Solving Einstein's Equation With Generalized Harmonic Gauge Conditions

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Collaborators: Larry Kidder, Robert Owen, Oliver Rinne, Harald Pfeiffer, Mark Scheel, Saul Teukolsky

New Frontiers in Numerical Relativity
AEI, Golm – 17 July 2006

- GH gauge conditions and constraint damping.
- Boundary conditions for the GH system.
- Dual-coordinate frame evolution method.

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Methods of Specifying Spacetime Coordinates

- The lapse N and shift N^i are generally used to specify how coordinates are layed out on a spacetime manifold: $\partial_t = N\vec{t} + N^k\partial_k$.
- An alternate way to specify the coordinates is through the generalized harmonic gauge source function H_a:
- Let H_a denote the function obtained by the action of the scalar wave operator on the coordinates x^b:

$$H_a \equiv \psi_{ab}\nabla^c\nabla_c x^b = -\Gamma_a,$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$.
- Specifying coordinates by the *generalized harmonic* (GH) method can be accomplished by choosing a gauge-source function $H_a(x,\psi)$, and requiring that $H_a(x,\psi) = -\Gamma_a$.

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Important Properties of the GH Method

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

$$R_{ab} = -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + F_{ab}(\psi, \partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$. The vacuum Einstein equation, $R_{ab} = 0$, has the same principal part as the scalar wave equation when $H_a(x, \psi) = -\Gamma_a$ is imposed.
- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by the derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv G_{ab}t^b = t^b\left(\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\right).$$

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Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 \;=\; R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[t_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\, t^c\, \mathcal{C}_c\right],$$

  where t^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

$$0 \;=\; \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[t_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \frac{1}{2}\gamma_0\, t_a\mathcal{C}^c\mathcal{C}_c.$$

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First Order Generalized Harmonic Evolution System

- Kashif Alvi (2002) derived a nice (symmetric hyperbolic) first-order form for the generalized-harmonic evolution system:

$$\begin{aligned} \partial_t\psi_{ab} - N^k\partial_k\psi_{ab} &= -N\,\Pi_{ab}, \\ \partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} &\simeq 0, \\ \partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} &\simeq 0, \end{aligned}$$

  where $\Phi_{kab} = \partial_k\psi_{ab}$.

- This system has two immediate problems:
  - This system has new constraints, $\mathcal{C}_{kab} = \partial_k\psi_{ab} - \Phi_{kab}$, that tend to grow exponentially during numerical evolutions.
  - This system is not linearly degenerate, so it is possible (likely) that shocks will develop (e.g. the shift evolution equation is of the form $\partial_t N^i - N^k\partial_k N^i \simeq 0$).

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A 'New' Generalized Harmonic Evolution System

- We can correct these problems by adding additional multiples of the constraints to the evolution system:

$$\partial_t \psi_{ab} - (1+\gamma_1) N^k \partial_k \psi_{ab} = -N\Pi_{ab} - \gamma_1 N^k \Phi_{kab},$$

$$\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + N g^{ki} \partial_k \Phi_{iab} - \gamma_1 \gamma_2 N^k \partial_k \psi_{ab} \simeq -\gamma_1\gamma_2 N^k \Phi_{kab},$$

$$\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} - \gamma_2 N \partial_i \psi_{ab} \simeq -\gamma_2 N \Phi_{iab}.$$

- This 'new' generalized-harmonic evolution system has several nice properties:
  - This system is linearly degenerate for $\gamma_1 = -1$ (and so shocks should not form from smooth initial data).
  - The Φ_{iab} evolution equation can be written in the form, $\partial_t \mathcal{C}_{iab} - N^k \partial_k \mathcal{C}_{iab} \simeq -\gamma_2 N \mathcal{C}_{iab}$, so the new constraints are damped when $\gamma_2 > 0$.
  - This system is symmetric hyperbolic for all values of γ_1 and γ_2.

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Constraint Evolution for the New GH System

- The evolution of the constraints, $c^A = \{\mathcal{C}_a, \mathcal{C}_{kab}, \mathcal{F}_a \approx t^c\partial_c\mathcal{C}_a, \mathcal{C}_{ka} \approx \partial_k\mathcal{C}_a, \mathcal{C}_{klab} = \partial_{[k}\mathcal{C}_{l]ab}\}$ are determined by the evolution of the fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$:

$$\partial_t c^A + A^{k\,A}{}_B(u)\partial_k c^B = F^A{}_B(u, \partial u)\, c^B.$$

- This constraint evolution system is symmetric hyperbolic with principal part:

$$\begin{aligned}
\partial_t\mathcal{C}_a &\simeq 0,\\
\partial_t\mathcal{F}_a - N^k\partial_k\mathcal{F}_a - Ng^{ij}\partial_i\mathcal{C}_{ja} &\simeq 0,\\
\partial_t\mathcal{C}_{ia} - N^k\partial_k\mathcal{C}_{ia} - N\partial_i\mathcal{F}_a &\simeq 0,\\
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\partial_t\mathcal{C}_{ijab} - N^k\partial_k\mathcal{C}_{ijab} &\simeq 0.
\end{aligned}$$

- An analysis of this system shows that all of the constraints are damped in the WKB limit when $\gamma_0 > 0$ and $\gamma_2 > 0$. So, this system has constraint suppression properties that are similar to those of the Pretorius (and Gundlach, et al.) system.

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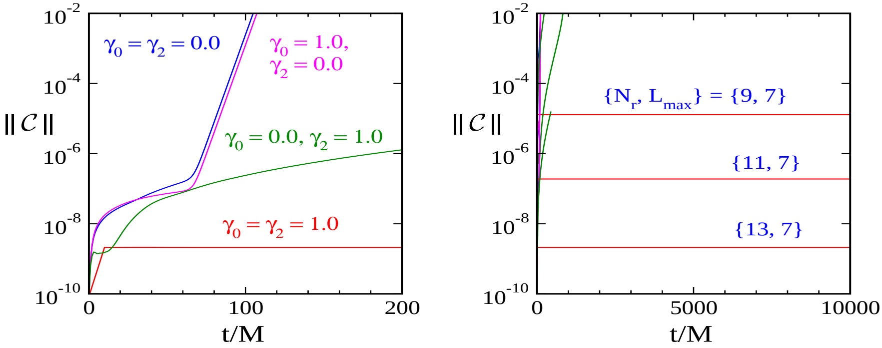
Numerical Tests of the New GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of our GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.



- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

Boundary Condition Basics

- Boundary conditions are imposed on first-order hyperbolic evolutions systems, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\partial_k u^\beta = F^\alpha(u)$ in the following way (where in our case $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$):
- Find the eigenvectors of the characteristic matrix $n_k A^{k\,\alpha}{}_\beta$ at each boundary point:

$$e^{\hat\alpha}{}_\alpha\, n_k A^{k\,\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta,$$

  where n_k is the outward directed unit normal.
- For hyperbolic evolution systems the eigenvectors $e^{\hat\alpha}{}_\alpha$ are complete: $\det e^{\hat\alpha}{}_\alpha \neq 0$. So we define the characteristic fields:

$$u^{\hat\alpha} = e^{\hat\alpha}{}_\alpha u^\alpha.$$

- A boundary condition must be imposed on every incoming characteristic field (*i.e.* every field with $v_{(\hat\alpha)} < 0$), and must not be imposed on any outgoing field (*i.e.* any field with $v_{(\hat\alpha)} > 0$).

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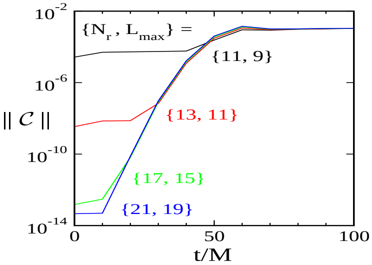
Evolutions of a Perturbed Schwarzschild Black Hole

- A black-hole spacetime is perturbed by an incoming gravitational wave that excites quasi-normal oscillations.
- Use boundary conditions that *Freeze* the remaining incoming characteristic fields.
- The resulting outgoing waves interact with the boundary of the computational domain and produce constraint violations.



Lapse Movie   Constraint Movie

Constraint Preserving Boundary Conditions

- Construct the characteristic fields, $\hat{c}^{\hat{A}} = e^{\hat{A}}{}_{A} c^A$, associated with the constraint evolution system, $\partial_t c^A + A^{k\,A}{}_B \partial_k c^B = F^A{}_B c^B$.
- Split the constraints into incoming and outgoing characteristics: $\hat{c} = \{\hat{c}^-, \hat{c}^+\}$.
- The incoming characteristic fields mush vanish on the boundaries, $\hat{c}^- = 0$, if the influx of constraint violations is to be prevented.
- The constraints depend on the primary evolution fields (and their derivatives). We find that $\hat{c}^-$ for the GH system can be expressed:

$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$

- Set boundary conditions on the fields $\hat{u}^-$ by requiring

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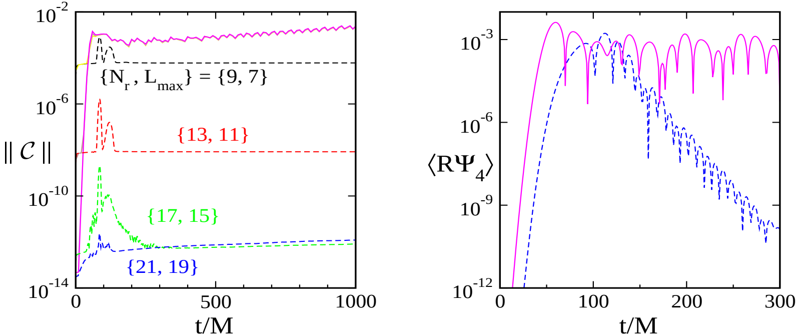
- Evolve the perturbed black-hole spacetime using the resulting constraint preserving boundary conditions for the generalized harmonic evolution systems.



- Evolutions using these new constraint-preserving boundary conditions are still stable and convergent.
- The Weyl curvature component Ψ_4 shows clear quasi-normal mode oscillations in the outgoing gravitational wave flux when constraint-preserving boundary conditions are used.

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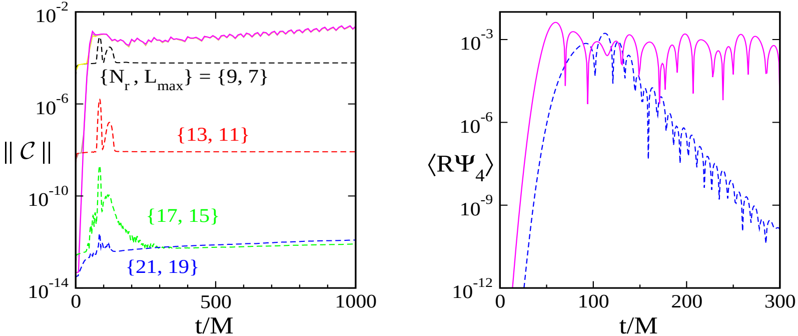

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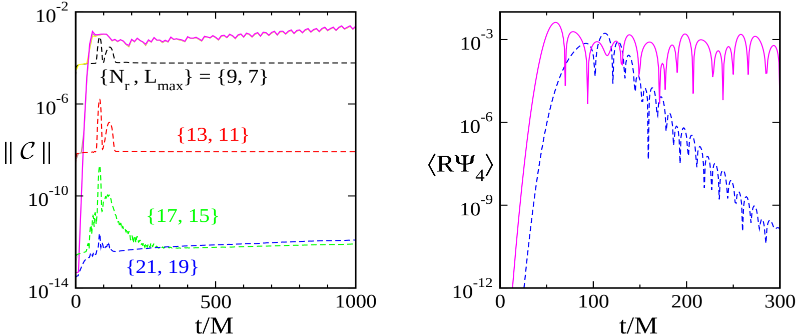
- The Weyl curvature component Ψ_4 shows clear quasi-normal mode oscillations in the outgoing gravitational wave flux when constraint-preserving boundary conditions are used.

Numerical Tests of Constraint Preserving BC

- Evolve the perturbed black-hole spacetime using the resulting constraint preserving boundary conditions for the generalized harmonic evolution systems.



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Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar{a}} = \{\bar{t}, x^{\bar{\imath}}\}$, to define field components, $u^{\bar{\alpha}} = \{\psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{\imath}\bar{a}\bar{b}}\}$, and the same coordinates to determine these components by solving Einstein's equation: $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^{\bar{a}})$.

- Dual-coordinate frame method uses a second set of coordinates, $x^{a} = \{t, x^{i}\} = x^{a}(x^{\bar{a}})$, to determine the original representation of the dynamical fields, $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^{a})$, by solving the transformed Einstein equation:

$$\partial_t u^{\bar{\alpha}} + \left[\frac{\partial x^i}{\partial \bar{t}} \delta^{\bar{\alpha}}{}_{\bar{\beta}} + \frac{\partial x^i}{\partial x^{\bar{k}}} A^{\bar{k}\,\bar{\alpha}}{}_{\bar{\beta}}\right] \partial_i u^{\bar{\beta}} = F^{\bar{\alpha}}.$$

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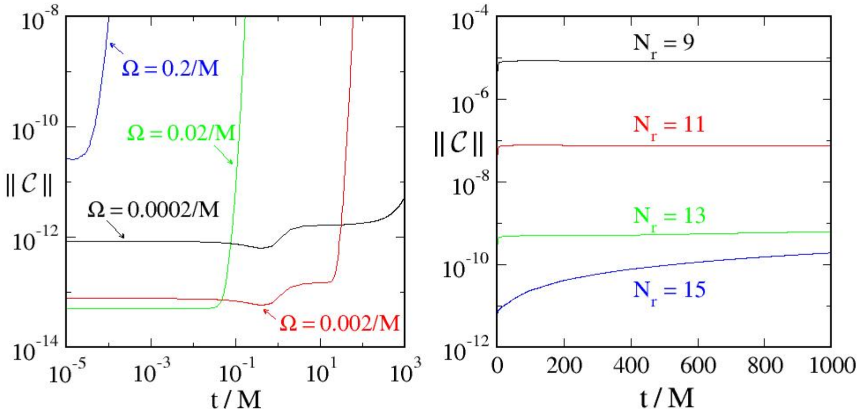
Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:

Single Frame Evolution | Dual Frame Evolution



- Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/M$ on a domain with outer radius $r = 1000M$.