Numerical Simulations of Black Hole Spacetimes

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Caltech-Cornell Numerical Relativity Collaboration

Group leaders: Lee Lindblom, Mark Scheel, and Harald Pfeiffer at Caltech; Saul Teukolsky and Larry Kidder at Cornell.

Kidder  Lindblom  Pfeiffer  Scheel  Teukolsky
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- Kidder
- Lindblom
- Pfeiffer
- Scheel
- Teukolsky

Caltech group: Michael Boyle, Jeandrew Brink, Duncan Brown, Tony Chu, Michael Cohen, Lee Lindblom, Geoffrey Lovelace, Keith Matthews, Robert Owen, Harald Pfeiffer, Oliver Rinne, Mark Scheel, Kip Thorne.

Motivation: Gravitational Wave Astronomy

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Why Is Numerical Relativity So Difficult?

- Dynamics of binary black hole problem is driven by delicate adjustments to orbit due to emission of gravitational waves.
- Very big computational problem:
  - Must evolve $\sim 50$ dynamical fields (spacetime metric plus all first derivatives).
  - Must accurately resolve features on many scales from black hole horizons $r \sim GM/c^2$ to emitted waves $r \sim 100GM/c^2$.
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- Most representations of the Einstein equations have mathematically ill-posed initial value problems.

- Constraint violating instabilities destroy stable numerical solutions in many well-posed forms of the equations.

Unstable BBH Movie
Recent Progress in Numerical Relativity

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- Penn State group begins the study of physical properties of BBH orbits in early 2006 by evolving unequal mass binaries and measuring the kick velocity using BSSN–puncture methods.

- ...
Outline of Remainder of Talk:

- Three technical issues:
  - Constraint Damping.
  - Spectral Methods.
  - Feedback Control Systems.

- Interesting binary black hole simulations.
The usual representation of the vacuum Maxwell equations split into evolution equations and constraints:

$$\begin{align*}
\partial_t \vec{E} &= \vec{\nabla} \times \vec{B}, \\
\nabla \cdot \vec{E} &= 0, \\
\partial_t \vec{B} &= -\vec{\nabla} \times \vec{E}, \\
\nabla \cdot \vec{B} &= 0.
\end{align*}$$

These equations are often written in the more compact 4-dimensional notation: $\nabla^a F_{ab} = 0$ and $\nabla [a F_{bc}] = 0$, where $F_{ab}$ has components $\vec{E}$ and $\vec{B}$. 
Gauge and Constraints in Electromagnetism

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- Maxwell’s equations are often re-expressed in terms of a vector potential \(F_{ab} = \nabla_a A_b - \nabla_b A_a\):
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This form of Maxwell’s equations is manifestly hyperbolic as long as the gauge is chosen correctly, e.g., let \( \nabla^a A_a = H(x, t) \), giving:

\[ \nabla^a \nabla_a A_b \equiv (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2) A_b = \nabla_b H. \]
Constraint Damping

Where are the constraints: $\nabla^a \nabla_a A_b = \nabla_b H$?
Constraint Damping

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- Gauge condition becomes a constraint: $0 = C \equiv \nabla^a A_a - H$.
- Maxwell’s equations imply that this constraint is preserved:
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  \[
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  \]
- Modify evolution equations by adding multiples of the constraints:
  \[
  \nabla^a \nabla_a A_b = \nabla_b H + \gamma_0 t_b C = \nabla_b H + \gamma_0 t_b (\nabla^a A_a - H).
  \]
- These changes also affect the constraint evolution equation,
  \[
  \nabla^a \nabla_a C - \gamma_0 t^b \nabla_b C = 0,
  \]
  so constraint violations are damped when \( \gamma_0 > 0 \).
Constraint Damped Einstein System

- “Generalized Harmonic” form of Einstein’s equations have properties similar to Maxwell’s equations:
  - Gauge (coordinate) conditions are imposed by specifying the divergence of the spacetime metric: $\partial_\alpha g^{\alpha\beta} = H^\beta + \ldots$
  - Evolution equations become manifestly hyperbolic: $\Box g_{\alpha\beta} = \ldots$
  - Gauge conditions become constraints.
  - Constraint damping terms can be added which make numerical evolutions stable.

![Graph showing effect of constraint damping on ||C|| over time](image)
Numerical Solution of Evolution Equations

\[ \partial_t u = F(u, \partial_x u, x, t). \]

- Choose a grid of spatial points, \( x_n \).

\[ x_{n-1} \quad x_n \quad x_{n+1} \]
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\[
\begin{array}{ccc}
  u_{n-1} & u_n & u_{n+1} \\
  x_{n-1} & x_n & x_{n+1}
\end{array}
\]

Approximate the spatial derivatives at the grid points

\[
\partial_x u(x_n) = \sum_k D_{n,k} u_k.
\]

Evaluate \( F \) at the grid points \( x_n \) in terms of the \( u_k \): \( F(u_k, x_n, t) \).

Solve the coupled system of ordinary differential equations,

\[
\frac{du_n}{dt} = F[u_k(t), x_n, t],
\]

using standard numerical methods (e.g., Runge-Kutta).
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Basic Numerical Methods

- Different numerical methods use different ways of choosing the grid points $x_n$, and different expressions for the spatial derivatives

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Most numerical groups use finite difference methods:

- Uniformly spaced grids: $x_n - x_{n-1} = \Delta x = \text{constant}$.
- Use Taylor expansions,

\[
\begin{align*}
  u_{n-1} &= u(x_n - \Delta x) = u(x_n) - \partial_x u(x_n) \Delta x + \partial^2_x u(x_n) \Delta x^2 / 2 + \mathcal{O}(\Delta x^3), \\
  u_{n+1} &= u(x_n + \Delta x) = u(x_n) + \partial_x u(x_n) \Delta x + \partial^2_x u(x_n) \Delta x^2 / 2 + \mathcal{O}(\Delta x^3),
\end{align*}
\]

To obtain the needed expressions for $\partial_x u$:

$$\partial_x u(x_n) = \frac{u_{n+1} - u_{n-1}}{2\Delta x} + \mathcal{O}(\Delta x^2).$$
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    to obtain the needed expressions for \( \partial_x u \):
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- Grid spacing decreases as the number of grid points \( N \) increases, \( \Delta x \sim 1/N \). Errors in finite difference methods scale as \( N^{-p} \).
A few groups (Caltech/Cornell, Meudon) use spectral methods.

Represent functions as finite sums:

$$u(x,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) e^{ikx}.$$ 

Choose grid points $x_n$ to allow efficient (and exact) inversion of the series:

$$\tilde{u}_k(t) = \sum_{n=0}^{N-1} w_n u(x_n,t) e^{-ikx_n}.$$ 

Obtain derivative formulas by differentiating the series:

$$\partial_x u(x_n,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) \partial_x e^{ikx_n} = \sum_{m=0}^{N-1} D_{nm} u(x_m,t).$$ 

Errors in spectral methods are dominated by the size of $\tilde{u}_N$.

Estimate the errors (for Fourier series of smooth functions):

$$\tilde{u}_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-iNx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} du(x) e^{-iNx} = \frac{1}{2\pi} \int_{p}^{p} \int_{-\pi}^{\pi} d\rho u(x) dx e^{-iN \rho} \leq \frac{1}{N \rho_{\max}} \left\| \partial_\rho u(x) \right\|.$$ 

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Lee Lindblom (Caltech)
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- A few groups (Caltech/Cornell, Meudon) use spectral methods.
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Errors in spectral methods decrease faster than any power of \( N \).
Comparing Different Numerical Methods

- Wave propagation with second-order finite difference method:

Figures from Hesthaven, Gottlieb, & Gottlieb (2007).
Comparing Different Numerical Methods

- Wave propagation with second-order finite difference method:

- Wave propagation with spectral method:

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Moving Black Holes

- Black hole interior is not in causal contact with exterior. Interior is removed, introducing an excision boundary.
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![Diagram of black holes and horizons]
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Problems:
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Solution:
Choose coordinates that smoothly track the motions of the centers of the black holes.
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Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- A coordinate transformation from “inertial” coordinates, \((\bar{x}, \bar{y}, \bar{z})\), to “co-moving” coordinates \((x, y, z)\), consisting of a rotation followed by an expansion,

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
\end{pmatrix}
= e^{a(\bar{t})}
\begin{pmatrix}
  \cos \varphi(\bar{t}) & -\sin \varphi(\bar{t}) & 0 \\
  \sin \varphi(\bar{t}) & \cos \varphi(\bar{t}) & 0 \\
  0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
  \bar{x} \\
  \bar{y} \\
  \bar{z} \\
\end{pmatrix},
\]

is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions \(a(\bar{t})\) and \(\varphi(\bar{t})\).

- Since the motions of the holes are not known \textit{a priori}, the functions \(a(\bar{t})\) and \(\varphi(\bar{t})\) must be chosen dynamically and adaptively as the system evolves.
Measure the co-moving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently

$$Q^x(t) = \frac{x_c(t) - x_c(0)}{x_c(0)},$$

$$Q^y(t) = \frac{y_c(t)}{x_c(t)}.$$
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$$Q^y(t) = \frac{y_c(t)}{x_c(t)}.$$ 

Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.

Changing the map parameters by the small amounts, $\delta a$ and $\delta \varphi$, results in associated small changes in $\delta Q^x$ and $\delta Q^y$:

$$\delta Q^x = -\delta a,$$

$$\delta Q^y = -\delta \varphi.$$
Horizon Tracking Coordinates III

- Measure the quantities $Q^y(t)$, $dQ^y(t)/dt$, $d^2Q^y(t)/dt^2$, and set

$$
\frac{d^3\varphi}{dt^3} = \lambda^3 Q^y + 3\lambda^2 \frac{dQ^y}{dt} + 3\lambda \frac{d^2Q^y}{dt^2} = -\frac{d^3Q^y}{dt^3}.
$$

The solutions to this “closed-loop” equation for $Q^y$ have the form

$Q^y(t) = (At^2 + Bt + C)e^{-\lambda t}$, so $Q^y$ always decreases as $t \to \infty$. 

This works! We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.
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- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.
Caltech/Cornell Spectral Einstein Code (SpEC):

- Multi-domain spectral method.
- State of the art elliptic solver for computing BBH initial data, etc.
- Constraint damped “generalized harmonic” Einstein equations:
  \[ \square g_{ab} = F_{ab}(g, \partial g). \]
- Constraint-preserving, physical and gauge boundary conditions.
Evolving Binary Black Hole Spacetimes

- We can now evolve binary black hole spacetimes with excellent accuracy and computational efficiency through many orbits.

Head-on Merger Movie
Gravitational Waveforms

- High precision gravitational waveform for equal mass non-spinning BBH system.
Black Hole Recoil

- Asymmetric binaries (unequal masses and/or non-aligned spins) emit linear momentum into GW; final merged hole recoils with non-zero velocity.
- Maximum recoil velocity for non-spinning holes is 175 km/sec.
- Recoil velocities of 2000 km/sec have been measured in spinning black hole simulations (estimated maximum $\sim 4000$ km/sec).
- Large recoils are probably rare (Schnittman & Buonanno 2007).

![Graph showing recoil velocities as a function of mass ratio, with data points from Baker et al., Campanelli, Damour and Gopakumar, Herrmann et al., and Sopuerta et al.](graph1.png)

![Graph showing recoil velocities as a function of spin parameter, with data points from Gonzalez et al. and Campanelli et al.](graph2.png)
Spin Dynamics in Binary Black Hole Systems

- Merger is delayed in BBHs with spins aligned with the orbital angular momentum; merger happens more quickly in binaries with anti-aligned spins.
- Complicated spin dynamics are observed in BBH mergers with non-aligned spins.

Campanelli, et al. (2007)
Post-Newtonian Waveform Calibration

- Preliminary comparisons of the numerical gravitational wave phase with predictions of various post-Newtonian orders.

![Graph showing PN-phase comparison and numerical error budget (May 2007)](image-url)
Advances in understanding the Einstein equations provide new formulations suitable for numerical evolutions: hyperbolic formulations with constraint damping and well posed initial-boundary value problems.

High accuracy multi-orbit binary black hole simulations are now routine (but not yet cheap).

Numerical waveforms suitable for LIGO data analysis are starting to be generated.

Interesting non-linear dynamics of binary black hole mergers are beginning to be investigated.