Solving Einstein's Equation Numerically III

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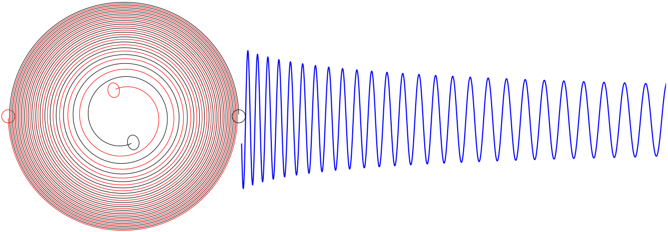
Mathematical Sciences Center Lecture Series
Tsinghua University – 21 November 2014



Gauge and Hyperbolicity in Electromagnetism

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$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0,$$
$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0.$$

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- This form of the equations can be made manifestly hyperbolic by choosing the gauge correctly, e.g., let $\nabla^a A_a = H(x, t, A)$, giving:

$$\nabla^a \nabla_a A_b = \left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right) A_b = \nabla_b H.$$

Gauge and Hyperbolicity in General Relativity

- The spacetime Ricci curvature tensor can be written as:

$$R_{ab} = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi, \partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi_{ad}\psi^{bc}\Gamma^d{}_{bc}$.

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- The gauge freedom in general relativity theory is the freedom to represent the equations using any coordinates x^a on spacetime.
- Solving the equations requires some specific choice of coordinates be made. Gauge conditions are used to impose the desired choice.
- One way to impose the needed gauge conditions is to specify H^a, the source term for a wave equation for each coordinate x^a:

$$H^a = \nabla^c\nabla_c x^a \;=\; \psi^{bc}(\partial_b\partial_c x^a - \Gamma^e{}_{bc}\partial_e x^a) = -\Gamma^a,$$

  where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

Gauge Conditions in General Relativity

- Specifying coordinates by the *generalized harmonic* (GH) method is accomplished by choosing a gauge-source function $H^a(x,\psi)$, e.g. $H^a = \psi^{ab}H_b(x)$, and requiring that

$$H^a(x,\psi) = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b\psi_{dc} + \partial_c\psi_{db} - \partial_d\psi_{bc}).$$

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$$R_{ab} = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi,\partial\psi).$$
- The Generalized Harmonic Einstein equation is obtained by replacing $\Gamma_a = \psi_{ab}\Gamma^b$ with $-H_a(x,\psi) = -\psi_{ab}H^b(x,\psi)$:
$$R_{ab} - \nabla_{(a}\left[\Gamma_{b)} + H_{b)}\right] = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(\psi,\partial\psi).$$

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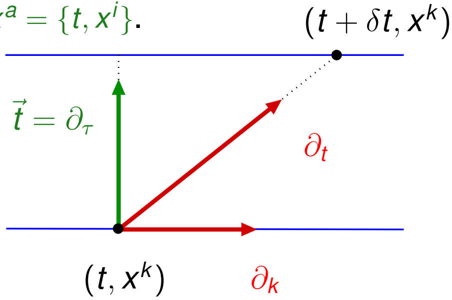
- The vacuum GH Einstein equation, $R_{ab} = 0$ with $\Gamma_a + H_a = 0$, is therefore manifestly hyperbolic, having the same principal part as the scalar wave equation:
$$0 = \nabla_a\nabla^a\psi = \psi^{ab}\partial_a\partial_b\psi + Q(\partial\psi).$$

ADM 3+1 Approach to Fixing Coordinates

- Coordinates must be chosen to label points in spacetime before the Einstein equations can be solved. For some purposes it is convenient to split the spacetime coordinates x^a into separate time and space components: $x^a = \{t, x^i\}$.
- Construct spacetime foliation by spacelike slices.
- Choose time function with t = const. on these slices.
- Choose spatial coordinates, x^k, on each slice.



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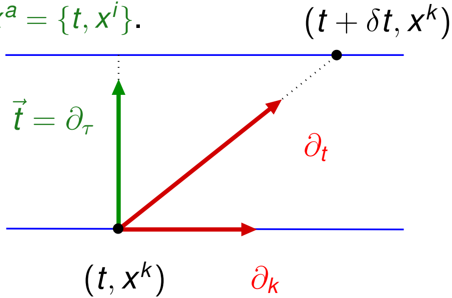

- Decompose the 4-metric ψ_{ab} into its 3+1 parts:

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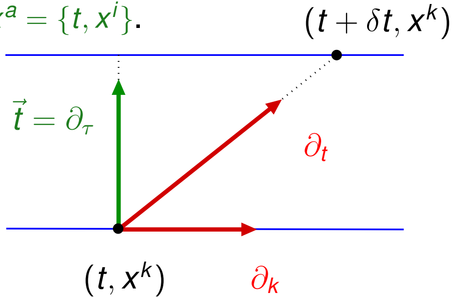

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- The unit vector t^a normal to the $t =$constant slices depends only on the lapse N and shift N^i: $\vec{t} = \partial_\tau = \frac{\partial x^a}{\partial \tau}\partial_a = \frac{1}{N}\partial_t - \frac{N^k}{N}\partial_k.$

ADM Approach to the Einstein Evolution System

- Decompose the Einstein equations $R_{ab} = 0$ using the ADM 3+1 coordinate splitting. The resulting system includes evolution equations for the spatial metric g_{ij} and extrinsic curvature K_{ij}:

$$\begin{aligned}
\partial_t g_{ij} - N^k\partial_k g_{ij} &= -2NK_{ij} + g_{jk}\partial_i N^k + g_{ik}\partial_j N^k, \\
\partial_t K_{ij} - N^k\partial_k K_{ij} &= NR^{(3)}_{ij} + K_{jk}\partial_i N^k + K_{ik}\partial_j N^k \\
&\quad -\nabla_i\nabla_j N - 2NK_{ik}K^k{}_j + NK^k{}_k K_{ij}.
\end{aligned}$$

- The resulting system also includes constraints:

$$\begin{aligned}
0 &= R^{(3)} - K_{ij}K^{ij} + (K^k{}_k)^2, \\
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- System includes no evolution equations for lapse N or shift N^i. These quanties can be specified freely to fix the gauge.
- Resolving the issues of hyperbolicity (i.e. well posedness of the initial value problem) and constraint stability are much more complicated in this approach. The most successful version is the BSSN evolution system used by many (most) codes.

Dynamical GH Gauge Conditions

- The spacetime coordinates x^b are fixed in the generalized harmonic Einstein equations by specifying H^b:

$$\nabla^a \nabla_a x^b \equiv H^b.$$

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- The simplest choice $H^b = 0$ (harmonic gauge) fails for very dynamical spacetimes, like binary black hole mergers.
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- This failure seems to occur because the coordinates themselves become very dynamical solutions of the wave equation $\nabla^a \nabla_a x^b = 0$ in these situations.
- Another simple choice – keeping H^b fixed in the co-moving frame of the black holes – also works well during the long inspiral phase, but fails when the black holes begin to merge.

Dynamical GH Gauge Conditions II

- Some of the extraneous gauge dynamics could be removed by adding a damping term to the harmonic gauge condition:

$$\nabla^a\nabla_a x^b = H^b = \mu t^a\partial_a x^b = \mu t^b = -\mu N\psi^{tb}.$$

- This works well for the spatial coordinates x^i, driving them toward solutions of the spatial Laplace equation on the timescale $1/\mu$.

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- For the time coordinate t, this damped wave condition drives t to a time independent constant, which is not a good coordinate.
- A better choice sets $t^a H_a = -\mu \log\sqrt{g/N^2}$. The gauge condition in this case becomes

$$t^a\partial_a \log\sqrt{g/N^2} = -\mu\log\sqrt{g/N^2} + N^{-1}\partial_k N^k$$

This coordinate condition keeps g/N^2 close to unity, even during binary black hole mergers (where it became of order 100 using simpler gauge conditions).

The Constraint Problem

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- In a well-posed representation, the constraints, $\mathcal{C} = 0$, remain satisfied for all time if they are satisfied initially.

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- Constraint violating instabilities were one of the major problems that made progress on solving the binary black hole problem so slow.
- Special representations of the Einstein equations are needed that control the growth of any constraint violations.

Constraint Damping in Electromagnetism

- Electromagnetism is described by the hyperbolic evolution equation $\nabla^a\nabla_a A_b = \nabla_b H$. Are there any constraints? Where have the usual $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ constraints gone?

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- Gauge condition becomes a constraint: $0 = \mathcal{C} \equiv \nabla^b A_b - H$.
- Maxwell's equations imply that this constraint is preserved:

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- Modify evolution equations by adding multiples of the constraints:

$$\nabla^a\nabla_a A_b = \nabla_b H + \gamma_0 t_b\,\mathcal{C} = \nabla_b H + \gamma_0 t_b\left(\nabla^a A_a - H\right).$$

- These changes effect the constraint evolution equation,

$$\nabla^a\nabla_a\mathcal{C} - \gamma_0 t^b\nabla_b\mathcal{C} = 0,$$

  so constraint violations are damped when $\gamma_0 > 0$.

Constraints in the GH Evolution System

- The GH evolution system has the form,

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

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  where $\mathcal{C}_a = H_a + \Gamma_a$ plays the role of a constraint. Without constraint damping, these equations are very unstable to constraint violating instabilities.
- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv \left[R_{ab} - \tfrac{1}{2}\psi_{ab}R\right]t^b = \left[\nabla_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\right]t^b.$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 \;=\; R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[t_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\, t^c\,\mathcal{C}_c\right],$$

where t^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

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- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[t_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \tfrac{1}{2}\gamma_0\; t_a\mathcal{C}^c\mathcal{C}_c.$$

  This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

Summary of the GH Einstein System

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$$H^a(x,\psi) = \nabla^c\nabla_c x^a = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b\psi_{dc} + \partial_c\psi_{db} - \partial_d\psi_{bc}).$$

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- Principal part of evolution system becomes manifestly hyperbolic:

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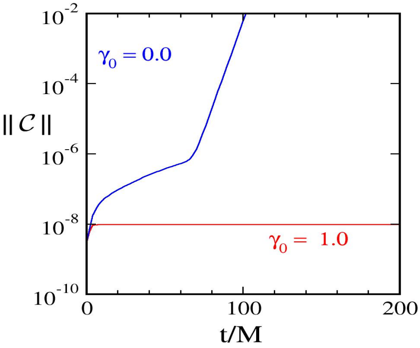
- Add constraint damping terms for stability:

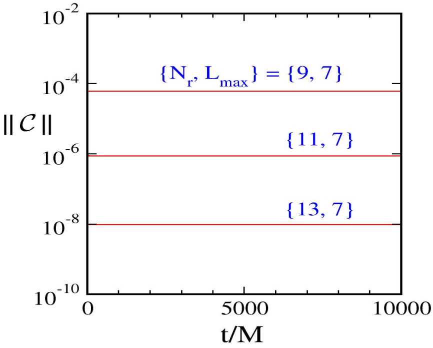
$$0 \;=\; R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[t_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\,t^c\,\mathcal{C}_c\right],$$

where t^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

Numerical Tests of the GH Evolution System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = 1$.





- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial analytical values.

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- We have argued that Einstein's equation is "manifestly hyperbolic" because its principal part is the same as the scalar wave equation.
- Exactly what does this mean? Does this make sense?
- From a pragmatic physicist's point of view, hyperbolic means anything that acts like the wave equation, i.e. any system of equations having a well posed initial-boundary value problem.
- Symmetric hyperbolic systems are one class of equations for which suitable well-posedness theorems exist, and which are general enough to include Einstein's equations together with most of the other dynamical field equations used by physicists.

What Do We Mean By Hyperbolic?

- We have argued that Einstein's equation is "manifestly hyperbolic" because its principal part is the same as the scalar wave equation.
- Exactly what does this mean? Does this make sense?
- From a pragmatic physicist's point of view, hyperbolic means anything that acts like the wave equation, i.e. any system of equations having a well posed initial-boundary value problem.
- Symmetric hyperbolic systems are one class of equations for which suitable well-posedness theorems exist, and which are general enough to include Einstein's equations together with most of the other dynamical field equations used by physicists.
- Evolution equations of the form,
$$\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u,x,t)\partial_k u^\beta = F^\alpha(u,x,t),$$
for a collection of dynamical fields u^α, are called symmetric hyperbolic if there exists a positive definite $S_{\alpha\beta}$ having the property that $S_{\alpha\gamma}A^{k\,\gamma}{}_\beta \equiv A^k_{\alpha\beta} = A^k_{\beta\alpha}$.

Example: Scalar Wave Equation

- Consider the scalar wave equation in flat space, expressed in terms of arbitrary spatial coordinates:
$$0 = -\partial_t^2\psi + \nabla^k\nabla_k\psi = -\partial_t^2\psi + g^{k\ell}(\partial_k\partial_\ell\psi - \Gamma^n_{k\ell}\partial_n\psi).$$

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- Define the first-order dynamical fields, $u^\alpha = \{\psi, \Pi, \Phi_k\}$, which satisfy the following evolution equations:
$$\partial_t\psi = -\Pi, \qquad \partial_t\Pi + \nabla^k\Phi_k = 0, \qquad \partial_t\Phi_k + \nabla_k\Pi = 0.$$

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- The principal part of this system, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta\partial_k u^\beta \simeq 0$, is given:

$$\partial_t \begin{pmatrix} \psi \\ \Pi \\ \Phi_x \\ \Phi_y \\ \Phi_z \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g^{xx} & g^{xy} & g^{xz} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \psi \\ \Pi \\ \Phi_x \\ \Phi_y \\ \Phi_z \end{pmatrix} + \ldots \simeq 0$$

Example: Scalar Wave Equation II

- The symmetrizer for the first-order scalar field system is:

$$dS^2 = S_{\alpha\beta}du^\alpha du^\beta = \Lambda^2 d\psi^2 + d\Pi^2 + g^{mn}d\Phi_m d\Phi_n.$$

Example: Scalar Wave Equation II

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- Check the symmetrization of the characteristic matrices:

$$S_{\alpha\gamma}A^{x\,\gamma}{}_{\beta} = \begin{pmatrix} \Lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & g^{xx} & g^{xy} & g^{xz} \\ 0 & 0 & g^{yx} & g^{yy} & g^{yz} \\ 0 & 0 & g^{zx} & g^{zy} & g^{zz} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g^{xx} & g^{xy} & g^{xz} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g^{xx} & g^{xy} & g^{xz} \\ 0 & g^{xx} & 0 & 0 & 0 \\ 0 & g^{yx} & 0 & 0 & 0 \\ 0 & g^{zx} & 0 & 0 & 0 \end{pmatrix} = A^x_{\alpha\beta}$$