suitable choice of the ratio of the strengths of the dissipative forces, the stabilizing terms dominate in both modes.This results in stable Maclaurin spheroids past the point of bifurcation.

To show precisely how the cancellation of instabilities occurs, we must examine the perturbations of the Maclaurin spheroids. This subject has been discussed at length by Chandrasekhar (1969, 1970*b*); we adopt the notation of that work, and refer the reader thereto for the method of derivation of the equations which we employ.

We have examined all of the "second harmonic modes" of the Maclaurin spheroids, and find that only the toroidal modes have instabilities which are induced by the dissipative effects. The perturbation is described (in the corotating frame of the fluid) by a Lagrangian displacement of the form $\xi_i(x)e^{i\sigma t}$. The equations for the toroidal modes are expressed in terms of the quantity

$$V_{ij} = \int_V \rho(\xi_i x_j + \xi_j x_i)d^3x \,. \tag{1}$$

It is a straightforward matter to generalize the work of Chandrasekhar to obtain the equations governing the toroidal modes, with the inclusion of the effects of viscosity and gravitational radiation reaction. These equations are

$$[-\sigma^2 + 2(2B_{11} - \Omega^2) + 10i\sigma\nu/a_1^2 + 2DQ_1](V_{11} - V_{22}) - 4[i\sigma\Omega - \tfrac{1}{2}DQ_2]V_{12} = 0\,, \tag{2a}$$

and

$$[-\sigma^2 + 2(2B_{11} - \Omega^2) + 10i\sigma\nu/a_1^2 + 2DQ_1]V_{12} + [i\sigma\Omega - \tfrac{1}{2}DQ_2](V_{11} - V_{22}) = 0. \tag{2b}$$

† Reprinted from "On the Secular Instabilities of the Maclaurin Spheroids" in the Astrophysical Journal <u>211</u>, 565-567 (1977).

In these equations we have used the symbols:

ν, viscosity;

$D = (\pi G\rho)^{3/2} G M a_1^2 / 5c^5$, gravitational radiation;

a_1, a_3, equatorial and polar radii of spheroid;

Ω, angular velocity of spheroid;

$$B_{11} = \int_0^\infty a_1^2 a_3 (a_1^2 + u)^{-3} (a_3^2 + u)^{-1/2} u\,du;$$

$$Q_1 = -2i\sigma(\sigma^2 + 12\Omega^2)(\Omega^2 - 2B_{11}) - \tfrac{3}{5} i\sigma^5 - 8i\sigma^3\Omega^2 + 16i\sigma\Omega^4;$$

$$Q_2 = 8\Omega(3\sigma^2 + 4\Omega^2)(\Omega^2 - 2B_{11}) + 8\sigma^4\Omega - 128\Omega^5/5.$$

Equations (2) have nontrivial solutions if and only if the frequency, σ, satisfies the characteristic equation,

$$0 = \sigma^2 - 2\sigma\Omega - 2(2B_{11} - \Omega^2) - 10i\nu\sigma/a_1^2 + 4iD(2\Omega - \sigma)^3[2B_{11} - \Omega^2 + \tfrac{1}{10}(2\Omega - \sigma)(4\Omega + 3\sigma)]. \quad (3)$$

In the nondissipative limit (i.e., $D = \nu = 0$) the solutions to equation (3) for the characteristic frequencies of the toroidal modes are given by

$$\sigma_0^{(1)} = \Omega - (4B_{11} - \Omega^2)^{1/2}, \quad (4a)$$

and

$$\sigma_0^{(2)} = \Omega + (4B_{11} - \Omega^2)^{1/2}. \quad (4b)$$

When the effects of viscosity and gravitational radiation reaction are small (an assumption used in the derivation of eq. [2]), equation (3) may be solved approximately. Let $\sigma \approx \sigma_0 + \Delta\sigma$ represent the solution to equation (3) where $\Delta\sigma$ is considered to be small. It follows that

$$i\Delta\sigma = \frac{2D(2\Omega - \sigma_0)^5}{5(\sigma_0 - \Omega)} - \frac{5\nu\sigma_0}{a_1^2(\sigma_0 - \Omega)}. \quad (5)$$

Equation (5) can be written in a more convenient form by defining X, the ratio of the strengths of the dissipative forces:

$$X = 25\nu(1 - e_0^2)^{2/3} / [2a_1^2 D\Omega_0^4 (\pi G\rho)^{-3/2} (1 - e^2)^{2/3}]. \quad (6)$$

The quantity Ω_0 is the angular velocity of the Maclaurin spheroid having $e_0 = 0.95289$, the point of dynamical instability ($\Omega_0^2/\pi G\rho = 0.44022$). When X is defined as in equation (6), it is a function only of the total mass, average radius, and average viscosity of the ellipsoid (see eq. [8]). Equation (5) can be rewritten as

$$i\Delta\sigma^{(1)} = -\frac{2D}{5(4B_{11} - \Omega^2)^{1/2}} \left[(\sigma_0^{(2)})^5 - X\Omega_0^4\sigma_0^{(1)} \frac{(1 - e^2)^{2/3}}{(1 - e_0^2)^{2/3}}\right], \quad (7a)$$

and

$$i\Delta\sigma^{(2)} = -\frac{2D}{5(4B_{11} - \Omega^2)^{1/2}} \left[(\sigma_0^{(1)})^5 - X\Omega_0^4\sigma_0^{(2)} \frac{(1 - e^2)^{2/3}}{(1 - e_0^2)^{2/3}}\right]. \quad (7b)$$

These equations (7a, b) give expressions for the imaginary part of the characteristic frequencies of the toroidal modes. If either expression is positive, an instability occurs.

We have evaluated these equations (3) and (7) numerically for various values of the eccentricity ($e^2 = 1 - a_3^2/a_1^2$) of the spheroids and the ratio of the dissipative strengths, X. Figure 1 illustrates the results of these computations. This figure depicts the critical eccentricity (where instability first sets in) as a function of X. The entire region below the curve is stable. Note that for very large or very small values of X the critical eccentricity approaches 0.81267, the bifurcation point; this corresponds to the limiting cases with pure viscosity or pure radiation reaction. Also note that for $X = 1$, the region of stability is extended all the way to $e = 0.95289$, the point of dynamical instability. For values of $X < 1$ the spheroid is radiation-reaction dominated, and it is the $\sigma_0^{(2)}$ mode which becomes unstable; for $X > 1$, viscosity dominates and the $\sigma_0^{(1)}$ mode is unstable.

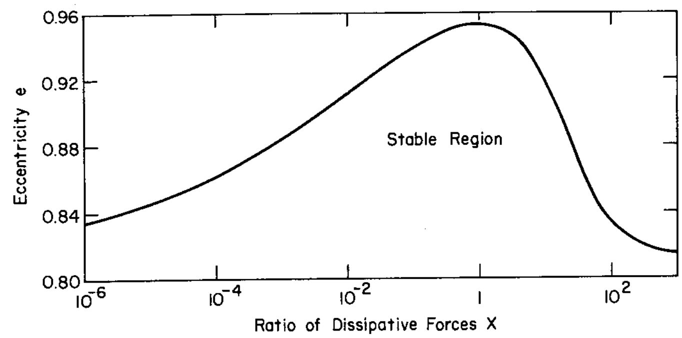

FIG. 1.—The eccentricity of the Maclaurin spheroid at the onset of secular instability is illustrated for given values of the ratio of the viscous force to the gravitational radiation reaction force.

To relate the scale of the parameter X to a more astrophysically relevant set of units, we note that equation (6) may be written in the form

$$X = 5.863 \times 10^{-3}\nu(\mathrm{cm^2\ s^{-1}})\left(\frac{R}{R_\odot}\right)^2\left(\frac{M_\odot}{M}\right)^3 . \tag{8}$$

The average radius of the ellipsoid is R and its mass is M. The maximum stabilizing effect occurs when $X = 1$. For a star of given mass and given average radius, we define the critical viscosity to be the one for which $X = 1$,

$$\nu_c = 170.6\left(\frac{R_\odot}{R}\right)^2\left(\frac{M}{M_\odot}\right)^3(\mathrm{cm^2\ s^{-1}}) . \tag{9}$$

Table 1 lists the estimated actual viscosity, ν, along with the critical viscosity ν_c for different types of stars. The actual viscosity of the Sun is estimated to be very close to the critical value; however, the time scales for both the viscous and the gravitational radiation induced evolution are so long in this case as to be ignorable. The compact objects (white dwarfs and neutron stars) are listed as having very small actual viscosities. Those estimates are based on the viscosity of a degenerate gas, and ignore the possibility of a crystalline structure which could increase the real values by many orders of magnitude.

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TABLE 1

CRITICAL VISCOSITIES FOR VARIOUS TYPES OF STARS

Type of Star	Estimated Actual Viscosity ($\mathrm{cm^2\ s^{-1}}$)	Critical Viscosity ($\mathrm{cm^2\ s^{-1}}$)
Sun*	$1 < \nu < 10^8$	$\nu_c = 170$
White dwarfs†	$10^{-1} < \nu < 10$	$10^4 < \nu_c < 10^8$
Neutron stars‡	$\nu \approx 1$	$10^7 < \nu_c < 10^{14}$

* Kopal 1968.
† Durisen 1973.
‡ Ruderman 1968.

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