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STABILITY IN DISSIPATIVE RELATIVISTIC FLUID THEORIES

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ABSTRACT. This paper examines the problem of finding a theory that describes the effects of dissipation (viscosity and thermal conductivity) in a fully relativistic fluid. Several of the proposed theories, including those of Eckart, Landau-Lifshitz, Havas-Swenson, and Israel-Stewart, are examined. A number of difficulties have been identified with these theories in the literature: non-causal propagation of signals, poorly posed dynamical evolution of initial data, and generic instability of the equilibrium states. This paper describes how the stability of the equilibrium states can be analyzed in this entire class of theories. It is shown that all of the "first-order" theories (Eckart, Landau-Lifshitz, and Havas-Swenson) have very short timescale instabilities in every equilibrium state. These first-order theories are consequently inadequate. The second-order theories (Israel-Stewart) in contrast can have stable equilibria. Furthermore, it is shown that the conditions needed for these theories to have stable equilibrium states are equivalent to the conditions needed to guarantee that perturbations propagate causally via hyperbolic differential equations. Thus the second-order theories appear to be promising candidates for an acceptable theory of dissipative relativistic fluids.

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1. INTRODUCTION

As is well-known, in the Fourier theory of heat flow, temperature fluctuations propagate via a parabolic equation. In a non-relativistic (Newtonian) theory this is acceptable, if somewhat of a curiosity (Maxwell[1] was already concerned about the resulting infinite propagation velocity in 1867), since Newtonian physics does not possess a maximum velocity for the transmission of information. It does hint, however, that it may be difficult to create an acceptable theory of dissipative relativistic fluids, in which all fluid variables obey hyperbolic equations, and all transmission of information through the fluid occurs inside the light cone (i.e., at velocities less than the speed of light). The purpose of this paper is to review recent work aimed at finding the simplest acceptable truly relativistic theory of dissipative fluids. The unifying theme of the present discussion is the question of the stability of the equilibrium states in various proposed theories. Many of the theories have *no* stable equilibrium states at all!

The first attempts at creating a theory of relativistic dissipative fluids are now called "first-order" theories; in these theories the definition of the entropy current contains no terms of higher than first order in the deviations from equilibrium (heat flow, viscous stresses, etc.). The simplest such theories are those of Eckart[2] and Landau and Lifshitz[3], which are presented in many textbooks on relativistic physics[3,4,5]. These theories are the simplest covariant generalizations of the Navier-Stokes-Fourier theory of Newtonian dissipative fluids. It is also possible to construct more complicated theories which are still first-order in this sense; the theory created by Havas and Swenson[6] may be the most general such theory possible.

It is also possible to create either Newtonian or relativistic second-order theories, where the entropy current definition is extended to include terms quadratic in the deviations from equilibrium. The kinetic theory version of such a Newtonian second-order theory was first studied in 1949 by Grad[7]; the associated second-order Newtonian phenomenological fluid theory was later developed by Müller[8]. Around 1970, a number of workers began to extend Grad's work to the relativistic regime[9-11]. The first relativistic second-order phenomenological fluid theory was described by Israel[12]; later work of Stewart[13] and Israel[14-16] clarified the relation between the kinetic theory and phenomenological approaches, and extended the theories to more complicated situations. Section 2 of this paper reviews how the first- and second-order theories are constructed.

It has been known for some time that there are definite problems with the first-order theories. Under certain special conditions, a parabolic equation may be obtained for the propagation of thermal fluctuations in the simpler (Eckart and Landau-Lifshitz) first-order theories[16]. This raised serious doubts as to whether the first-order theories should be considered truly "relativistic", but in itself is not conclusive, for the following reasons. First, an analysis of the propagation of fluctuations for the complete theory (with both viscosity and thermal conductivity nonzero) has not been completed. Second, in the more complicated, but still first-order, Havas-Swenson theory, thermal fluctuations obey a hyperbolic equation[17], at least in the Newtonian limit.

The question of the stability of equilibrium states in the Eckart theory has also been raised[18]. It was shown that instabilities driven by thermal conductivity exist in the Eckart theory, but only for perturbations with length scales much smaller than is physically meaningful (i.e., smaller than the interparticle separation in the fluid).

While the results mentioned above are suggestive of serious problems in the first-order theories, they are not conclusive. These potential problems (the stability of equilibrium states, and the causality of linear wave propagation) may be addressed by studying the linearized equations of motion for small perturbations about an equilibrium state of the fluid. Determining the speed at which information can be transmitted through the fluid is a very complicated problem. For all of the first-order theories, the system of equations which govern the evolution of linear perturbations is neither purely hyperbolic, nor parabolic, nor elliptic. Little seems to be known about the properties of such complicated systems[19]. In particular, it is not clear whether a well-posed initial value problem exists for this type of system of equations. As a result, the analysis of signal propagation to determine the causal properties of the first-order fluids is even more complicated than in the dispersive, dissipative electromagnetic case[20]. The stability of equilibrium states, on the other hand, can be studied by a simple plane-wave normal mode analysis of the perturbation equations. Such an analysis has been performed and it has been found that all of these first-order theories predict rapid evolution away from an arbitrary equilibrium state[21,22]. In other words, *every* equilibrium state in *every* first-order theory is unstable in the sense that small spatially bounded departures from equilibrium at one instant in time grow exponentially with time. The time scales for these instabilities can be ridiculously short: for example, water at room temperature and pressure has an instability with a growth time scale of about 10^{-34} seconds in these theories. Since these theories predict nonsensical behavior for phenomena which should be well within their range of applicability, we feel that these first-order theories are unacceptable. Section 3 of this paper reviews the stability analysis of these theories.

The stability of equilibrium states and the causality of signal propagation in the second-order Israel-Stewart[23] theory have also been investigated. The major result of this study was that the conditions necessary for equilibrium states to be stable in the second-order theories are equivalent to the conditions needed to guarantee that perturbations propagate causally and obey a hyperbolic set of equations. Thus, stability implies causality and hyperbolicity for perturbative waves. The converse is also true (although a particular definition of hyperbolicity must be imposed); any Israel-Stewart fluid with perturbations which propagate causally and obey a set of hyperbolic equations will possess stable equilibrium states. Section 4 of this paper reviews the stability and causality analysis of the second-order theories.

In addition, several new results not contained in Ref. (23) are discussed in Section 4 and an accompanying appendix. In Section 4 we show that causality and a naive version of hyperbolicity do not imply stability. In particular, it is not sufficient to simply assume that the characteristic velocities are all real (a more naive notion of hyperbolicity) and less than the speed of light; this is demonstrated by explicitly constructing a counterexample in which all the characteristic velocities are real and bounded between zero and one (the speed of light in our units), yet the equilibrium states are unstable. In an appendix we present a new result which widens the class of equilibrium states of Israel-Stewart fluids for which stability can be rigorously analyzed. Stability is analyzed via an energy functional, quadratic in the perturbation variables, which has a non-positive time derivative. In the appendix, we show that any perturbation which conserves the overall particle number of the fluid, and the total momenta associated with any background Killing vector fields of the spacetime, will grow without bound if the energy functional for that perturbation is negative on some spacelike surface. This result eliminates some assumptions about the properties of the equilibrium states made in Ref. (23).

The combination of the main results, that the first-order theories are necessarily unstable, while the second-order theories can be both stable and causal, strongly suggests to us that the second-order theories should replace the first-order theories as the standard theory of relativistic dissipative fluid mechanics. The first-order theories can probably not be made causal; their equilibrium states are all unstable; and it appears that the first-order theories have no well-posed initial value problem. The Israel-Stewart theory, on the other hand, possesses stable equilibrium states, and has perturbations which propagate causally according to hyperbolic equations.

2. THEORIES OF RELATIVISTIC DISSIPATIVE FLUIDS

(a) *Constructing the Theories.* The fundamental variables of a relativistic theory of fluids are the stress-energy tensor, T^{ab}, and the particle number current N^a. The fundamental equations of motion are the conservation laws for these quantities:

(1) $$\nabla_a T^{ab} = 0,$$

(2) $$\nabla_a N^a = 0.$$

The derivative operators in Eqs. (1-2) are four-dimensional covariant derivatives; no specialization to a fixed background spacetime is to be assumed. In an equilibrium state these fundamental tensors can be decomposed in terms of other familiar fluid variables,

(3) $$T^{ab} = \rho u^a u^b + p q^{ab},$$

(4) $$N^a = n u^a,$$

where u^a is the four-velocity of the fluid, ρ is the energy density (as measured by an observer co-moving with the fluid), p is the pressure, n is the number density, and q^{ab} is the projection tensor orthogonal to u^a:

(5) $$q^{ab} = g^{ab} + u^a u^b.$$

For a fluid which is not in an equilibrium state, it is also customary to introduce a four-velocity vector field u^a and a set of thermodynamic variables ρ, p, n, s (entropy per particle), T (temperature), μ (chemical potential), etc. which are used to supplement the description of the fluid contained in the fundamental tensors T^{ab} and N^a. There are, however, a variety of ways of introducing these auxiliary fields. Different theories adopt different rules for identifying these extra fields.

The four-velocity u^a of a fluid which is not in equilibrium can be defined in terms of the fundamental tensors of the theory in a variety of reasonable ways. In the Eckart[2] theory the four-velocity is identified with the direction in spacetime in which the particles of the fluid move, i.e., the four-velocity is parallel to the particle number current:

(6) $$u^a = (-N^b N_b)^{-1/2} N^a.$$

In the Landau-Lifshitz[3] theory, on the other hand, the four-velocity is identified with the spacetime direction of energy flow. Thus u^a is taken as the timelike eigenvector of T^{ab} in the Landau-Lifshitz theory:

$$q_{ab}T^{bc}u_c = 0. \tag{7}$$

Other choices are possible. In the Havas-Swenson theory[6], and also in a general class of theories which we have studied[21], the initial choice of u^a is only restricted by demanding that it be a unit timelike vector field.

A great deal of freedom is also available in introducing the thermodynamic variables ρ, p, n, etc. into the description of a fluid which is not in equilibrium. It is customary to constrain the thermodynamic variables by the "first law of thermodynamics",

$$d\rho = nTds + \frac{\rho + p}{n}\,dn \tag{8}$$

where $d\rho$, ds, and dn are one-forms. It is also customary to take over to the nonequilibrium theory the equation of state which describes the equilibrium states of the fluid. The equation of state defines the value of one of the thermodynamic variables (often s, which is not generally considered to be directly observable) as a function of two other observable thermodynamic variables, e.g., $s = s(\rho,n)$. The given equation of state and the first law of thermodynamics [Eq. (8)] then reduce the thermodynamic variables to a space containing only two independent functions. These two independent thermodynamic variables may be identified with the fundamental tensors of the nonequilibrium theory in a variety of ways. The thermodynamic energy density, ρ, could be defined as the physical energy density,

$$\rho = T^{ab}u_a u_b, \tag{9}$$

the thermodynamic pressure, p, could be defined as the average physical pressure,

$$p = \tfrac{1}{3} T^{ab} q_{ab}, \tag{10}$$

and the thermodynamic particle number density, n, could be defined as the physical particle number density:

$$n = - N^{a} u_{a}. \tag{11}$$

Since the space of thermodynamic variables is only two-dimensional, one cannot make all of these identifications [Eqs. (9)-(11)] for an arbitrary nonequilibrium state of the fluid. The most common choice, made in the Eckart and Landau-Lifshitz theories, takes ρ and n to be related to the fundamental tensors by Eqs. (9) and (11). The thermodynamic pressure, p, is then determined by Eq. (8) and the equation of state. It will in general not satisfy Eq. (10). A nonequilibrium scalar stress, τ, may then be defined as the difference between the average physical pressure and the thermodynamic pressure:

$$\tau = \tfrac{1}{3} T^{ab} q_{ab} - p. \tag{12}$$

This is, however, not the only possible choice. One could take ρ and p, or n and p, as being defined by Eqs. (9)-(11) and then introduce a nonequilibrium correction field to modify the third equation [Eq. (11) or (9) respectively]. The Havas-Swenson theory allows an even more general identification of the thermodynamic variables with the fundamental tensors. In this type of theory one assumes that a procedure exists to measure two of the thermodynamic functions (not necessarily n, p, or ρ), for example the temperature and the chemical potential, in an arbitrary nonequilibrium state of the fluid. Given the values of these two functions, the rest of the thermodynamic variables (in particular n, p, and ρ) are determined by the equation of state and Eq. (8). In this case additional nonequilibrium correction terms must be added to each of Eqs. (9)-(11). Finally, note that once a set of rules for identifying the thermodynamic variables with the fundamental tensors is chosen, the particular values of the thermodynamic variables for a given nonequilibrium state still depend strongly on the choice of four-velocity definition.

To simplify the presentation in this paper, we will restrict further consideration to theories of the Eckart type. This implies that the four-velocity of the fluid be identified with the particle number current vector as in Eq. (6). It further implies that ρ and n be chosen as the fundamental observable thermodynamic variables, to be identified with the physical energy density and particle number density as in Eqs. (9) and (11), even for states which are not in equilibrium. Readers desiring more details concerning the Landau-Lifshitz theory, or other more general first-order theories, such as the Havas-Swenson theory, are referred to Refs. (3), (6), (21), and (22). The results concerning the stability of equilibrium states are not qualitatively different for those other theories.

Using the Eckart choice of four-velocity, u^a, and thermodynamic variables ρ and n, the stress-energy tensor and the number current vector can be decomposed in the following manner for an arbitrary nonequilibrium state

(13) $$T^{ab} = \rho u^a u^b + (p + \tau)q^{ab} + q^a u^b + q^b u^a + \tau^{ab},$$

(14) $$N^a = nu^a,$$

where

(15) $$u^a q_a = u^a \tau_{ab} = \tau^a{}_a = \tau^{ab} - \tau^{ba} = 0.$$

The three fields τ, q^a, and τ^{ab} describe the deviations from local equilibrium in the fluid. The vector field q^a describes the heat flow, and τ and τ^{ab} are nonequilibrium stresses in the fluid.

To complete the construction of the theory, equations to determine τ, q^a, and τ^{ab} must be given. The definitions of these variables will be based on the need to satisfy the second law of thermodynamics. The total entropy associated with a spacelike surface Σ (i.e., at one instant in time) is obtained by integrating the entropy current vector field over the surface:

(16) $$S(\Sigma) = \int_\Sigma s^a \, d\Sigma_a.$$

The second law of thermodynamics requires this total entropy to be a non-decreasing function of time for isolated systems. If another spacelike surface Σ' lies to the future of Σ, then the second law requires that

$$S(\Sigma') - S(\Sigma) = \int \nabla_a s^a \, dV \geqslant 0, \tag{17}$$

where the two surface integrals implied by Eq. (16) have been converted into a volume integral by Gauss' theorem. If the second law holds in the form of Eq. (17) for all surfaces Σ' to the future of Σ, then it is clear that the following inequality (a local form of the second law) must also hold:

$$\nabla_a s^a \geqslant 0. \tag{18}$$

The first-order (Eckart) theory of relativistic dissipative fluids is obtained by modeling the entropy current, s^a, by a sum of terms no higher than first order in the deviations from equilibrium, τ, q^a, and τ^{ab}. The entropy current must therefore have the form

$$s^a = sn\, u^a + \beta q^a, \tag{19}$$

where β is an as yet unconstrained (zeroth-order) thermodynamic function. A term linear in τ multiplying u^a has been omitted as it inevitably leads to a defining equation for τ which is nonlinear. The divergence of this entropy current can now be evaluated, using the equations of motion for the fluid [Eqs. (1)-(2)] to simplify the resulting expressions; the following expression results:

$$T\nabla_a s^a = -\tau \nabla_a u^a + q^a(T\nabla_a \beta - u^b \nabla_b u_a) - \tau^{ab}\langle \nabla_a u_b \rangle + (T\beta - 1)\nabla_a q^a, \tag{20}$$

where the brackets $\langle\ \rangle$ which appear in Eq. (20) are defined by:

$$\langle A_{ab} \rangle = \tfrac{1}{2}\, q_a{}^c q_b{}^d (A_{cd} + A_{dc}) - \tfrac{1}{3}\, q_{ab} q^{cd} A_{cd} \tag{21}$$

for any second rank tensor. The simplest way to guarantee that

Eq. (20) is consistent with the second law, Eq. (18), is to require that the deviations from equilibrium be defined as follows:

(22) $$\beta = 1/T,$$

(23) $$\tau = -\zeta\nabla_a u^a,$$

(24) $$q^a = -\kappa q^{ab}(\nabla_b T + Tu^c\nabla_c u_b),$$

(25) $$\tau_{ab} = -2\eta\langle\nabla_a u_b\rangle.$$

With these definitions the divergence of the entropy current takes on its familiar quadratic form:

(26) $$T\nabla_a s^a = \frac{\tau^2}{\zeta} + \frac{q^a q_a}{\kappa T} + \frac{\tau^{ab}\tau_{ab}}{2\eta}.$$

which is manifestly non-negative if the three thermodynamic coefficients ζ, κ, and η are required to be positive. These coefficients may be identified with the familiar Newtonian dissipation coefficients by examining the theory in the Newtonian limit; one finds that ζ is the bulk viscosity, η is the shear viscosity, and κ is the thermal conductivity. The far more general Havas-Swenson theory results from assuming that all of the non-equilibrium fields (including non-equilibrium contributions to the energy and number densities) are allowed to depend on all first derivatives of the background equilibrium fields which have the correct tensor rank. This results in eleven additional coefficients in the theory; the constraints on these coefficients which result from enforcing the second law of thermodynamics are quite complicated and are described in the paper of Havas and Swenson[6].

Equations (1), (2), and (22)-(25) form a complete set of equations of motion for the dynamical variables n, ρ, u^a, τ, q^a, and τ^{ab} of the first-order Eckart theory constructed here. Gravitational interactions may be taken into account (if desired) by including the Einstein equations,

(27) $$G_{ab} = 8\pi\, T_{ab}$$

for the spacetime metric, g_{ab}, coupled to the stress-energy tensor of the fluid.

The second-order Israel-Stewart theory results from adopting an expression for the entropy current [Eq. (19)] which includes all possible terms through second order in the deviations from equilibrium. Specifically,

$$s^a = snu^a + \frac{q^a}{T} - \frac{1}{2}(\beta_0\tau^2 + \beta_1 q^b q_b + \beta_2 \tau^{bc}\tau_{bc})\,\frac{u^a}{T} + \alpha_0\tau\,\frac{q^a}{T} + \alpha_1\tau^a{}_b\,\frac{q^b}{T}. \tag{28}$$

The three new thermodynamic coefficients β_i model the deviations of the physical entropy density from the thermodynamic entropy density, sn. The other two new coefficients, α_i, represent changes in the entropy current due to possible viscous-heat flux couplings. The divergence of Eq. (28) may be computed and simplified, in analogy to the treatment of Eq. (19), to yield

$$\begin{aligned} T\nabla_a s^a = &-\tau\left[\nabla_a u^a+\beta_0 u^a\nabla_a\tau-\alpha_0\nabla_a q^a-\gamma_0 Tq^a\nabla_a\left(\frac{\alpha_0}{T}\right) + \frac{1}{2}\tau T\nabla_a\left(\frac{\beta_0}{T}u^a\right)\right] \\ &- q^a\left[\frac{1}{T}\nabla_a T+u^b\nabla_b u_a+\beta_1 u^b\nabla_b q_a-\alpha_0\nabla_a\tau-\alpha_1\nabla_b\tau^b_a + \frac{1}{2}Tq_a\nabla_b\left(\frac{\beta_1}{T}u^b\right)\right. \\ &\left.-(1-\gamma_0)\tau T\nabla_a\left(\frac{\alpha_0}{T}\right) - (1-\gamma_1)T\tau_a{}^b\nabla_b\left(\frac{\alpha_1}{T}\right) + \gamma_2\nabla_{[a}u_{b]}q^b\right] \\ &-\tau^{ab}\langle\nabla_a u_b+\beta_2 u^c\nabla_c\tau_{ab}-\alpha_1\nabla_a q_b + \frac{1}{2}T\tau_{ab}\nabla_c\left(\frac{\beta_2}{T}u^c\right) \\ &-\gamma_1 Tq_a\nabla_b\left(\frac{\alpha_1}{T}\right) + \gamma_3\nabla_{[a}u_{c]}\tau^c{}_b\rangle. \end{aligned} \tag{29}$$

As in the first-order case, τ, q^a, and τ^{ab} are defined in the simplest fashion which will ensure that the second law holds in its divergence form [Eq. (18)]. The simplest definitions which will guarantee the quadratic form [Eq. (26)] of the divergence of the entropy current are

$$\tau = -\zeta\left[\nabla_a u^a+\beta_0 u^a\nabla_a\tau-\alpha_0\nabla_a q^a-\gamma_0 Tq^a\nabla_a\left(\frac{\alpha_0}{T}\right) + \frac{1}{2}\tau T\nabla_a\left(\frac{\beta_0}{T}u^a\right)\right], \tag{30}$$

$$(31)\quad q^a = -\kappa T q^{ab}\Big[\frac{1}{T}\nabla_b T + u^c\nabla_c u_b + \beta_1 u^c\nabla_c q_b - \alpha_0\nabla_b \tau - \alpha_1\nabla_c \tau^c{}_b$$

$$+\frac{1}{2}Tq_b\nabla_c\left[\frac{\beta_1}{T}u^c\right] - (1-\gamma_0)\tau T\nabla_b\left[\frac{\alpha_0}{T}\right]$$

$$-(1-\gamma_1)T\tau_b{}^c\nabla_c\left[\frac{\alpha_1}{T}\right] + \gamma_2\nabla_{[b}u_{c]}q^c\Big],$$

$$(32)\qquad \tau^{ab} = -2\eta\langle\nabla^a u^b + \beta_2 u^c\nabla_c\tau^{ab} - \alpha_1\nabla^a q^b + \frac{1}{2}T\tau^{ab}\nabla_c\left[\frac{\beta_2}{T}u^c\right]$$

$$-\gamma_1 T q^a\nabla^b\left[\frac{\alpha_1}{T}\right] + \gamma_3\nabla^{[a}u^{c]}\tau_c{}^b\rangle.$$

There are two new coefficients present in Eqs. (30-32), γ_0 and γ_1, that appear because of ambiguity in factoring the cross-terms on the right hand side of Eq. (29) which involve τq^a and $\tau^a{}_b q^b$. There are also two new coefficients, γ_2 and γ_3, which couple the heat flow vector and stress tensor to the vorticity of the fluid. Within the phenomenological fluid theory, the magnitudes of the γ_i are unknown and could in principle be large compared to unity.

Equations (1), (2), and (30-32) form the complete set of equations of motion for the second-order Israel-Stewart theory, written in the so-called "Eckart frame" in which the four-velocity is chosen to be parallel to the particle number current. Israel and Stewart[12-15] have also given the equations of motion for the second-order theory in a frame with arbitrary four-velocity. Their more general theory still adopts Eqs. (9) and (11) to define ρ and n; no one has yet had the temerity to construct the analogue of the Havas-Swenson theory at second order.

(b) *Equilibrium states.* Before the propagation of perturbations on an equilibrium background, or the stability of the equilibrium states can be studied, the nature of the equilibrium states themselves must be determined. Equilibrium is defined by the condition that the entropy of the fluid must not change with time; this implies that the divergence of the entropy current must be zero. Since the divergence of the entropy current is the sum of positive terms [Eq. (26)], each of these terms must vanish in equilibrium. Thus, in both the first- and second-order theories, the heat flow and viscous stresses must vanish in equilibrium:

$$\tau = q^{a} = \tau^{ab} = 0. \tag{33}$$

The vanishing of these deviations from equilibrium, applied to the defining equations for these quantities [Eqs. (23-25) for the first-order theory, Eqs. (30-32) for the second-order theory] imply the following additional conditions on an equilibrium state:

$$\nabla_a u^a = 0, \tag{34}$$

$$\langle \nabla_a u_b \rangle = 0, \tag{35}$$

$$q^{ab}(\nabla_b T + T u^c \nabla_c u_b) = 0. \tag{36}$$

Applying the conservation laws [Eqs. (1) and (2)] to a fluid which satisfies these constraints yields

$$u^a \nabla_a n = 0, \tag{37}$$

$$u^a \nabla_a \rho = 0, \tag{38}$$

$$q^{ab}[\nabla_b p + (\rho + p) u^c \nabla_c u_b] = 0. \tag{39}$$

Equations (37) and (38) imply that all of the equilibrium thermodynamic variables (e.g., s, T, p) must be constant along the fluid flow lines (integral curves of u^a), since each of these variables depends only on ρ and n through the equation of state. This result and Eqs. (34) - (36) imply that the vector field u^a/T is a Killing vector field, i.e.,

$$\nabla_a \left(\frac{u_b}{T} \right) + \nabla_b \left(\frac{u_a}{T} \right) = 0. \tag{40}$$

The final equation, Eq. (39), combined with these other results, is equivalent to the requirement that a certain thermodynamic potential,

(41) $$\Theta = \frac{\rho + p}{nT} - s$$

have vanishing gradient.

Notice that the conditions satisfied by the equilibrium states are identical in the simple first- and second-order theories. The more general first-order Havas-Swenson theory allows the possibility of more general "equilibrium" states; e.g., states with nonzero heat flow or viscous stresses which possess unchanging entropy.

3. STABILITY OF FIRST-ORDER DISSIPATIVE RELATIVISTIC FLUIDS

In this section the dynamics of small perturbations about an equilibrium state of the first-order (Eckart) fluid theory will be studied. First the set of equations governing linear perturbations about equilibrium in the Eckart theory are determined. Then the exponential plane wave solutions to these equations of motion are examined. At least one transverse and one longitudinal solution are found to be growing exponentially in time. Finally, physically acceptable perturbations are expressed as Fourier transforms of the exponential plane waves. This shows that in the first-order Eckart theory, equilibrium states are always unstable to physically reasonable perturbations.

The perturbations about an equilibrium state will be analyzed in the Eulerian framework in order to avoid the gauge ambiguities present in the Lagrangian approach[18,24]. The difference between the actual nonequilibrium value of a field Q at a spacetime point and the value of Q in the fiducial background equilibrium state will be denoted by δQ. Any field which does not include the prefix δ (e.g., ρ, n, u^a, ...) will henceforth refer to the fiducial equilibrium state which satisfies the equilibrium conditions outlined above in Eqs. (33) - (41). In order to simplify the analysis, only perturbations which leave the gravitational field fixed, i.e.,

$\delta g_{ab} = 0$, will be considered. This is appropriate for special relativistic fluids or for short-wavelength perturbations of any equilibrium state.

The perturbations δQ are assumed to be small enough that their evolution is adequately described by the equations of motion [Eqs. (1), (2), and (23) - (25)] linearized about the fiducial background equilibrium state. For the purposes of deriving the linearized equations of motion for the perturbations, no special symmetry of the background equilibrium state is assumed; in particular, it could include rapid rotation and/or strong gravitational fields. The linearized equations of motion are then:

(42) $$\nabla_a \delta T^{ab} = 0,$$

(43) $$\nabla_a \delta N^a = 0,$$

(44) $$\delta\tau = -\zeta \nabla_a \delta u^a,$$

(45) $$\delta q^a = -\kappa T q^{ab}\left[\nabla_b\left(\frac{\delta T}{T}\right) + u^c\nabla_c \delta u_b + \delta u^c \nabla_c u_b\right],$$

(46) $$\delta\tau^{ab} = -2\eta\langle\nabla^a \delta u^b + \delta u^a u^c \nabla_c u^b\rangle,$$

where the linearly perturbed stress-energy tensor and particle number current are given by

(47) $$\delta T^{ab} = (\rho+p)(\delta u^a u^b + u^a \delta u^b) + \delta\rho u^a u^b + (\delta p + \delta\tau) q^{ab} + u^a \delta q^b + u^b \delta q^a + \delta\tau^{ab},$$

(48) $$\delta N^a = \delta n u^a + n \delta u^a.$$

The derivatives which appear in the linearized equations of motion are covariant derivatives compatible with the background spacetime metric g_{ab}; indices are also raised and lowered with the background metric. The linear perturbations satisfy the linearized versions of the constraints outlined in Eq. (15):

(49) $$u^a \delta q_a = \delta\tau_{ab} - \langle\delta\tau_{ab}\rangle = u^a \delta u_a = 0.$$

It will be sufficient to look for solutions to these equations possessing the following properties:

(1) The background spacetime is assumed to be flat Minkowski space,

(2) The fiducial background equilibrium state is assumed to be homogeneous, so that all background field variables have vanishing gradients,

(3) The solutions represent exponential plane waves, i.e., they have the form

$$\delta Q = \delta Q_0 \exp(\Gamma t + ikx), \tag{50}$$

where δQ_0 is constant, t and x are two of the coordinates of the background Minkowski space. The case where the background equilibrium state is at rest in this coordinate system, i.e.,

$$u^a \nabla_a = \partial_t, \tag{51}$$

will be considered first. After making these simplifying assumptions, the equations of motion may be put into matrix form,

$$M^A{}_B \delta Y^B = 0, \tag{52}$$

where δY^B represents the list of linear perturbation fields, and $M^A{}_B$ is a 14 × 14 complex valued matrix which describes the linearized equations of motion, specialized to plane wave solutions with the restrictions outlined above. The matrix takes on a particularly nice block-diagonal form when the following 14 fields are chosen as the perturbation variables (note that the order of the fields in this equation defines the columns of the matrix $M^A{}_B$);

$$\delta Y^B = \{\delta\rho,\ \delta n,\ \delta u^x,\ \delta\tau,\ \delta q^x,\ \delta\tau^{xx},\ \delta u^y,\ \delta q^y,\ \delta\tau^{xy},\ \delta u^z,\ \delta q^z,\ \delta\tau^{xz},\ \delta\tau^{yz},\ \delta\tau^{yy} - \delta\tau^{zz}\}. \tag{53}$$

The matrix $M^A{}_B$ then block diagonalizes as follows:

(54)
$$\mathbf{M} = \begin{bmatrix} \mathbf{Q} & 0 & 0 & 0 \\ 0 & \mathbf{R} & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix},$$

where the submatrices **Q**, **R**, and **I** are defined as follows:

(55)
$$\mathbf{Q} = \begin{bmatrix} 0 & \Gamma & ink & 0 & 0 & 0 \\ \Gamma & 0 & i(\rho+p)k & 0 & ik & 0 \\ i(\partial p/\partial \rho)_n k & i(\partial p/\partial n)_\rho k & (\rho+p)\Gamma & ik & \Gamma & ik \\ 0 & 0 & ik & 1/\zeta & 0 & 0 \\ \frac{i}{T}(\partial T/\partial \rho)_n k & \frac{i}{T}(\partial T/\partial n)_\rho k & \Gamma & 0 & 1/\kappa T & 0 \\ 0 & 0 & ik & 0 & 0 & 3/4\eta \end{bmatrix},$$

(56)
$$\mathbf{R} = \begin{bmatrix} (\rho+p)\Gamma & \Gamma & ik \\ \Gamma & 1/\kappa T & 0 \\ ik & 0 & 1/\eta \end{bmatrix},$$

and **I** is the 2×2 unit matrix. It is immediately obvious that the two components of the shear stress, $\delta\tau^{yy} - \delta\tau^{zz}$ and $\delta\tau^{yz}$, vanish identically at linear order.

There exist exponential plane-wave solutions of Eq. (52) whenever Γ and k have values which satisfy the dispersion relation,

(57)
$$\det \mathbf{M} = 0.$$

The determinant of **M** is simply the product of the determinants of its diagonal blocks, and thus the roots of Eq. (57) are simply the collection of roots obtained by separately setting the determinants of **Q** and **R** equal to zero. The roots obtained by setting the determinant of **Q** equal to zero are referred to as longitudinal modes since the matrix **Q** involves only scalars and the components of the perturbation fields which are parallel to the direction of spatial variation (x). The roots obtained by setting the determinant of **R** equal to zero describe the propagation of transverse modes of the fluid, since the matrix **R** involves only the components of the perturbation fields which are orthogonal to the direction of spatial variation (x) of a perturbation.

The determinant of the matrix **R** is

$$-\eta\kappa T \det \mathbf{R} = \kappa T\Gamma^2 - (\rho + p)\Gamma - \eta k^2 = 0, \tag{58}$$

which can be solved for the frequency, Γ,

$$\Gamma_{\pm} = \frac{1}{2\kappa T}\{(\rho+p) \pm [(\rho+p)^2 + 4\eta\kappa T k^2]^{1/2}\}. \tag{59}$$

The frequencies of these transverse modes, given by Eq. (59), are purely real for real wave numbers k, and hence these modes do not propagate. An observer at fixed coordinate x would observe only a monotonically growing or decaying perturbation; there would be no superimposed oscillation. The existence of a positive real root (Γ_+) implies the existence of a growing mode, and hence of an instability in the fluid (unless the thermal conductivity is zero). Since the frequency Γ_+ is positive for all real wave numbers k, the fluid is unstable to a growing transverse mode at all wavelengths. Note that as the thermal conductivity approaches zero, the growth timescale associated with the growing mode also approaches zero; the dissipative fluid thus becomes more unstable the less dissipative it is.

The dispersion relation for the longitudinal modes, obtained by setting det **Q** = 0, is a quartic polynomial in Γ:

$$\begin{aligned} F(\Gamma) &= \frac{4}{3}\eta\zeta\kappa T \det \mathbf{Q} \\ &= \kappa T\Gamma^4 - (\rho+p)\Gamma^3 + \left\{\kappa T\left[\left(\frac{\partial p}{\partial \rho}\right)_s + \left(\frac{\partial \Theta}{\partial s}\right)_\rho\right] - \left(\zeta + \frac{4}{3}\eta\right)\right\}k^2\Gamma^2 \\ &\quad - \left\{(\rho+p)\left(\frac{\partial p}{\partial \rho}\right)_s + \kappa\left(\zeta + \frac{4}{3}\eta\right)k^2\left(\frac{\partial T}{\partial \rho}\right)_n\right\}k^2\Gamma \\ &\quad + \kappa T\left(\frac{\partial p}{\partial \rho}\right)_s\left(\frac{\partial \Theta}{\partial s}\right)_p k^4. \end{aligned} \tag{60}$$

In the special case of a spatially homogeneous perturbation ($k = 0$), the only nonzero root of Eq. (60) is

$$\Gamma = \frac{\rho + p}{\kappa T}, \tag{61}$$

a growing spatially homogeneous perturbation. Note that as the limit of zero thermal conductivity is taken, $\kappa T \to 0$, the growth timescale of this mode ($1/\Gamma$) approaches zero, as in the transverse case. It is therefore clear that there exist growing longitudinal modes for at least some neighborhood in k about $k = 0$.

In the exceptional case where the thermal conductivity is zero, it is possible to show that the transverse modes are still unstable as seen by an observer moving with a nonzero constant velocity relative to the background equilibrium state[21], as long as the shear viscosity coefficient is not also equal to zero. In the even more special case when both the thermal conductivity and shear viscosity are zero (leaving the bulk viscosity as the only source of dissipation), it is possible to show that at least long wavelength longitudinal modes will be unstable as seen by an observer moving relative to the background state. A weaker version of this result, which assumed the positivity of certain thermodynamic derivatives, can be found in Ref. (21).

This analysis has shown that the equilibrium states of the first-order Eckart theory of dissipative relativistic fluids always possess linear plane-wave perturbations which grow exponentially in time; does this necessarily imply that physically reasonable perturbations will grow exponentially in time also? The answer to this question clearly depends to some extent on how one defines "physically reasonable" perturbations. If square-integrable initial data (i.e., in L^2) are defined as being physically reasonable, then these perturbations do grow exponentially in time. Since the initial data is taken to be in L^2, it has a well-defined spatial Fourier transform; if it is a generic perturbation then its spatial Fourier transform will have non-zero support on the set of growing plane-wave modes. It is then possible to show that the L^2 norm of the perturbation will in fact diverge exponentially in time[21].

Finally, it is important to estimate the growth timescales associated with these instabilities to determine their relevance to physics. If, for all imaginable conditions, the growth timescales were absurdly large (e.g., greater than the age of the universe), then one could claim that the instabilities were only of academic interest; in any conceivable application the instabilities would be undetectable in the dynamics of the fluid. In fact, the growth timescales for these instabilities are absurdly *short* for nearly Newtonian, weakly dissipative systems. For example, the frequency for the transverse modes described by Eq. (59) is bounded below by

$$\Gamma_+ \geqslant \frac{(\rho c^2 + p)c^2}{\kappa T}, \tag{62}$$

where the speed of light (c) has been explicitly inserted into this equation. The characteristic growth time is given by $\tau = \Gamma_+^{-1}$, so

$$\tau \leqslant \frac{\kappa T}{(\rho c^2 + p)c^2}. \tag{63}$$

This timescale is ridiculously short for everyday fluids; for example, water at room temperature ($\approx$ 300 K) and pressure ($\approx$ 1 bar) has a growth timescale given by

$$\tau \lesssim 2 \times 10^{-34} \text{ sec.} \tag{64}$$

4. STABILITY AND CAUSALITY IN SECOND-ORDER DISSIPATIVE RELATIVISTIC FLUIDS

In this section the stability of equilibrium states in the second-order Israel-Stewart theory of dissipative relativistic fluids is examined. This analysis reveals that it is possible for all equilibria in this theory to be stable, provided certain constraints on the thermodynamic derivatives and second-order coefficients (the α_i and β_i) are satisfied. These stability conditions are shown to imply that all linear perturbations propagate causally (i.e., subluminally) and obey a symmetric hyperbolic system of equations. The converse theorem is also shown to be true: if the linear perturbations propagate causally and obey a symmetric hyperbolic (in a particular sense) system of equations, then all equilibria are stable. The definition of hyperbolicity needed to establish the converse theorem is rather strict and cannot be replaced by a more naive definition. A new result presented in this section is an example of a fluid system in which all the characteristic velocities are real and less than the speed of light, but which violates the stability conditions, so that no equilibrium state is stable. This shows that simple, naive ideas about hyperbolicity are insufficient to establish stability.

(a) *Stability.* As in the previous section, the perturbations are analyzed here within the Eulerian framework, and only linear departures from equilibrium are considered. The equations of motion for the perturbations are obtained by linearizing the fluid's full set of equations of motion [Eqs. (1), (2), and (30)-(32)] about a fiducial equilibrium background state; the resulting equations of motion for the linearized perturbations are given by Eq. (42), (43), and

$$(65)\quad \delta\tau = -\zeta\left[\nabla_a \delta u^a + \beta_0 u^a \nabla_a \delta\tau - \alpha_0 \nabla_a \delta q^a - \gamma_0 T \delta q^a \nabla_a \left(\frac{\alpha_0}{T}\right)\right],$$

$$(66)\quad \delta q^a = -\kappa T q^{ab}\left[\nabla_b \left(\frac{\delta T}{T}\right) + u^c \nabla_c \delta u_b + \delta u^c \nabla_c u_b + \beta_1 u^c \nabla_c \delta q_b - \alpha_0 \nabla_b \delta\tau - \alpha_1 \nabla_c \delta\tau^c_{\ b} - (1-\gamma_0) T \delta\tau \nabla_b \left(\frac{\alpha_0}{T}\right) - (1-\gamma_1) T \delta\tau^c_{\ b} \nabla_c \left(\frac{\alpha_1}{T}\right) + \gamma_2 \nabla_{[b} u_{c]} \delta q^c\right],$$

$$(67)\quad \delta\tau^{ab} = -2\eta\langle \nabla^a \delta u^b + \delta u^a u^c \nabla_c u^b + \beta_2 u^c \nabla_c \delta\tau^{ab} - \alpha_1 \nabla^a \delta q^b - \gamma_1 T \delta q^a \nabla^b \left(\frac{\alpha_1}{T}\right) + \gamma_3 \nabla^{[a} u^{c]}\, \delta\tau_c^{\ b}\rangle,$$

The approach to analyzing the stability of equilibrium states in the second-order theory is rather different than that used in the previous section. In that case, the equilibrium states were unstable and the aim was to identify some particular unstable perturbations.

In this case, the aim is to find the necessary and sufficient set of conditions which will imply the stability of all equilibrium states. This will be done by constructing an energy functional for the perturbations, i.e., a monotonically decreasing function of time which depends quadratically on the perturbation variables. Such a functional can be constructed for Israel-Stewart fluids in terms of an energy current vector defined by

$$(68)\quad TE^a = \delta T^a{}_b \delta u^b - \frac{1}{2}(\rho+p)u^a\delta u^b\delta u_b - \alpha_0\delta\tau\delta q^a - \alpha_1\delta\tau^a{}_b\delta q^b + \frac{\delta T}{T}\delta q^a$$
$$+ \frac{1}{2}(\rho+p)^{-1}\left[\left[\frac{\partial\rho}{\partial p}\right]_s(\delta p)^2 + \left[\frac{\partial\rho}{\partial s}\right]_p\left[\frac{\partial p}{\partial s}\right]_\Theta(\delta s)^2\right]u^a$$
$$+ \frac{1}{2}\left[\beta_0(\delta\tau)^2 + \beta_1\delta q^b\delta q_b + \beta_2\delta\tau^{bc}\delta\tau_{bc}\right]u^a .$$

The total energy associated with a spacelike surface Σ (i.e., the energy at one instant of time) is given by the integral of the energy current over the surface:

$$(69)\qquad E(\Sigma) = \int_\Sigma E^a d\Sigma_a.$$

From an argument analogous to that given for the entropy of the fluid in Section 2, this energy will be a decreasing function of time (for fluids with compact spatial support) as long as the divergence of the energy current is negative. The divergence of this energy current can be computed, and the resulting expression simplified using the perturbation equations to yield:

$$(70)\qquad \nabla_a E^a = -\left[\frac{(\delta\tau)^2}{T\zeta} + \frac{\delta q^a\delta q_a}{\kappa T^2} + \frac{\delta\tau^{ab}\delta\tau_{ab}}{2\eta T}\right].$$

The energy functional is thus a monotonically decreasing function of time.

The motivation for constructing this energy functional for Israel-Stewart fluids was to study their stability. Such a functional is expected to provide a useful indication of stability because of the following qualitative argument. If the energy functional were non-negative for all possible values of the perturbation variables, then it would suggest that the equilibrium state is stable. The energy of any perturbation would be bounded above by its initial value and below by zero in this case. If, on the other hand, perturbations exist having negative energy, then the energy is unbounded below and could evolve towards negative infinity, suggesting the presence of an instability. To establish a rigorous connection between the sign of the energy functional and stability, a detailed analysis of the equations of motion for the perturbations is needed; such an analysis is supplied in the Appendix of Ref. (23), where we prove the following two propositions:

PROPOSITION A. *The perturbations of an Israel-Stewart fluid will not grow without bound (as measured by a square integral norm) if the energy functional is non-negative for all perturbations.*

PROPOSITION B. *Consider an equilibrium state of an Israel-Stewart fluid in which the thermodynamic inequalities,*

$$0 < \left(\frac{\partial p}{\partial \rho}\right)_s < 1, \tag{71}$$

and

$$\left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta > 0. \tag{72}$$

are satisfied. If there exist perturbations having negative energy, then they will grow without bound as they evolve in time.

These two propositions demonstrate that a positive energy functional is a sufficient condition for the stability of Israel-Stewart fluids, and that it is a necessary condition for the stability of those equilibrium configurations that satisfy Eqs. (71) and (72). It has not yet been proven that a positive energy functional is strictly necessary for the stability of these fluids; however, another result which indicates that this is probably the case has been established. In the appendix the following proposition is proven:

PROPOSITION C. *Any solution of the linearized Israel-Stewart fluid perturbation equations which has vanishing variation in the conserved particle number,* $\delta N = 0$, *and vanishing variation of the conserved momenta,* $\delta P(k^a) = 0$, *for each Killing vector field* k^a *of the background spacetime, will grow without bound if the energy functional for this solution is negative on some spacelike surface.*

Proposition C applies to the stability of any equilibrium configuration, including those which violate Eqs. (71) and (72). It is limited, however, to the class of perturbations that leave the conserved particle number and conserved momenta of the background equilibrium configuration unchanged. Thus, Propositions B and C are complementary; they establish that a positive energy functional is a necessary condition for stability under different circumstances.

The energy functional defined by Eqs. (68) and (69) is a complicated function of the perturbation variables. Since the sign of the energy determines which equilibrium states are stable, it is desirable to factor the energy into a form which makes the conditions necessary for it to be positive self-evident.

Let the vector t^a be the future-directed unit normal to the spacelike surface Σ upon which E is defined. Associated with t^a are several useful tensors. The vector λ^a is defined to be the velocity of observers moving along t^a relative to the fluid

$$\lambda^a = q^a{}_b t^b / u^c t_c. \tag{73}$$

It is easy to see that the norm of λ^a is bounded between zero and one. The projection tensor associated with λ^a will also be useful:

$$\gamma^a{}_b = q^a{}_b - \lambda^{-2}\lambda^a\lambda_b. \tag{74}$$

The expression for the energy can now be written in terms of an energy density e, defined by

$$e = TE^a t_a / u^b t_b, \tag{75}$$

so that the energy is given by

$$E(\Sigma) = \int_{\Sigma} e \frac{u^{a}}{T} d\Sigma_{a}. \tag{76}$$

The total energy E will then be positive for all possible perturbations if and only if the energy density is also positive for all perturbations at every point in the fluid. Fortunately, it is possible to factor the energy density functional into the following form:

$$e = \frac{1}{2} \sum_{A} \Omega_{A}(\delta Z_{A})^{2}, \tag{77}$$

where the Ω_A are the following functions of the thermodynamic variables:

$$\Omega_1 = (\rho + p)^{-1} \left(\frac{\partial \rho}{\partial p}\right)_{s}, \tag{78}$$

$$\Omega_2 = (\rho + p)^{-1} \left(\frac{\partial \rho}{\partial s}\right)_{p} \left(\frac{\partial p}{\partial s}\right)_{\Theta}, \tag{79}$$

$$\Omega_3 = (\rho + p)\left[1 - \lambda^2 \left(\frac{\partial p}{\partial \rho}\right)_{s}\right] - \left[\frac{1}{\beta_0} + \frac{2}{3\beta_2} + \frac{K^2}{\Omega_6}\right]\lambda^2, \tag{80}$$

$$\Omega_4 = (\rho + p) - \frac{2\beta_2 + (\beta_1 + 2\alpha_1)\lambda^2}{2\beta_1\beta_2 - \alpha_1^2\lambda^2}, \tag{81}$$

$$\Omega_5 = \beta_0, \tag{82}$$

$$\Omega_6 = \frac{\beta_1}{\lambda^2} - \left[\frac{\alpha_0^2}{\beta_0} + \frac{2\alpha_1^2}{3\beta_2} + \frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_{n}\right], \tag{83}$$

$$\Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2}\lambda^2, \tag{84}$$

$$\Omega_8 = \beta_2, \tag{85}$$

and the δZ_A represent certain linearly independent combinations of the perturbation functions, given by

(86) $$\delta Z_1 = \delta p + (\rho + p)\left(\frac{\partial p}{\partial \rho}\right)_s\left[\lambda_a \delta u^a + \frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_s \lambda_a \delta q^a\right],$$

(87) $$\delta Z_2 = \delta s + \frac{1}{T}(\rho + p)\left(\frac{\partial T}{\partial \rho}\right)_p\left(\frac{\partial s}{\partial p}\right)_\Theta \lambda_a \delta q^a,$$

(88) $$\delta Z_3 = \lambda^{-1}\lambda_a \delta u^a,$$

(89) $$\delta Z^a{}_4 = \gamma^a{}_b \delta u^b,$$

(90) $$\delta Z_5 = \delta\tau + \frac{1}{\beta_0}\lambda_a \delta u^a - \frac{\alpha_0}{\beta_0}\lambda_a \delta q^a,$$

(91) $$\delta Z_6 = \lambda_a \delta q^a + \frac{K}{\Omega_6}\lambda_a \delta u^a,$$

(92) $$\delta Z_7{}^a = \gamma^a{}_b \delta q^b + \frac{2\beta_2 + \alpha_1\lambda^2}{2\beta_1\beta_2 - \alpha_1^2\lambda^2}\gamma^a{}_b \delta u^b,$$

(93) $$\delta Z_8{}^{ab} = \delta\tau^{ab} + \frac{1}{\beta_2}\langle\lambda^a \delta u^b\rangle - \frac{\alpha_1}{\beta_2}\langle\lambda^a \delta q^b\rangle,$$

where

(94) $$K = \frac{1}{\lambda^2} + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\left(\frac{\partial T}{\partial n}\right)_s.$$

The conditions necessary for E to be positive are then simply that the Ω_A must all be positive. A number of the Ω_A depend on the choice of spacelike surface because of their dependence on the parameter λ^2. There is enough freedom in the choice of spacelike surface so that λ^2 can take on any value between zero and one at any point in the star. The positivity of the Ω_A must therefore be imposed for all values of λ^2 in order to assure the positivity of the energy for all perturbations and all choices of spacelike surface. It is possible to show[23] that the most restrictive case is when $\lambda^2 = 1$, so that the conditions $\Omega_A\ (\lambda^2 = 1) > 0$ imply $\Omega_A > 0$ for all $0 \leq \lambda^2 \leq 1$.

Before proceeding further, it is worthwhile to note several implications of the stability conditions $\Omega_A > 0$. The first two conditions are the usual stability conditions for a relativistic perfect fluid; the positivity of Ω_1 guarantees that the square of the "adiabatic sound speed" will be positive (note that there is no *a priori* theoretical reason to believe that any perturbation will propagate at the adiabatic sound speed in the *dissipative* fluid theory); the positivity of Ω_2 is just the relativistic Schwarzchild criterion for stability against convection[25]. The three new second-order thermodynamic coefficients, β_i are required to be positive by the conditions on Ω_5, Ω_7, and Ω_8; and are further bounded from below by $\beta_0 > (\rho + p)^{-1}$, $\beta_1 > (\rho + p)^{-1}$, and $\beta_2 > \frac{2}{3}(\rho + p)^{-1}$ as a result of the positivity of Ω_3. That the β_i must be positive confirms the expectation that the nonequilibrium entropy density will be smaller in magnitude than the equilibrium value, sn.

(b) *Causality and Hyperbolicity.* The original motivation for constructing the second-order theory of relativistic dissipative fluids[12] was the desire to obtain a theory in which all perturbations propagated causally. Stewart and Israel[13,14] have investigated the extent to which the Israel theory succeeds in this respect. They derived expressions for the characteristic velocities for the system of perturbation equations [essentially, Eqs. (42), (43), and (65) - (67)]. These expressions are so complicated in form that it is not possible to determine by inspection whether the velocities are necessarily less than the speed of light, or whether they are even all real. In order to proceed, Israel and Stewart then specialized to the dilute gas limit where relativistic kinetic theory can be used to obtain explicit expressions for the α_i and β_i. In this limit, they were able to conclude that the characteristic velocities for this system of equations were less than the speed of light. The following analysis shows that perturbations propagate causally and obey a set of hyperbolic equations in a far wider range of circumstances, namely whenever the equilibrium states are stable.

The system of equations for the perturbations of an equilibrium state of an Israel second-order fluid, Eqs. (42), (43), and (65) - (67), have the following general form:

$$A^{A}{}_{B}{}^{a}\nabla_a \delta Y^B + B^{A}{}_{B}\delta Y^B = 0, \qquad (95)$$

where δY^B represents the list of the fourteen perturbation variables, i.e., $\delta\rho$, δn, δu^a, $\delta\tau$, δq^a, $\delta\tau^{ab}$. The index B runs over these fourteen fields, while the index A runs over the fourteen equations of motion for the perturbations. The matrices $A^A{}_B{}^a$ and $\mathcal{B}^A{}_B$ are functions of the unperturbed equilibrium fluid configuration.

A three dimensional surface is a characteristic surface for these equations if the initial values of the fields δY^B cannot be freely specified on that surface. The characteristic surfaces coincide with the level surfaces of a scalar function φ which satisfies the equation (see, e.g., ref. (19), page 170):

$$\det(A^A{}_B{}^a \nabla_a \varphi) = 0. \tag{96}$$

The characteristic velocities are the slopes of these surfaces. To solve the characteristic equation [Eq. (96)], a Cartesian coordinate system is chosen which is at some point in the fluid momentarily co-moving with the fluid; further, the scalar field φ is chosen to vary spatially only in the (arbitrarily chosen) x^1 direction. These conditions may be summarized as follows:

$$g^{ab}\partial_a\partial_b = -(\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2, \tag{97}$$

$$u^a\partial_a = \partial_0, \tag{98}$$

and

$$\varphi = \varphi(x^0, x^1). \tag{99}$$

These conditions do not restrict the background equilibrium state in any way; it can be rotating, inhomogeneous, and in a curved spacetime.

In this coordinate system the characteristic equation takes the form

$$\det(v\, A^A{}_B{}^0 - A^A{}_B{}^1) = 0, \tag{100}$$

where the characteristic velocity, v, is defined by

(101) $$\nu = -\partial_0\varphi/\partial_1\varphi.$$

In analogy with the calculation in Section 3, the characteristic equation matrices may be simplified by choosing the following set of perturbation variables:

(102) $$\delta Y^B = \left\{T\delta\Theta,\ \frac{\delta T}{T},\ \delta\tau,\ \delta u^1,\ \delta q^1,\ \delta\tau^{11},\ \delta u^2,\ \delta q^2,\ \delta\tau^{21},\ \delta u^3,\ \delta q^3,\ \delta\tau^{31},\ \delta\tau^{22}-\delta\tau^{33},\ \delta\tau^{23}\right\}.$$

The characteristic matrix then block diagonalizes:

(103) $$\nu A^0 - A^1 = \begin{bmatrix} \mathbf{Q} & 0 & 0 & 0 \\ 0 & \mathbf{R} & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & \mathbf{S} \end{bmatrix}.$$

where the submatrices **Q**, **R**, and **S** are defined as follows:

(104) $$\mathbf{Q} = \begin{bmatrix} \frac{\nu}{T}\left(\frac{\partial n}{\partial\Theta}\right)_T & \frac{\nu}{T}\left(\frac{\partial\rho}{\partial\Theta}\right)_T & 0 & -n & 0 & 0 \\ \frac{\nu}{T}\left(\frac{\partial\rho}{\partial\Theta}\right)_T & \nu T\left(\frac{\partial\rho}{\partial T}\right)_\Theta & 0 & -(\rho+p) & -1 & 0 \\ 0 & 0 & \beta_0\nu & -1 & \alpha_0 & 0 \\ -n & -(\rho+p) & -1 & \nu(\rho+p) & \nu & -1 \\ 0 & -1 & \alpha_0 & \nu & \beta_1\nu & \alpha_1 \\ 0 & 0 & 0 & -1 & \alpha_1 & \frac{3}{2}\beta_2\nu \end{bmatrix},$$

(105) $$\mathbf{R} = \begin{bmatrix} \nu(\rho+p) & \nu & -1 \\ \nu & \beta_1\nu & \alpha_1 \\ -1 & \alpha_1 & 2\beta_2\nu \end{bmatrix},$$

and

(106)
$$\mathbf{S} = \begin{bmatrix} 2\beta_2 v & 0 \\ 0 & \beta_2 v \end{bmatrix}.$$

The determinants of the submatrices are given by

(107)
$$\det \mathbf{Q} = \frac{3}{2} v^2[Av^4 + Bv^2 + C]\left[\left(\frac{\partial n}{\partial \theta}\right)_T\left(\frac{\partial \rho}{\partial T}\right)_\theta - \left(\frac{\partial n}{\partial T}\right)_\theta\left(\frac{\partial \rho}{\partial \theta}\right)_T\right],$$

(108)
$$\det \mathbf{R} = v\{2\beta_2[\beta_1(\rho + p) - 1]v^2 - [(\rho + p)\alpha_1{}^2 + 2\alpha_1 + \beta_1]\},$$

(109)
$$\det \mathbf{S} = 2(\beta_2 v)^2.$$

The functions A, B, and C which appear in Eq. (107) are defined by

(110)
$$A = \beta_0\beta_2[\beta_1(\rho + p) - 1],$$

(111)
$$B = -(\rho + p)D - \beta_1 E - 2F,$$

(112)
$$C = (DE - F^2)/\beta_0\beta_2,$$

where D, E, and F are given by

(113)
$$D = \beta_0\beta_2\left[\frac{\alpha_0{}^2}{\beta_0} + \frac{2\alpha_1{}^2}{3\beta_2} + \frac{1}{nT^2}\left(\frac{\partial T}{\partial s}\right)_n\right],$$

(114)
$$E = \beta_0\beta_2\left[(\rho + p)\left(\frac{\partial p}{\partial \rho}\right)_s + \frac{1}{\beta_0} + \frac{2}{3\beta_2}\right],$$

(115)
$$F = \beta_0\beta_2\left[\frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\left(\frac{\partial T}{\partial n}\right)_s\right].$$

The characteristic velocities are obtained by setting the determinants of the submatrices separately to zero and solving for v. The two characteristic velocities corresponding to the zeros of det(**S**) are both zero. The matrix **R**, which describes transverse perturbations of the fluid, has one characteristic velocity which is zero, and two non-zero characteristic velocities, given by

$$v_T{}^2 = \frac{(\rho + p)\alpha_1{}^2 + 2\alpha_1 + \beta_1}{2\beta_2[\beta_1(\rho + p) - 1]}. \tag{116}$$

The longitudinal matrix, **Q**, has two characteristic velocities which are zero, and four nonzero characteristic velocities which are the roots of the quartic polynomial

$$P(v^2) = Av^4 + Bv^2 + C = 0. \tag{117}$$

One of the pairs of roots of Eq. (117) should correspond in some sense to the propagation of sound in an Israel-Stewart fluid, and the other should correspond to the propagation of temperature fluctuations (second sound).

The system of perturbation equations for the Israel-Stewart fluid is called a symmetric system because the matrices $\mathbf{A}^a$ are all symmetric. The system would also be *symmetric hyperbolic* (see Ref. (19), p. 593) if some linear combination of the $\mathbf{A}^a$ is (positive) definite. Other definitions of hyperbolicity, which impose conditions on the characteristic velocities (such as reality) fail when there are multiple characteristics having the same velocity. Since this is always the case for the Israel-Stewart fluids, the matrix condition given above is used to determine the hyperbolicity of the system.

The following may be shown to be the necessary and sufficient conditions for the matrix $\mathbf{A}^0$ to be positive definite:

$$\left(\frac{\partial n}{\partial \theta}\right)_T > 0, \tag{118}$$

$$\left(\frac{\partial \rho}{\partial T}\right)_\theta > 0, \tag{119}$$

$$\beta_i > 0, \tag{120}$$

for $i = 1,2,3$,

$$\beta_1(\rho + p) - 1 > 0, \tag{121}$$

and

$$\left(\frac{\partial \rho}{\partial T}\right)_n > 0. \tag{122}$$

The perturbation equations for an Israel fluid will then form a symmetric hyperbolic system if Eqs. (118) - (122) are satisfied. It is not known whether these are the weakest conditions that imply that the system is symmetric hyperbolic or not.

The stability conditions, $\Omega_A > 0$ [Eqs. (78) - (85)] and the conditions for the system of perturbation equations to be symmetric hyperbolic [Eqs. (118) - (122)] are clearly related to one another. Both sets of conditions imply that the β_i must be positive, for example. The final portions of this section establish the relationships that exist between these conditions together with the conditions that guarantee that the characteristic velocities are subluminal.

(c) *Stability implies causality and hyperbolicity.* One relationship that exists between the stability and causality conditions is that perturbations will propagate causally according to a symmetric hyperbolic system of equations in any Israel-Stewart fluid that satisfies the stability conditions. The proof of this relationship follows.

Using the expression for the $\Omega_A(\lambda^2)$ [Eqs. (78) - (85)], it is easy to verify that the transverse characteristic velocities [given by Eq. (116)] are constrained by

$$1 - v_T{}^2 = \frac{\Omega_4(1)\Omega_7(1)}{\Omega_4(0)\Omega_7(0)} > 0, \tag{123}$$

and

$$v_T{}^2 = \frac{1}{2\beta_2}\left[\frac{\rho + p}{\Omega_4(0)\Omega_7(0)}\left(\alpha_1 + \frac{1}{\rho + p}\right)^2 + \frac{1}{\rho + p}\right] > 0, \tag{124}$$

which together guarantee that $0 < v_T{}^2 < 1$.

The longitudinal velocities are slightly more complicated to handle. The longitudinal velocities will be real only if their squares are real; their squares can be real only if the discriminant of the quartic polynomial in Eq. (117) is nonnegative. The discriminant can be put into the following form using Eqs. (110) - (115):

$$(125)\quad B^2 - 4AC = \frac{1}{\beta_1(\rho + p)}\{[(\rho + p)D + \beta_1 E + 2\beta_1(\rho + p)F]^2 + [\beta_1(\rho + p) - 1][(\rho + p)D - \beta_1 E]^2\} > 0.$$

Thus, the squares of the longitudinal velocities are real if the stability conditions are satisfied. The longitudinal velocities will then be real and less than the speed of light if the zeroes of the quartic polynomial lie between zero and one. The zeroes of the quartic polynomial can be located by using a geometrical argument. The coefficient of the quartic term, A, is positive if the stability conditions are satisfied; therefore $P(v^2)$ will be positive for large v^2. It is possible, after a fair amount of algebra, to show that the stability conditions [Eqs. (78) - (85)] imply the following conditions on P:

$$(126)\quad P(0) = C > 0,$$

$$(127)\quad \frac{dP}{dv^2}(0) = B < 0,$$

$$(128)\quad P(1) = A + B + C > 0,$$

$$(129)\quad \frac{dP}{dv^2}(1) = 2A + B > 0.$$

These four conditions imply that the zeroes of P lie between zero and one; i.e.,

$$(130)\quad 0 < {v_L}^2 < 1.$$

The conditions under which the perturbation equations form a symmetric hyperbolic system [Eqs. (118) - (122)] are easily shown to be consequences of the stability conditions. Thus, we have shown that the perturbations of a second-order Israel-Stewart fluid propagate causally according to a symmetric hyperbolic set of equations if the stability criteria ($\Omega_A > 0$) are satisfied.

(d) *Causality and hyperbolicity imply stability.* A fairly tedious and complicated argument can also be made to prove the converse theorem, namely that an Israel-Stewart fluid must be stable if perturbations propagate causally and the perturbation equations form a symmetric hyperbolic system [in the sense of Eqs. (118) - (122)]. The details of the proof of this theorem are given in Ref. (23). The discussion here will be confined to an examination of the necessity of making such a strong assumption about hyperbolicity in this theorem.

It is interesting that it appears to be necessary to use the definition of a hyperbolic system given in Eqs. (118) - (122) in order to prove that causality plus hyperbolicity implies stability. One might guess that causality plus the weaker (and more familiar) requirement that the characteristic velocities all be real might be enough to guarantee stability. A counterexample, however, exists to that conjecture, in which the characteristic velocities are all real and bounded between zero and one, so causality is assured, yet the stability conditions are *not* satisfied.

Consider a fluid which has an equation of state such that

(131) $$0 < \left(\frac{\partial p}{\partial \rho}\right)_s < 1$$

and

(132) $$0 < \left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta .$$

These two thermodynamic constraints then imply that $(\partial p/\partial n)_s > 0$, $(\partial T/\partial s)_p > 0$, and $(\partial \Theta/\partial s)_\rho < 0$, as can be shown using the thermodynamic identities derived in Ref. (23), Sec. III(c). Now choose the second order coefficients to have the following values:

(133) $$\beta_0 = -\frac{3}{2}\beta_2 = -1/(\rho + p),$$

(134) $$\alpha_0 = \alpha_1 = -1/(\rho + p),$$

(135) $$\beta_1 = \left\{1 + \left[1 - \left(\frac{\partial \Theta}{\partial s}\right)_p\right]\left[1 - \left(\frac{\partial p}{\partial \rho}\right)_s\right]^{-1}\right\}(\rho + p)^{-1} > (\rho + p)^{-1}.$$

It is clear that the stability criteria are not all satisfied with these choices; in particular, $\Omega_5 = \beta_0$ is negative. Nevertheless, it is easy to show that the transverse characteristic velocity, defined in Eq. (116), is given by $v_T^2 = 3/4$, so that $0 < v_T^2 < 1$. With the above choices for the α_i, β_i, and thermodynamic derivative signs, it is also possible to show that the coefficients A, B, and C, defined in Eqs. (110) - (113), which determine the longitudinal velocities through Eq. (117), satisfy the inequalities given in Eqs. (125) - (129), so that the longitudinal velocities also all satisfy $0 < v_L^2 < 1$.

This then is a specific example demonstrating that it is possible to have a second order Israel-Stewart fluid whose characteristic velocities are all real and less than the speed of light, but which nonetheless contains unstable equilibria. It does seem necessary, therefore, to make the stronger assumption that the perturbation equations form a symmetric hyperbolic system in the specific sense of Eqs. (118) - (122) in order to conclude that causality plus hyerbolicity implies stability.

APPENDIX

This appendix presents the proof of the following proposition [see Ref. (26) for an analogous Newtonian result]:

PROPOSITION. *Any solution of the linearized Israel fluid perturbation equations which has vanishing variation in the conserved particle number,* $\delta N = 0$, *and vanishing variation of the conserved momenta,* $\delta P(k^a) = 0$, *for each Killing vector field* k^a *of the background spacetime, will grow without bound if the energy functional for this solution is negative on some spacelike surface.*

Proof. Consider an equilibrium solution of the Israel-Stewart fluid equations on a spacetime manifold $\mathcal{M}$. Assume that $\mathcal{M}$ can be foliated by spacelike surfaces Σ: $\mathcal{M} = \{t\} \times \Sigma$. Define the energy $E(t)$ associated with the solutions to the linearized perturbation equations by

(A1) $$E(t) = -\int_{\Sigma(t)} E^a t_a d\Sigma ,$$

where E^a is the energy current defined in Eq. (68), and t^a is the unit normal vector to the surface $\Sigma(t)$. The derivative of this energy is determined by Eq. (70) to be:

(A2) $$\frac{dE}{dt} = -\int_{\Sigma(t)} \left[\frac{(\delta\tau)^2}{\zeta T} + \frac{\delta q^a \delta q_a}{\kappa T^2} + \frac{\delta\tau^{ab}\delta\tau_{ab}}{2\eta T}\right](-\nabla^c t \nabla_c t)^{-1/2} d\Sigma.$$

First show that the set of perturbations for which dE/dt vanishes is a subset of the perturbations for which E itself vanishes. From Eq. (A2) it follows that $dE/dt = 0$ implies that

(A3) $$\delta\tau = \delta q^a = \delta\tau^{ab} = 0.$$

Using these conditions, and the linearized perturbation equations [Eqs. (42) - (43), and (65) - (67)], it is straightforward to show that any such solution also has

(A4) $$\nabla_a \delta\Theta = 0,$$

and the vector

(A5) $$\xi^a = \delta u^a/T - \delta T u^a/T^2$$

must be a Killing vector field. The energy functional, Eq. (A1), for this set of perturbations can now be evaluated. Using Eq. (81) it follows that

(A6) $$E^a = \tfrac{1}{2}\delta T^a{}_b \xi^b + \tfrac{1}{2}\delta\Theta[n\delta u^a + \delta n u^a],$$

so that

(A7) $$E(t) = \tfrac{1}{2}\, \delta P(\xi^a) + \tfrac{1}{2}\, \delta\Theta\delta N$$

where

(A8) $$\delta N = -\int_{\Sigma(t)} [n\delta u^a + \delta n u^a] t_a d\Sigma$$

is the variation in the conserved particle number, and

(A9) $$\delta P(\xi^a) = -\int_{\Sigma(t)} \delta T^a{}_b \xi^b t_a d\Sigma$$

is the variation in the conserved momentum associated with the Killing vector field ξ^a. For those perturbations where the variations in these conserved quantities vanish, it follows that $E(t) = 0$ from Eq. (A7). Thus, perturbations that are constrained by $\delta N = \delta P(\xi^a) = 0$, which are elements of the kernel of dE/dt (i.e., $dE/dt = 0$) are also elements of the kernel of E (i.e., $E = 0$).

Next consider initial values of the perturbation functions on some surface $\Sigma(t)$ for which the energy $E < 0$. This energy is monotonically decreasing from Eq. (A2). Since the functionals E and dE/dt are continuous (in an appropriate square integrable norm) functionals of the perturbation solutions, it follows that any perturbation having initially negative E must remain outside an open set containing the kernel of E. Since the kernel of dE/dt is a subset of the kernel of E, it follows that dE/dt will also be bounded away from zero as the fluid perturbations evolve. It follows that E will decrease without bound. Since E can be diagonalized as in Eq. (88), it follows that one of the perturbation functions δZ_A associated with a negative eigenvalue, $\Omega_A < 0$, of the energy must grow without bound in this case.

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