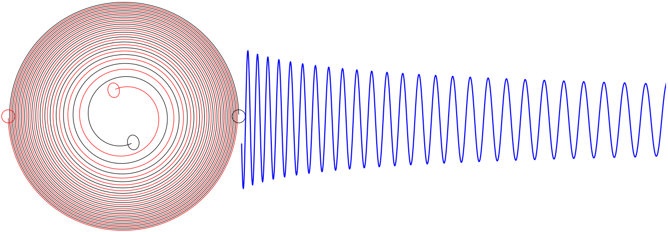
New Insights Into Gauge Freedom and Constraints in Numerical Relativity

Lee Lindblom
Caltech

Collaborators: Michael Boyle, Duncan Brown, Michael Cohen, Larry Kidder, Goeffrey Lovelace, Keith Matthews, Robert Owen, Harald Pfeiffer, Oliver Rinne, Mark Scheel, Saul Teukolsky

CIMMS Seminar :: Caltech :: 26 April 2007



Recent Progress in Numerical Relativity

- Frans Pretorius performed first numerical BBH inspiral, merger and ringdown calculations in the spring of 2005 using a generalized harmonic (GH) formulation of the Einstein equations.
  Pretorius Inspiral Movies

Recent Progress in Numerical Relativity

- Frans Pretorius performed first numerical BBH inspiral, merger and ringdown calculations in the spring of 2005 using a generalized harmonic (GH) formulation of the Einstein equations.
  Pretorius Inspiral Movies
- Groups at NASA Goddard and U. Texas–Brownsville simultaneously announce similar BBH simulations in the fall of 2005 using very different methods (BSSN–puncture).
  LSU/AEI collaboration obtains similar results in Dec. 2005.
- Penn State group begins the study of physical properties of BBH orbits in early 2006 by evolving unequal mass binaries and measuring the kick velocity using BSSN–puncture methods.

Recent Progress in Numerical Relativity

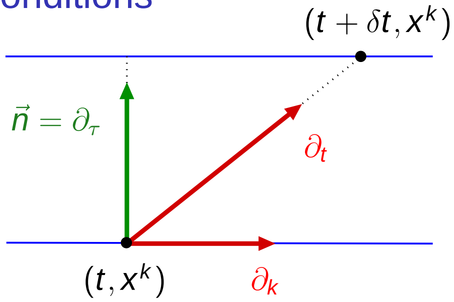
- Frans Pretorius performed first numerical BBH inspiral, merger and ringdown calculations in the spring of 2005 using a generalized harmonic (GH) formulation of the Einstein equations.
  Pretorius Inspiral Movies
- Groups at NASA Goddard and U. Texas–Brownsville simultaneously announce similar BBH simulations in the fall of 2005 using very different methods (BSSN–puncture).
  LSU/AEI collaboration obtains similar results in Dec. 2005.
- Penn State group begins the study of physical properties of BBH orbits in early 2006 by evolving unequal mass binaries and measuring the kick velocity using BSSN–puncture methods.
- Caltech/Cornell collaboration and the AEI/Pittsburgh collaboration perform successful BBH simulations in 2006 using GH methods.
- ...

Outline of Talk:

- Methods of Specifying Gauge (Coordinates).
- Generalized Harmonic (GH) Einstein Equations.
- Constraint Damping.
- Moving Black Holes in a Spectral Code.
- Dual Coordinate Frame Evolution.
- Choosing Coordinates by Feedback Control.
- Recent GH Binary Black Hole Results.

Traditional ADM Gauge Conditions

- Construct a foliation of spacetime by spatial slices.
- Choose a time function with t = const. on these slices.
- Choose spatial coordinates, x^k, on each slice.



- Decompose the 4-metric ψ_{ab} into its 3+1 parts: $ds^2 = \psi_{ab}dx^a dx^b = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$.
- The lapse N and shift N^i measure how coordinates are laid out on spacetime:

$$\vec{n} = \partial_\tau = \frac{\partial t}{\partial \tau}\partial_t + \frac{\partial x^k}{\partial \tau}\partial_k, = \frac{1}{N}\partial_t - \frac{N^k}{N}\partial_k.$$

- Spacetime coordinates are determined in the traditional ADM method by specifying the lapse N and shift N^i.

ADM Evolution System

- When the gauge is determined by specifying the lapse N and shift N^k, the Einstein system becomes a set of evolution equations for the spatial metric g_{ij} and extrinsic curvature K_{ij}:

$$\begin{aligned}\partial_t g_{ij} - N^k\partial_k g_{ij} &= -2NK_{ij} + \nabla_i N_j + \nabla_j N_i,\\ \partial_t K_{ij} - N^k\partial_k K_{ij} &= N\left(R_{ij} - 2K_i{}^k K_{kj} + KK_{ij}\right)\\ &\quad -\nabla_i\nabla_j N + K_{ik}\partial_j N^k + K_{kj}\partial_i N^k.\end{aligned}$$

- The Einstein equations also include constraints:

$$\begin{aligned}0 &= \mathcal{M}_{\hat{t}} \equiv R - K_{ij}K^{ij} + K^2,\\ 0 &= \mathcal{M}_i \equiv \nabla^k K_{ki} - \nabla_i K.\end{aligned}$$

- This traditional form of the Einstein equations is not hyperbolic, and numerical solutions are found to suffer from generic constraint violating instabilities.

Generalized Harmonic Gauge Conditions

- An alternate way to specify the coordinates is through the gauge source function H^a:
- Let H^a denote the function obtained by the action of the covariant scalar wave operator on the coordinates x^a:

$$H^a \equiv \nabla^c \nabla_c x^a = \psi^{bc}(\partial_b \partial_c x^a - \Gamma^e_{bc} \partial_e x^a) = -\Gamma^a,$$

where $\Gamma^a = \psi^{bc} \Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

Generalized Harmonic Gauge Conditions

- An alternate way to specify the coordinates is through the gauge source function H^a:
- Let H^a denote the function obtained by the action of the covariant scalar wave operator on the coordinates x^a:

$$H^a \equiv \nabla^c \nabla_c x^a = \psi^{bc}(\partial_b \partial_c x^a - \Gamma^e_{bc} \partial_e x^a) = -\Gamma^a,$$

where $\Gamma^a = \psi^{bc} \Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

- Specifying coordinates by the *generalized harmonic* (GH) method can be accomplished by choosing a gauge-source function $H_a(x, \psi) = \psi_{ab} H^b$, and requiring that

$$H_a(x, \psi) = -\Gamma_a = -\psi_{ab} \psi^{cd} \Gamma^b_{cd}.$$

Important Properties of the GH Method

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

$$R_{ab} = -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + F_{ab}(\psi, \partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$. The vacuum Einstein equation, $R_{ab} = 0$, has the same principal part as the scalar wave equation when $H_a(x, \psi) = -\Gamma_a$ is imposed.

Generalized Harmonic Evolution System

- Frans Pretorius wrote a very nice second order finite difference AMR code to solve the generalized harmonic Einstein equations:

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

  where $\mathcal{C}_a = H_a + \Gamma_a$. Unfortunately initial code was very unstable.

Generalized Harmonic Evolution System

- Frans Pretorius wrote a very nice second order finite difference AMR code to solve the generalized harmonic Einstein equations:

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

where $\mathcal{C}_a = H_a + \Gamma_a$. Unfortunately initial code was very unstable.

- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by the derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv \left[R_{ab} - \frac{1}{2}\psi_{ab}R\right]n^b = \left[\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\right]n^b.$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0 \left[n_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\, n^c\, \mathcal{C}_c \right],$$

where n^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0 \left[n_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\, n^c\, \mathcal{C}_c \right],$$

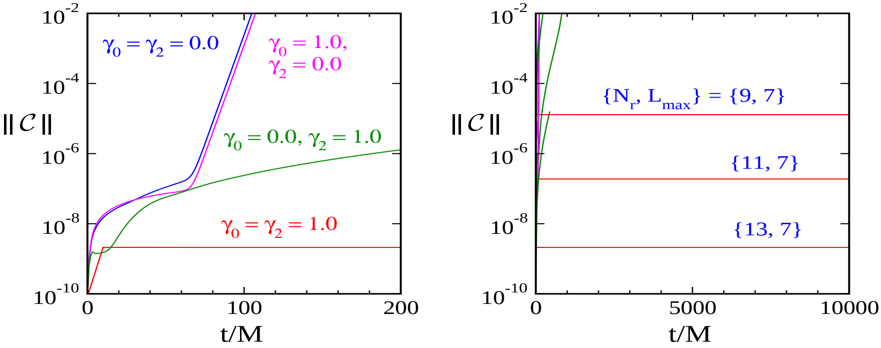
  where n^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.
- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[n_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \frac{1}{2}\gamma_0\, n_a\mathcal{C}^c\mathcal{C}_c.$$

  This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

Numerical Tests of the First-Order GH System

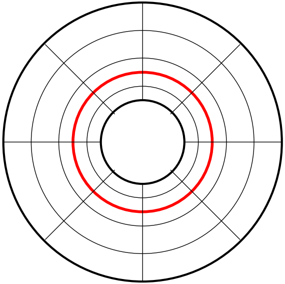
- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.



- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

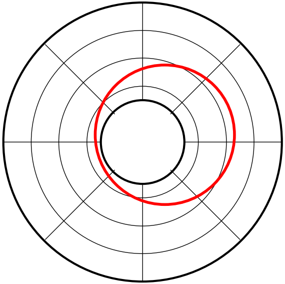
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.



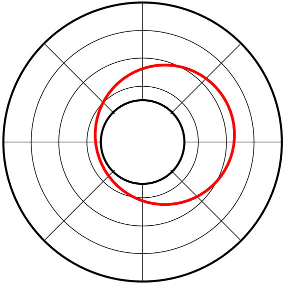
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.



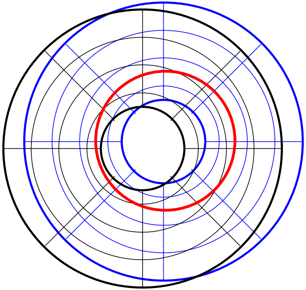
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.



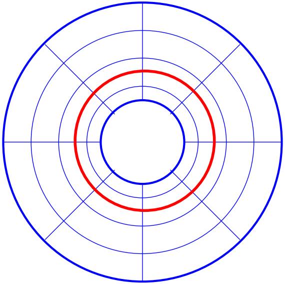
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
    - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.



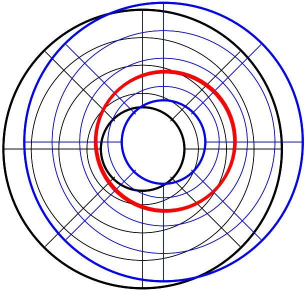
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.



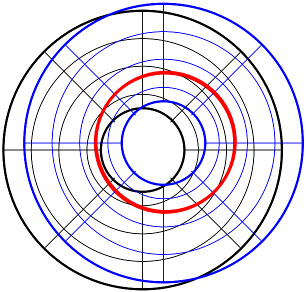
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.
- Problems:
  - Re-gridding/interpolation is expensive.



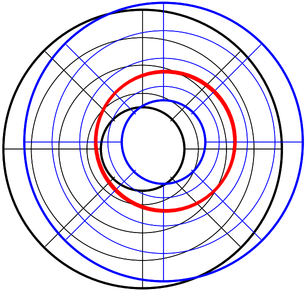
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.
- Problems:
  - Re-gridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.



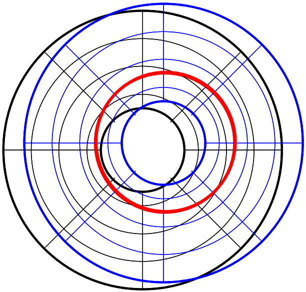
Moving Black Holes in a Spectral Code

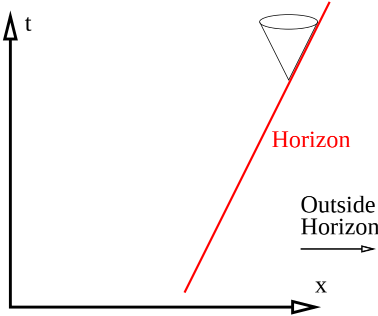
- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.
- Problems:
  - Re-gridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.



Moving Black Holes in a Spectral Code

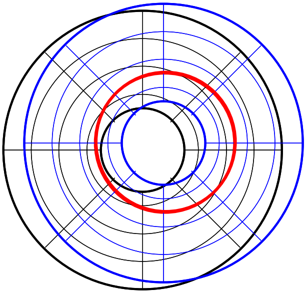
- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.
- Problems:
  - Re-gridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.

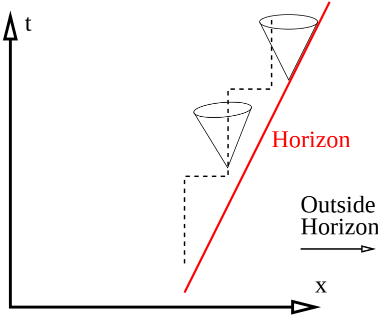




Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.
- Problems:
  - Re-gridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.



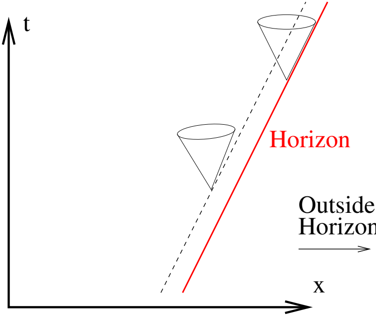


Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.
- Problems:
  - Re-gridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.
- Solution:

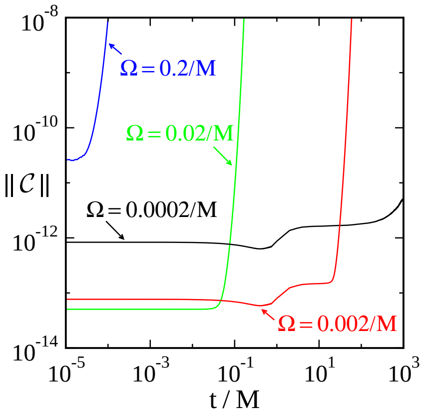
  Choose coordinates that smoothly track the location of the black hole.

  For a black hole binary this means using coordinates that rotate with respect to inertial frames at infinity.



Evolving Black Holes in Rotating Frames

- Coordinates that rotate with respect to the inertial frames at infinity are needed to track the horizons of orbiting black holes.
- Evolutions of Schwarzschild in rotating coordinates are unstable.



- Evolutions shown use a computational domain that extends to $r = 1000M$.
- Angular velocity needed to track the horizons of an equal mass binary at merger is about $\Omega \approx 0.2/M$.
- Problem caused by asymptotic behavior of metric in rotating coordinates: $\psi_{tt} \sim \rho^2\Omega^2$, $\psi_{ti} \sim \rho\Omega$, $\psi_{ij} \sim 1$.

Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar{a}} = \{\bar{t}, x^{\bar{\imath}}\}$, to define field components, $u^{\bar{\alpha}} = \{\psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{\imath}\bar{a}\bar{b}}\}$, and the same coordinates to determine these components by solving Einstein's equation for $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^{\bar{a}})$:

$$\partial_{\bar{t}} u^{\bar{\alpha}} + A^{\bar{k}\,\bar{\alpha}}{}_{\bar{\beta}} \partial_{\bar{k}} u^{\bar{\beta}} = F^{\bar{\alpha}}.$$

Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar a} = \{\bar t, x^{\bar\imath}\}$, to define field components, $u^{\bar\alpha} = \{\psi_{\bar a\bar b}, \Pi_{\bar a\bar b}, \Phi_{\bar\imath\bar a\bar b}\}$, and the same coordinates to determine these components by solving Einstein's equation for $u^{\bar\alpha} = u^{\bar\alpha}(x^{\bar a})$:

$$\partial_{\bar t} u^{\bar\alpha} + A^{\bar k\,\bar\alpha}{}_{\bar\beta}\partial_{\bar k} u^{\bar\beta} = F^{\bar\alpha}.$$

- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^{\bar a})$, to represent these components as functions, $u^{\bar\alpha} = u^{\bar\alpha}(x^a)$.

Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar{a}} = \{\bar{t}, x^{\bar{\imath}}\}$, to define field components, $u^{\bar{\alpha}} = \{\psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{\imath}\bar{a}\bar{b}}\}$, and the same coordinates to determine these components by solving Einstein's equation for $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^{\bar{a}})$:

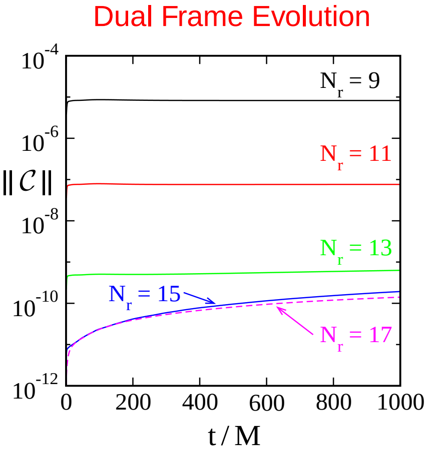
$$\partial_{\bar{t}} u^{\bar{\alpha}} + A^{\bar{k}\,\bar{\alpha}}{}_{\bar{\beta}} \partial_{\bar{k}} u^{\bar{\beta}} = F^{\bar{\alpha}}.$$

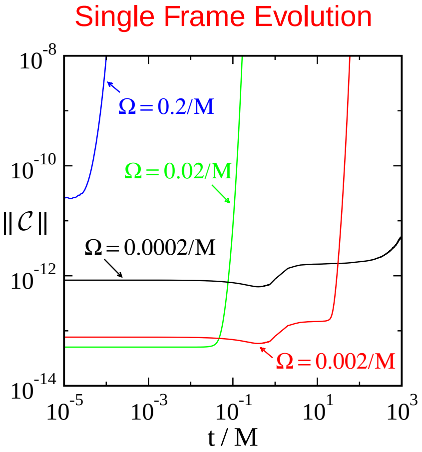
- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^{\bar{a}})$, to represent these components as functions, $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^a)$.
- These functions are determined by solving the transformed Einstein equation:

$$\partial_t u^{\bar{\alpha}} + \left[\frac{\partial x^i}{\partial \bar{t}} \delta^{\bar{\alpha}}{}_{\bar{\beta}} + \frac{\partial x^i}{\partial x^{\bar{k}}} A^{\bar{k}\,\bar{\alpha}}{}_{\bar{\beta}}\right] \partial_i u^{\bar{\beta}} = F^{\bar{\alpha}}.$$

Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:





- Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/M$ on a domain with outer radius $r = 1000M$.

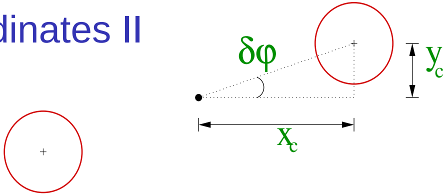
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- A coordinate transformation from inertial coordinates, $(\bar{x}, \bar{y}, \bar{z})$, to co-moving coordinates (x, y, z), consisting of a rotation followed by an expansion,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{a(\bar{t})} \begin{pmatrix} \cos\varphi(\bar{t}) & -\sin\varphi(\bar{t}) & 0 \\ \sin\varphi(\bar{t}) & \cos\varphi(\bar{t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix},$$

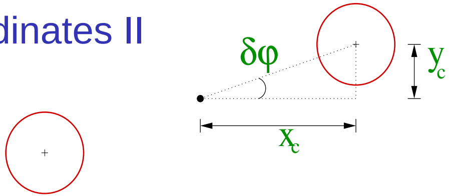
  with $t = \bar{t}$, is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions $a(\bar{t})$ and $\varphi(\bar{t})$.
- Since the motions of the holes are not known *a priori*, the functions $a(\bar{t})$ and $\varphi(\bar{t})$ must be chosen dynamically and adaptively as the system evolves.

Horizon Tracking Coordinates II



- Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.
- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.

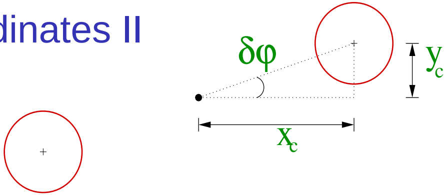
Horizon Tracking Coordinates II



- Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.
- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.
- Changing the map parameters by the small amounts, δa and $\delta\varphi$, results in associated small changes in δQ^x and δQ^y:

$$\delta Q^x = -\delta a, \qquad \delta Q^y = -\delta\varphi.$$

Horizon Tracking Coordinates II



- Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.
- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.
- Changing the map parameters by the small amounts, δa and $\delta\varphi$, results in associated small changes in δQ^x and δQ^y:

$$\delta Q^x = -\delta a, \qquad \delta Q^y = -\delta\varphi.$$

- Measure the quantities $Q^y(t)$, $dQ^y(t)/dt$, $d^2Q^y(t)/dt^2$, and set

$$\frac{d^3\varphi}{dt^3} = \lambda^3 Q^y + 3\lambda^2\frac{dQ^y}{dt} + 3\lambda\frac{d^2Q^y}{dt^2} = -\frac{d^3Q^y}{dt^3}.$$

  The solutions to this “closed-loop” equation for Q^y have the form $Q^y(t) = (At^2 + Bt + C)e^{-\lambda t}$, so Q^y always decreases as $t \to \infty$.

Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times $t = t_i$.
- In the time interval $t_i < t < t_{i+1}$ we set:

$$
\begin{aligned}
\varphi(t) &= \varphi_i + (t - t_i)\frac{d\varphi_i}{dt} + \frac{(t-t_i)^2}{2}\frac{d^2\varphi_i}{dt^2} \\
&\quad + \frac{(t-t_i)^3}{2}\left(\lambda\frac{d^2Q_i^y}{dt^2} + \lambda^2\frac{dQ_i^y}{dt} + \lambda^3\frac{Q_i^y}{3}\right),
\end{aligned}
$$

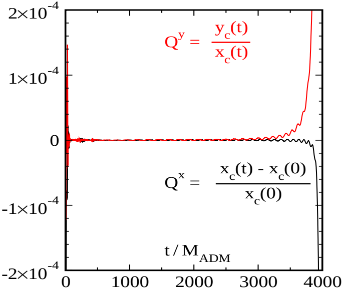
where Q^x, Q^y, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop
equation at $t = t_i$.

Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times $t = t_i$.
- In the time interval $t_i < t < t_{i+1}$ we set:

$$\varphi(t) = \varphi_i + (t - t_i)\frac{d\varphi_i}{dt} + \frac{(t-t_i)^2}{2}\frac{d^2\varphi_i}{dt^2} + \frac{(t-t_i)^3}{2}\left(\lambda\frac{d^2Q_i^y}{dt^2} + \lambda^2\frac{dQ_i^y}{dt} + \lambda^3\frac{Q_i^y}{3}\right),$$

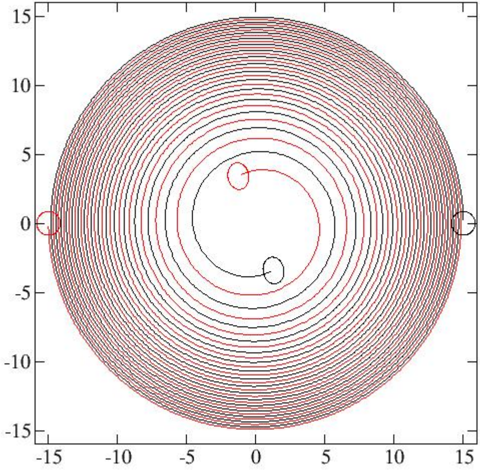
where Q^x, Q^y, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop equation at $t = t_i$.



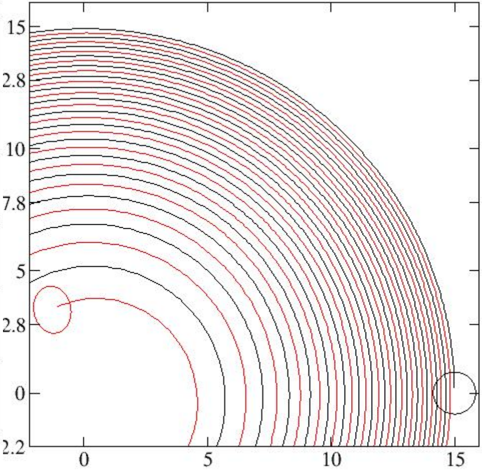
- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.

Evolving Binary Black Hole Spacetimes

- We can now evolve binary black hole spacetimes with excellent accuracy and computational efficiency through many orbits.



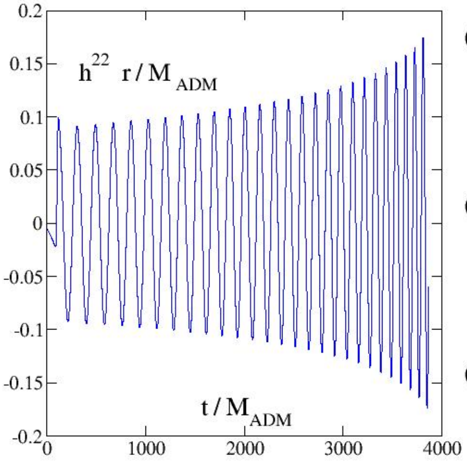
Head-on Merger Movie

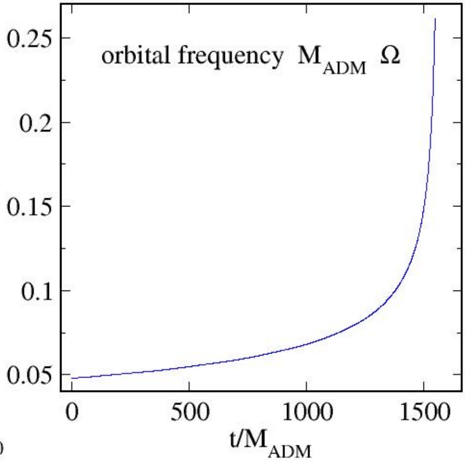


Lapse-Ψ_4 Movie

Evolving Binary Black Hole Spacetimes II

- Gravitational waveform and frequency evolution for the equal mass non-spinning BBH evolution.





Summary

- Generalized Harmonic method produces manifestly hyperbolic representations of the Einstein equations for any choice of coordinates.
- Constraint damping makes the modified GH equations stable for numerical simulations.
- Dual coordinate frame evolution makes evolutions stable in coordinates that track the black hole motions.
- Feedback control systems can be used to construct co-moving coordinates that accurately track the black hole motions.