THE PROBLEM OF DISSIPATION IN RELATIVISTIC FLUIDS

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ABSTRACT

This paper describes the properties of the Israel-Stewart theory of dissipative relativistic fluids. The conditions needed to guarantee the stability, hyperbolicity and causality of the evolution of fluctuations about an equilibrium state are described. An unexpected relationship between these conditions is revealed: the stability conditions are satisfied if and only if the hyperbolicity and causality conditions hold. The manner in which the complicated (14 degrees of freedom) dynamics of an Israel-Stewart fluid reduces in appropriate limits to the familiar (5 degrees of freedom) dynamics of a relativistic ideal fluid or a Navier-Stokes-Fourier fluid is described.

I. INTRODUCTION. The original attempts to include the effects of viscosity and thermal conductivity in a relativistic fluid theory were made by Eckart[1)] and Landau and Lifshitz[2)]. These theories are the simplest Lorentz covariant generalizations of the Navier-Stokes-Fourier theory. Unfortunately, these theories are extremely pathological[3)]. They admit no stable equilibrium states; they do not have hyperbolic evolution equations; and they violate causality[4)].

In this paper we describe the properties of a more complicated theory of dissipative relativistic fluids proposed by Israel and Stewart[5-8)] that overcomes many of the problems found in the original theories.

II. THE ISRAEL-STEWART THEORY. The dynamical variables of an Israel-Stewart fluid include the familiar variables of an ideal relativistic fluid: the particle number density n, the energy density ρ, the pressure p, the temperature T, the entropy per particle s, and the four-velocity u^a (with $u^a u_a = -1$). In addition, however, these fluids also have as dynamical variables the scalar stress τ, the heat flux vector q^a (with $q^a u_a = 0$) and the spatial stress tensor τ^{ab} (with τ^{ab}

symmetric, traceless and $\tau^{ab}u_b = 0$). These variables are related to the particle number current N^a and the stress energy tensor T^{ab} by:

$$N^a = n\, u^a, \tag{1}$$

$$T^{ab} = \rho\, u^a u^b + (p + \tau) q^{ab} + \tau^{ab} + q^a u^b + q^b u^a, \tag{2}$$

where $q^{ab} = g^{ab} + u^a u^b$ and g^{ab} is the metric tensor which is used to raise and lower indices. The evolution equations for these fluids include the conservation laws for these quantities:

$$\nabla_a N^a = \nabla_a T^{ab} = 0. \tag{3}$$

In an ideal fluid (where $\tau = q^a = \tau^{ab} = 0$) these conservation laws together with an equation of state, $s = s(\rho, n)$, and the first law of thermodynamics,

$$d\rho = nT\, ds + [(\rho + p)/n]\, dn, \tag{4}$$

would be sufficient to determine the evolution of the fluid. In an Israel–Stewart fluid, however, these equations must be supplemented with evolution equations for the new dynamical variables τ, q^a, and τ^{ab}. Their equations for these quantities are the following:

$$\tau = -\varsigma\Big[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a q^a - \gamma_0 T q^a \nabla_a(\alpha_0/T) + \tfrac{1}{2}\tau T \nabla_a(\beta_0 u^a/T)\Big], \tag{5}$$

$$\begin{aligned} q^a = -\kappa T q^{ab}\Big[&(\nabla_b T)/T + u^c\nabla_c u_b + \beta_1 u^c \nabla_c q_b - \alpha_0 \nabla_b \tau - \alpha_1 \nabla_c \tau^c{}_b + \gamma_2 \nabla_{[b}u_{c]} q^c \\ &- (1-\gamma_0) T\tau \nabla_b(\alpha_0/T) - (1-\gamma_1) T\tau^c{}_b \nabla_c(\alpha_1/T) + \tfrac{1}{2} T q_b \nabla_c(\beta_1 u^c/T)\Big], \end{aligned} \tag{6}$$

$$\begin{aligned} \tau^{ab} = -\eta(q^{ac}q^{bd} + q^{ad}q^{bc} - \tfrac{2}{3}q^{ab}q^{cd})\Big[&\nabla_c u_d + \beta_2 u^e \nabla_e \tau_{cd} - \alpha_1 \nabla_c q_d \\ &- \gamma_1 T q_c \nabla_d(\alpha_1/T) + \tfrac{1}{2} T\tau_{cd}\nabla_e(\beta_2 u^e/T) + \gamma_3 \nabla_{[c}u_{e]}\tau^e{}_d\Big]. \end{aligned} \tag{7}$$

In these equations ς, η, and κ are the viscosity coefficients and the thermal conductivity; the α_i, β_i, and γ_i are new "second order" coefficients to be computed from a microscopic theory of the fluid (e.g., kinetic theory[6,7,9]) or determined empirically. This choice of equations is motivated by the need to enforce the second law of thermodynamics. If the entropy current, s^a, is defined by,

$$Ts^a = snTu^a + q^a - \tfrac{1}{2}(\beta_0\tau^2 + \beta_1 q^b q_b + \beta_2 \tau^{bc}\tau_{bc})u^a + \alpha_0 \tau q^a + \alpha_1 \tau^a{}_b q^b, \tag{8}$$

then it follows from the evolution equations (3) – (7) that the total entropy of the fluid will be a non-decreasing function of time, since

$$TV_a s^a = \tau^2/\varsigma + q^a q_a/(\kappa T) + \tau^{ab}\tau_{ab}/(2\eta). \qquad (9)$$

The equilibrium states (those satisfying $V_a s^a = 0$) of a fluid satisfying these equations are identical to the equilibrium states of the simpler Eckart and Landau-Lifshitz theories. In particular, the vector field u^a/T must be a Killing vector field, each of the thermodynamic variables must be constant along the integral curves of u^a, and the thermodynamic variable, $\theta = -s + (\rho + p)/nT$, must have vanishing gradient.

III. STABILITY, CAUSALITY AND HYPERBOLICITY. To determine whether or not the Israel-Stewart theory suffers from the same problems found in the Eckart and the Landau-Lifshitz theories, we undertook a systematic study of the evolution of small perturbations about an arbitrary equilibrium state[10)]. To investigate the stability of these perturbations, an energy functional was constructed. By diagonalizing this functional, we determined that certain conditions, $\Phi_A > 0$, are necessary and sufficient for the stability of these perturbations; the Φ_A are defined by:

$$\Phi_1 = (\partial p/\partial \rho)_s, \qquad (10) \qquad\qquad \Phi_2 = (\partial \rho/\partial s)_p(\partial p/\partial s)_\theta, \qquad (11)$$

$$\Phi_3 = \beta_0, \qquad (12) \qquad\qquad \Phi_4 = \beta_2, \qquad (13)$$

$$\Phi_5 = (\rho + p)(1 - \Phi_1) - [1/\beta_0 + 2/(3\beta_2) + K^2/\Phi_7], \qquad (14)$$

$$\Phi_6 = \rho + p - [2\beta_2 + 2\alpha_1 + \beta_1]/[2\beta_1\beta_2 - (\alpha_1)^2], \qquad (15)$$

$$\Phi_7 = \beta_1 - [(\alpha_0)^2/\beta_0 + 2(\alpha_1)^2/(3\beta_2) + (\partial T/\partial s)_n/(nT^2)], \qquad (16)$$

where $$K = 1 + \alpha_0/\beta_0 + 2\alpha_1/(3\beta_2) - (n/T)(\partial T/\partial n)_s. \qquad (17)$$

Therefore, by restricting the values of the second order coefficients α_i and β_i and by imposing the usual thermodynamic constraints on the specific heats, etc., implied by eqs. (10) and (11) it is possible to have stable equilibrium configurations in the Israel-Stewart theory.

We also investigated the conditions under which the equations governing the evolution of the perturbations are hyperbolic. The following conditions are sufficient (perhaps not necessary) to assure that the perturbation equations are a symmetric hyperbolic system:

$$(\partial n/\partial\theta)_T > 0, \quad (18) \qquad (\partial\rho/\partial T)_\theta > 0, \quad (19)$$

$$(\partial\rho/\partial T)_n > 0, \quad (20) \qquad \beta_0 > 0, \quad (21)$$

$$\beta_2 > 0, \quad (22) \qquad \beta_1(\rho + p) > 1, \quad (23)$$

For a hyperbolic system of equations, the propagation of information is controlled by the characteristic velocities of the differential operator. We computed the characteristic velocities for the equations governing the perturbations away from an equilibrium state. Six characteristic velocities are zero, four characteristic velocities are given by (each occurs twice),

$$(v_T)^2 = [(\rho + p)(\alpha_1)^2 + 2\alpha_1 + \beta_1]/\{2\beta_2[\beta_1(\rho + p) - 1]\}, \tag{24}$$

and four more are given by the roots of the quartic equation,

$$A(v_L)^4 + B(v_L)^2 + C = 0, \tag{25}$$

where

$$A = \beta_0\beta_2[\beta_1(\rho + p) - 1], \tag{26}$$

$$B = A(\Phi_7/\beta_1 - 2) + \beta_0\beta_2\Big[\beta_1\Phi_5 + (\beta_1K - \Phi_7)^2/(\beta_1\Phi_7)\Big], \tag{27}$$

$$C = - A - B + \beta_0\beta_2\Phi_5\Phi_7. \tag{28}$$

The perturbations are guaranteed to propagate causally as long as these characteristic velocities are less than the speed of light (one in our units) and the system of equations is symmetric hyperbolic.

A remarkable relationship exists between the conditions for the stability, causality and hyperbolicity of these perturbation equations. The stability conditions for these perturbations, eqs. (10) – (16), are satisfied if and only if both the hyperbolicity conditions, eqs. (18)–(23), are satisfied and the characteristic velocities are real and less than the speed of light[10)]. A slightly stronger proposition is false. The stability conditions are not equivalent to the characteristic velocities being real and less than the speed of light[11)].

IV. <u>THE CLASSICAL FLUID LIMIT</u>. A potential difficulty for the Israel-Stewart theory is the embarrassing complexity of its dynamical structure. These fluids have fourteen dynamical fields, while an ideal relativistic fluid or a non-relativistic Navier-Stokes-Fourier fluid have only five. If this theory is capable of describing ordinary laboratory fluids, how do these additional degrees of freedom disappear? To understand this question we have analyzed the dispersion relations for the plane wave perturbation solutions to the Israel-Stewart theory[4]. When these dispersion relations are examined in the "classical" (i.e., long wavelength compared to the mean free path) limit we find that nine of the modes are strongly damped at lowest order in the wavenumber k:

$$\omega_1 = -\, i/(\varsigma\beta_0), \qquad (29) \qquad \omega_{2-6} = -\, i/(2\eta\beta_2), \qquad (30)$$

$$\omega_{7-9} = -\, i(\rho + p)/\{\kappa T[\beta_1(\rho + p) - 1]\}. \qquad (31)$$

The remaining five modes have dispersion relations which are simply the relativistic generalizations of the five modes of a Navier-Stokes-Fourier fluid in this long wavelength limit:

$$\omega_{10,11} = \pm k(\partial p/\partial\rho)_s^{1/2} - \frac{1}{2}ik^2\left[\frac{4}{3}\eta + \varsigma + \kappa(\partial p/\partial\rho)_s(\partial\rho/\partial s)_p^2/n^2T\right](\rho+p)^{-1}, \qquad (32)$$

$$\omega_{12} = -\, i\kappa k^2(\partial T/\partial s)_p/(nT), \qquad (33)$$

$$\omega_{13,14} = -\, i\eta k^2/(\rho + p). \qquad (34)$$

Thus the complicated dynamical structure of an Israel-Stewart fluid does reduce to the familiar dynamics of ordinary fluid mechanics in the regime where experimental data are most prevalent.

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