- There is a choice of projective parameter on these conformal geodesics so that the singularity is at a finite distance.
- The conformal factor dictated by this choice makes the rescaled fluid expansion finite.
- The conformal (tractor) curvatures are finite in the conformally-propagated frame.

Then the initial singularity is a conformal gauge singularity.

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A new generalized harmonic evolution system

Lee Lindblom

(joint work with M. A. Scheel, L. E. Kidder, R. Owen, and O. Rinne)

This report describes recent work on finding a formulation of the Einstein equations suitable for constructing stable numerical evolutions. The formulation studied here specifies the coordinate degrees of freedom with a generalized harmonic gauge source function rather than with the usual lapse and shift. This type of formulation appears to have played a critical role in the very impressive binary black hole evolutions performed recently by Pretorius [1, 2]. This report analyzes why this type of formulation is so effective for numerical work, describes a recent extension of the system that makes it possible to construct boundary conditions which prevent the influx of constraint violations, and describes numerical tests that demonstrate the effectiveness of the new equations and boundary conditions.

The gauge source function H_a is defined as the action of the scalar-wave operator on the coordinate functions x^a:

$$H_a(x) \equiv \psi_{ab}\nabla^c\nabla_c x^b = -\psi^{bc}\Gamma_{abc} \equiv -\Gamma_a, \tag{1}$$

where ψ_{ab} is the spacetime metric and Γ_{abc} is the usual Christoffel symbol. The coordinates are fixed in this approach by requiring that $\Gamma_a = -H_a$, for a prescribed

H_a. The existence of solutions to the inhomogeneous wave Eq. (1) guarantees the existence of such coordinates. Choosing the coordinates in this way has two important consequences. The first is well known: the vacuum Einstein equations,

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \tag{2}$$

where $\mathcal{C}_a = H_a + \Gamma_a$, are manifestly hyperbolic since the principal part of the equations is just $\psi^{cd}\partial_c\partial_d\psi_{ab}$ for any value of the gauge source function [3]. The second consequence is less widely appreciated: The constraints of the system are profoundly transformed. The condition $\mathcal{C}_a = 0$ is the primary constraint of this system, while the standard Hamiltonian and momentum constraints $\mathcal{M}_a = G_{ab}t^b$ (where t^a is the unit normal to a Cauchy surface) are determined by the derivatives of $\mathcal{C}_a$: $\mathcal{M}_a = t^b(\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c)$. This means that the primary constraints depend on the first but not the second derivatives of the metric.

Adding multiples of the constraints to the Einstein equations is known to have a significant effect on the growth rates of constraint-violating solutions [4]. However, multiples of the Hamiltonian and momentum constraints can be added only in very restricted ways consistent with the hyperbolic structure of the equations; this is because the addition of these constraints changes the principal part of the equations. In contrast, arbitrary multiples of the gauge constraint $\mathcal{C}_a$ can be added to the system, Eq. (2), without effecting the hyperbolic structure at all. Pretorius [2], based on the suggestion of Gundlach, et al. [5], used a modified evolution system that included the following additional gauge constraint terms designed to suppress the growth of the constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\big[t_{(a}\mathcal{C}_{\,b)} - \frac{1}{2}\psi_{ab}\, t^c\mathcal{C}_c\big]. \tag{3}$$

The Bianchi identities then imply that $\mathcal{C}_a$ satisfies the damped wave equation,

$$0 \;=\; \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^b[\,t_{(b}\mathcal{C}_{a)}] + \mathcal{C}^{\,b}\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\gamma_0\, t_a\mathcal{C}^{\,b}\mathcal{C}_b, \tag{4}$$

which exponentially suppresses all small short-wavelength constraint violations when the parameter γ_0 is positive [5]. This constraint-suppressing feature of the modified generalized harmonic system, Eq. (3), contributed significantly to the success of Pretorius' impressive binary black-hole evolutions [2].

We have recently extended the modified generalized harmonic evolution system, Eq. (3), to a first-order symmetric-hyperbolic form. (See Ref. [6] for the details.) This new system is linearly degenerate, so shocks do not form from smooth initial data. This system also includes new constraints that arise when additional fields are added to make the system first order. Appropriate terms (proportional to the constraints times a second constraint-damping parameter γ_2) are added to suppress the growth of these new constraints. Constraint-preserving and physical boundary conditions are also presented, and the well-posedness of the new evolution system with these boundary conditions is analyzed.

We tested the new evolution system by evolving initial data for a Schwarzschild black hole. In these evolutions we "freeze" the values of the incoming characteristic fields on the boundaries. We performed these numerical evolutions using spectral

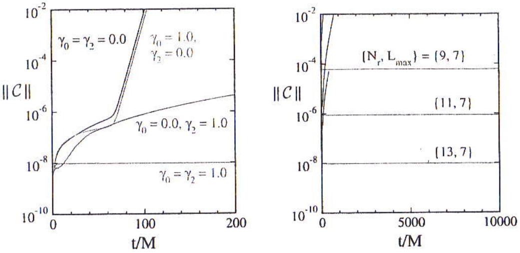

FIGURE 1. Evolution of Schwarzschild initial data.

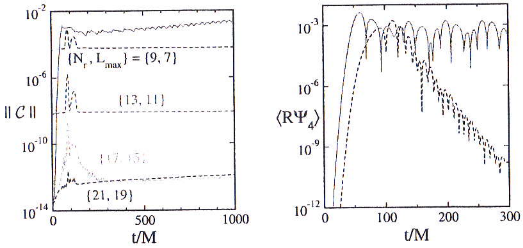

FIGURE 2. Evolution of perturbed Schwarzschild.

methods as described in Ref. [7] for a range of numerical resolutions specified by N_r (the highest order radial basis function) and $L_{\max}$ (the highest order spherical-harmonic). Figure 1 shows the time dependence of the constraint norm $||\mathcal{C}||$ for several values of the constraint-damping parameters γ_0 and γ_2. These tests show that without constraint damping the extended evolution system is extremely unstable, but with constraint damping the evolutions of the Schwarzschild spacetime are completely stable up to $t = 10,000M$ (and forever, we presume). These tests also illustrate that both the γ_0 and the γ_2 constraint damping terms are essential.

We also tested our new boundary conditions by evolving a black hole perturbed by an incoming gravitational wave (GW) pulse. We perturb Schwarzschild initial data by injecting a GW pulse through the boundary with time profile $f(t) = \mathcal{A}\, e^{-(t-t_p)^2/w^2}$ and $\mathcal{A} = 10^{-3}$, $t_p = 60M$, and $w = 10M$. Figure 2 shows the evolution of $||\mathcal{C}||$ for both constraint-preserving boundary conditions (dashed curves) and simple boundary conditions that freeze all the incoming characteristic fields (solid curves). These results illustrate that the new boundary conditions indeed prevent the influx of constraint violations. Figure 2 also illustrates the time dependence of the Weyl tensor component Ψ_4 averaged over the outer boundary of the computational domain. The dashed curve (using constraint-preserving boundary conditions) shows black-hole quasi-normal oscillations with the correct complex frequency, while the solid curve (using freezing boundary conditions) is completely unphysical. These results show that proper constraint-preserving boundary conditions are essential if accurate gravitational waveforms are needed.

This work was supported in part by a grant from the Sherman Fairchild Foundation to Caltech and Cornell, by NSF grants PHY-0099568, PHY-0244906 and NASA grants NAG5-10707, NAG5-12834 at Caltech, and by NSF grants PHY-0312072, PHY-0354631, and NASA grant NNG05GG51G at Cornell.

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The redshift effect and decay rates for the wave equation on a Schwarzschild exterior

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(joint work with Igor Rodnianski)

We consider the following problem: Let $(\mathcal{M}, g)$ denote the maximally extended Schwarzschild spacetime [7] with parameter $M > 0$. Let S denote a complete Cauchy surface, and consider locally C^6 solutions of the wave equation

$$\Box_g \phi = 0 \tag{1}$$

on $\mathcal{M}$, such that $\phi|_S$ and $\nabla\phi|_S$ decay sufficiently rapidly at spatial infinity. (We do *not* assume ϕ vanishes at the sphere of bifurcation of the event horizon.) The main result presented in this talk is a set of decay rates for ϕ and its energy flux in the closure of the domain of outer communications $\overline{J^+(\mathcal{I}^-) \cap J^-(\mathcal{I}^+)} \subset \mathcal{M}$. In particular, the decay rates apply along the event horizon $\mathcal{H}^+$.

To state precisely the result, let us introduce some notation: By u and v, we mean standard Eddington-Finkelstein retarded and advanced time coordinates on