Solving Einstein’s Equation Numerically V

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Characteristic Fields for the Einstein System

The characteristic fields $u^\hat{\alpha} = e^\hat{\alpha}_\beta u^\beta$ for the generalized harmonic version of the Einstein evolution equations look very much like their scalar field counterparts: $u^\hat{\alpha} = \{u^\hat{0}_{ab}, u^\hat{1}_{\pm ab}, u^\hat{2}_{iab}\}$, given by

\[
\begin{align*}
u^\hat{0}_{ab} & = \psi_{ab}, \\
u^\hat{1}_{\pm ab} & = \Pi_{ab} \pm n^k \Phi_{kab} - \gamma 2 \psi_{ab}, \\
u^\hat{2}_{iab} & = (\delta^k_i - n^k n_i) \Phi_{kab},
\end{align*}
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  \]

- The coordinate characteristic speeds associated with these fields also have the same forms as those for the scalar field system:
  \[
  v(\hat{0}) = -(1 + \gamma_1)n_k N^k \quad \text{for the fields } u^\hat{0}_{ab},
  \]
  \[
  v(\hat{1}_{\pm}) = -n^k N_k \pm N \quad \text{for the fields } u^\hat{1}_{\pm ab}, \quad \text{and}
  \]
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  v(\hat{2}) = -n_k N^k \quad \text{for the fields } u^\hat{2}_{iab}.\]
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A boundary condition must be imposed on each characteristic field whose characteristic speed is negative on that boundary.

A boundary condition may not be imposed on any characteristic field whose characteristic speed is non-negative on that boundary.
Construct the characteristic fields, $\hat{c}^A = e^A_A c^A$, associated with the constraint evolution system, 

$$\partial_t c^A + A^k_A B \partial_k c^B = F^A_B c^B.$$
Constraint Preserving Boundary Conditions

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- The constraints depend on the primary evolution fields (and their derivatives). We find that $\hat{c}^-$ for the GH system can be expressed:

$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$
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\[
\hat{c}^- = d_{\bot} \hat{u}^- + \hat{F}(u, d_{\parallel} u).
\]

Set boundary conditions on the fields \( \hat{u}^- \) by requiring

\[
d_{\bot} \hat{u}^- = -\hat{F}(u, d_{\parallel} u).
\]
Constraint Characteristic Fields

The characteristic fields associated with the constraint evolution system, and their associated characteristic speeds for the first-order Einstein system are:

\[ c_0^\pm = \mathcal{F}_a \mp n^k c_{ka} \approx t^c \partial_c c_a \mp n^k \partial_k c_a, \quad v_{(\pm)} = -n_k N^k \pm N, \]

\[ c_1 = c_a, \quad v_{(1)} = 0, \]

\[ c_i^2 = P^k i c_{ka} \approx (\delta^k_i - n^k n_i) \partial_k c_a, \quad v_{(2)} = -n_k N^k, \]

\[ c_i^3 = c_{iab}, \quad v_{(3)} = -(1 + \gamma_1) n_k N^k, \]

\[ c_i^4 = c_{ijab} = 2 \partial_{[j} c_{i]ab}, \quad v_{(4)} = -n_k N^k. \]
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\]

\[
c_{\hat{1}} = C_a, \quad v_{(\hat{1})} = 0,
\]

\[
c_{\hat{2}} = P^k_i C_{ka} \approx (\delta^k_i - n^k n_i) \partial_k C_a, \quad v_{(\hat{2})} = -n_k N^k,
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The constraint characteristic fields \(c_{\hat{0}^-}, c_{\hat{3}}\) and \(c_{\hat{4}}\) have the same characteristic speeds as the principal dynamical fields \(u_{\hat{1}^-}, u_{\hat{0}}\) and \(u_{\hat{2}}\) respectively. These constraint fields will be incoming under the same conditions as these dynamical fields.
Fortunately, the incoming constraint characteristic fields, $c_{a}^{0-}$, $c_{iab}^{3}$, and $c_{ikab}^{4}$, can be expressed in terms of the corresponding principal dynamical characteristic fields:

\[
\begin{align*}
    c_{a}^{0-} & \approx \sqrt{2} \left[ (c_{\psi}^{d})_{a} - \frac{1}{2} k_{a} \psi^{cd} \right] d_{\perp} u_{cd}^{1-}, \\
    n^{k} c_{kab}^{3} & \approx d_{\perp} u_{ab}^{0}, \\
    n^{k} c_{kiab}^{4} & \approx d_{\perp} u_{iab}^{2},
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where $k^{a} = (t^{a} - n^{a})/\sqrt{2}$ is the ingoing null vector.
Fortunately, the incoming constraint characteristic fields, $c_{a}^{0-}$, $c_{iab}^{3}$ and $c_{ikab}^{4}$, can be expressed in terms of the corresponding principal dynamical characteristic fields:

$$c_{a}^{0-} \approx \sqrt{2} \left[ k^{(c_{d} \psi)} a - \frac{1}{2} k_{a} \psi^{cd} \right] d_{\perp} u_{cd}^{1-},$$

$$n^{k} c_{kab}^{3} \approx d_{\perp} u_{ab}^{0},$$

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Setting these incoming characteristic constraint fields to zero therefore provides boundary conditions on the normal derivatives $d_{\perp} u^{\hat{\alpha}} = e^{\hat{\alpha} \beta} n^{k} \partial_{k} u^{\beta}$ of some of the primary dynamical characteristic fields.
Physical Boundary Conditions

- The Weyl curvature tensor $C_{abcd}$ satisfies a system of evolution equations from the Bianchi identities: $\nabla[a C_{bc}]de = 0$.
- The characteristic fields of this system corresponding to physical gravitational waves are the quantities:
  \[
  \hat{w}^\pm_{ab} = (P^c_a P^d_b - \frac{1}{2} P_{ab} P^{cd}) (t^e \mp n^e)(t^f \mp n^f) C_{cedf},
  \]

  where $t^a$ is a unit timelike vector, $n^a$ a unit spacelike vector (with $t^a n_a = 0$), and $P_{ab} = \psi_{ab} + t_a t_b - n_a n_b$. 

  The incoming field $\hat{w}^-_{ab}$ can be expressed in terms of the characteristic fields of the primary evolution system:
  \[
  \hat{w}^-_{ab} = d_\perp u \hat{w}^1_{ab} + \hat{F}_{ab}(u, d_\parallel u).
  \]

  We impose boundary conditions on the physical gravitational wave degrees of freedom then by setting:
  \[
  d_\perp u \hat{w}^1_{ab} = -\hat{F}_{ab}(u, d_\parallel u) + \hat{w}^-_{ab}|_{t = 0}.
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Imposing Neumann-like Boundary Conditions

Consider Neumann-like boundary conditions of the form
\[ e^{\hat{\alpha} \beta} n^k \partial_k u^\beta \equiv d_\bot u^{\hat{\alpha}} = d_\bot u^{\hat{\alpha}}|_{BC}. \]
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- The characteristic field projections of the evolution equations are:
  \[ d_t u^{\hat{\alpha}} \equiv e^{\hat{\alpha} \beta} \partial_t u^\beta = e^{\hat{\alpha} \beta} (-A^{k \gamma} \partial_k u^\gamma + F^\beta) \equiv D_t u^{\hat{\alpha}}. \]
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The spatial derivatives of \( u^\gamma \) in this expression can be re-written:
\[ e^{\hat{\alpha} \beta} A^{k \beta \gamma} \partial_k u^\gamma = v(\hat{\alpha}) e^{\hat{\alpha} \gamma} n^k \partial_k u^\gamma + e^{\hat{\alpha} \beta} A^{\ell \beta \gamma} (\delta^k_\ell - n^k n_\ell) \partial_k u^\gamma. \]
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\[ e^{\hat{\alpha}}_{\beta} A^{k\beta}_{\gamma} \partial_k u^{\gamma} = v(\hat{\alpha}) e^{\hat{\alpha}}_{\gamma} n^k \partial_k u^{\gamma} + e^{\hat{\alpha}}_{\beta} A^{\ell\beta}_{\gamma} (\delta^k_{\ell} - n^k n_{\ell}) \partial_k u^{\gamma}. \]

We impose these Neumann-like boundary conditions by changing the appropriate components of the evolution equations at the boundary to:
\[ d_t u^{\hat{\alpha}} = D_t u^{\hat{\alpha}} + v(\hat{\alpha}) (d_\perp u^{\hat{\alpha}} - d_\perp u^{\hat{\alpha}}|_{BC}). \]
Gauge Boundary Conditions

- Constraint preserving and physical boundary conditions discussed above place conditions on some (but not all) of the components of the incoming characteristic field $u_{ab}^\hat{1}$:

Constraint preserving boundary conditions place conditions on

$$P_{ab}^C d_\perp u_{cd}^\hat{1} \equiv \left( \frac{1}{2} P_{ab} P^{cd} - 2l_{(a} P_{b)}^{(c} k^{d)} + l_{a} l_{b} k^{c} k^{d} \right) d_\perp u_{cd}^\hat{1}.$$

Physical boundary conditions place conditions on

$$P_{ab}^P d_\perp u_{cd}^\hat{1} \equiv \left( P_{a}^{c} P_{b}^{d} - \frac{1}{2} P_{ab} P^{cd} \right) d_\perp u_{cd}^\hat{1}.$$
Gauge Boundary Conditions

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$$P^C_{ab} d_{\perp} u^{\hat{1}}_{cd} \equiv \left( \frac{1}{2} P_{ab} P^{cd} - 2 I_{(aP_{b)}^{(ckd)}} + I_{aI_{b}^{ckd}} \right) d_{\perp} u^{\hat{1}}_{cd}.$$ 

  - Physical boundary conditions place conditions on

$$P^P_{ab} d_{\perp} u^{\hat{1}}_{cd} \equiv \left( P^c_{ab} P^{d} - \frac{1}{2} P_{ab} P^{cd} \right) d_{\perp} u^{\hat{1}}_{cd}.$$ 

- Additional “gauge” boundary conditions are needed for the remaining components of $u^{\hat{1}}_{ab}$:

$$P^G_{ab} u^{\hat{1}}_{cd} \equiv \left( \delta_{a}^{c} \delta_{b}^{d} - P^C_{ab} - P^P_{ab} \right) u^{\hat{1}}_{cd}.$$
Gauge Boundary Conditions II

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- We first tried just freezing these boundary degrees of freedom:

\[ P^{G cd}_{ab} d_t u^i_{cd} = 0. \]

These conditions are mathematically well posed, but they tend to generate a lot of boundary reflections of the gauge degrees of freedom. These conditions do not produce solutions that converge rapidly with increasing numerical resolution.
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- We first tried just freezing these boundary degrees of freedom:

\[ P_{ab}^{Gcd} d_t \hat{u}_{cd}^1 = 0. \]

These conditions are mathematically well posed, but they tend to generate a lot of boundary reflections of the gauge degrees of freedom. These conditions do not produce solutions that converge rapidly with increasing numerical resolution.

- Better, less reflective, gauge boundary conditions can be obtained by imposing what amount to Sommerfeld conditions on these gauge degrees of freedom:

\[ P_{ab}^{Gcd} d_t \left[ u_{cd}^1 + (\gamma_2 - r^{-1}) u_{cd}^0 \right] = 0. \]
The combined constraint preserving boundary conditions, plus the simple “no incoming $\Psi_0$” physical gravitational wave boundary conditions for the first-order generalized harmonic Einstein evolution system are given by:

\[
\begin{align*}
  d_t u_{ab}^0 &= D_t u_{ab}^0 - (1 + \gamma_1) n_j N^j n^k c_{kab}^3, \\
  d_t u_{ab}^1 &= P_{ab}^{P cd} \left[ D_t u_{cd}^1 - (N + n_j N^j)(\hat{w}_{cd}^0 - \gamma_2 n^i c_{icd}^3) \right] \\
  &- (\gamma_2 - r^{-1}) P_{ab}^G cd d_t u_{cd}^0 + P_{ab}^C cd D_t u_{cd}^1 \\
  &+ \sqrt{2}(N + n_j N^j) \left[ l_a P_b^c - \frac{1}{2} P_{ab} l^c - \frac{1}{2} l_a l_b k^c \right] c_{c}^0, \\

  d_t u_{kab}^2 &= D_t u_{kab}^2 - n_l N^l n^i P_{ij k}^l c_{ijab}^4.
\end{align*}
\]

The quantity $P_{ab}$ in these expressions is the projection tensor, $P_{ab} = \psi_{ab} + t_a t_b - n_a n_b$, the incoming null vector $k^a$ is defined by $k^a = (t^a - n^a)/\sqrt{2}$, and the outgoing null vector $l^a$ is defined by $l^a = (t^a + n^a)/\sqrt{2}$. 
Tests of Constraint Preserving and Physical BC

- Evolve the perturbed black-hole spacetime using the resulting constraint preserving boundary conditions for the generalized harmonic evolution systems.
  - $\psi_4$ Freezing BC  $||C||$ Freezing BC  $\psi_4$ Better BC

Evolutions using new BC are stable and convergent.

The Weyl curvature $\Psi_4$ shows quasi-normal mode oscillations when new BC are used.
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- Evolutions using new BC are stable and convergent.
- The Weyl curvature \( \psi_4 \) shows quasi-normal mode oscillations when new BC are used.
Evolve a perturbed black-hole spacetime using the resulting constraint preserving boundary conditions for the generalized harmonic evolution systems. Compare results of evolutions on a small domain with $R = 41.9M$ with results from a large domain with $R = 961.9M$. Compare results using constraint preserving BC (dotted lines), with other possible outer boundary treatments (solid lines), see Rinne, et. al (2007) for details.
Spectral: Excision boundary is a smooth analytic surface.

*Problems:*
- Re-gridding/interpolation is expensive.
- Difficult to get smooth extrapolation at trailing edge of horizon.
- Causality trouble at leading edge of horizon.

*Solution:*
Choose coordinates that smoothly track the location of the black hole.
For a black hole binary this means using coordinates that rotate with respect to inertial frames at infinity.
Moving Black Holes Using Spectral Methods

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Evolving Black Holes in Rotating Frames

- Coordinates that rotate with respect to the inertial frames at infinity are needed to track the horizons of orbiting black holes.
- Evolutions of Schwarzschild in rotating coordinates are unstable.

Evolutions shown use a computational domain that extends to $r = 1000M$.

Angular velocity needed to track the horizons of an equal mass binary at merger is about $\Omega \approx 0.2/M$.

Problem caused by asymptotic behavior of metric in rotating coordinates: $\psi_{tt} \sim \varpi^2\Omega^2$, $\psi_{ti} \sim \varpi\Omega$, $\psi_{ij} \sim 1$. 
Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, \( \bar{x}^a = \{ \bar{t}, \bar{x}^i \} \), to define field components, \( \bar{u}^\alpha = \{ \bar{\psi}^a_b, \bar{\Pi}^a_b, \Phi_{\bar{i}a\bar{b}} \} \), and the same coordinates to determine these components by solving Einstein’s equation for \( \bar{u}^\alpha = \bar{u}^\alpha(\bar{x}^\bar{a}) \):

\[
\partial_{\bar{t}} \bar{u}^\alpha + \bar{A}^k_{\bar{a}\bar{b}} \partial_{\bar{k}} \bar{u}^\beta = F^\alpha.
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\[
\partial_{\bar{t}} u^{\bar{a}} + A^{\bar{k}}_{\bar{a} \beta} \partial_{\bar{k}} u^{\bar{\beta}} = F^{\bar{a}}.
\]

- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, \( x^{a} = \{ t, x^{i} \} = x^{a}(x^{\bar{a}}) \), to represent these components as functions, \( u^{\bar{a}} = u^{\bar{a}}(x^{a}) \).
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$$
\partial_t u^\alpha + A^k_{\bar{\alpha}\bar{\beta}} \partial_k u^\beta = F^\alpha.
$$

- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^\bar{a})$, to represent these components as functions, $u^\alpha = u^\alpha(x^a)$.

- These functions are determined by solving the transformed Einstein equation:

$$
\partial_t u^\bar{\alpha} + \left[ \frac{\partial x^i}{\partial t} \delta^\bar{\alpha}_{\bar{\beta}} + \frac{\partial x^i}{\partial x^k} A^k_{\bar{\alpha}\bar{\beta}} \right] \partial_i u^\beta = F^\bar{\alpha}.
$$
Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:

Dual Frame Evolution: $\Omega = 0.2/\mu$

Single Frame Evolution: $\Omega = 0.02/\mu$, $\Omega = 0.002/\mu$, $\Omega = 0.0002/\mu$

- Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/\mu$ on a domain with outer radius $r = 1000M$. 
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- The coordinate transformation from inertial coordinates, \((\bar{x}, \bar{y}, \bar{z})\), to co-moving coordinates \((x, y, z)\),

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = e^{a(\bar{t})}
\begin{pmatrix}
    \cos \varphi(\bar{t}) & -\sin \varphi(\bar{t}) & 0 \\
    \sin \varphi(\bar{t}) & \cos \varphi(\bar{t}) & 0 \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    \bar{x} \\
    \bar{y} \\
    \bar{z}
\end{pmatrix},
\]

with \(t = \bar{t}\), is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions \(a(\bar{t})\) and \(\varphi(\bar{t})\).

- Since the motions of the holes are not known \textit{a priori}, the functions \(a(\bar{t})\) and \(\varphi(\bar{t})\) must be chosen dynamically and adaptively as the system evolves.
Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$. Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.
Measure the comoving centers of the holes: \(x_c(t)\) and \(y_c(t)\), or equivalently \(Q^x(t) = \frac{x_c(t) - x_c(0)}{x_c(0)}\) and \(Q^y(t) = \frac{y_c(t)}{x_c(t)}\).

Choose the map parameters \(a(t)\) and \(\varphi(t)\) to keep \(Q^x(t)\) and \(Q^y(t)\) small.

Changing the map parameters by the small amounts, \(\delta a\) and \(\delta \varphi\), results in associated small changes in \(\delta Q^x\) and \(\delta Q^y\):

\[
\delta Q^x = -\delta a, \quad \delta Q^y = -\delta \varphi.
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Measure the comoving centers of the holes: \( x_c(t) \) and \( y_c(t) \), or equivalently \( Q^x(t) = [x_c(t) - x_c(0)]/x_c(0) \) and \( Q^y(t) = y_c(t)/x_c(t) \).

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\[
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\]

Measure the quantities \( Q^y(t) \), \( dQ^y(t)/dt \), \( d^2 Q^y(t)/dt^2 \), and set

\[
\frac{d^3 \varphi}{dt^3} = \lambda^3 Q^y + 3\lambda^2 \frac{dQ^y}{dt} + 3\lambda \frac{d^2 Q^y}{dt^2} = -\frac{d^3 Q^y}{dt^3}.
\]

The solutions to this “closed-loop” equation for \( Q^y \) have the form \( Q^y(t) = (At^2 + Bt + C)e^{-\lambda t} \), so \( Q^y \) always decreases as \( t \to \infty \).
Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times $t = t_i$.
- In the time interval $t_i < t < t_{i+1}$ we set:

$$\varphi(t) = \varphi_i + (t - t_i) \frac{d\varphi_i}{dt} + \frac{(t - t_i)^2}{2} \frac{d^2\varphi_i}{dt^2} + \frac{(t - t_i)^3}{2} \left( \lambda \frac{d^2Q_i^y}{dt^2} + \lambda^2 \frac{dQ_i^y}{dt} + \lambda^3 \frac{Q_i^y}{3} \right),$$

where $Q^x$, $Q^y$, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop equation at $t = t_i$. 

This works! We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.
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- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- This can be implemented by using a coordinate transformation from inertial coordinates, $\bar{x}^i$, to co-moving coordinates $x^i$, consisting of a translation followed by a rotation followed by an expansion:

$$x^i = e^{a(\bar{t})} R^{(z)}_{ij}[\varphi(\bar{t})] R^{(y)}_{jk}[\xi(\bar{t})] \left[ \bar{x}^k - c^k(\bar{t}) \right],$$

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- This transformation keeps the holes fixed in co-moving coordinates for suitably chosen $a(\bar{t}), \varphi(\bar{t}), \xi(\bar{t})$, and $c^k(\bar{t})$.
- Motions of the holes are not known \textit{a priori}, so $a(\bar{t}), \varphi(\bar{t}), \xi(\bar{t})$, and $c^k(\bar{t})$ must be chosen dynamically and adaptively.
- A simple feedback-control system has been used to choose $a(\bar{t}), \varphi(\bar{t}), \xi(\bar{t})$, and $c^k(\bar{t})$ by fixing the black-hole positions, even in evolutions with precession and “kicks”.

Horizon Distortion Maps

- Tidal deformation, along with kinematic and gauge effects cause the shapes of the black holes to deform:

![Diagram showing distorted horizon shapes compared to a spherical excision surface.](image)
Horizon Distortion Maps

- Tidal deformation, along with kinematic and gauge effects cause the shapes of the black holes to deform:

- If the holes become significantly distorted – relative to the spherical excision surface – bad things happen:
  - Some points on the excision boundary are much deeper inside the singular black hole interior. Numerical errors and constraint violations are largest there, sometimes leading to instabilities.
Tidal deformation, along with kinematic and gauge effects cause the shapes of the black holes to deform:

If the holes become significantly distorted – relative to the spherical excision surface – bad things happen:

- Some points on the excision boundary are much deeper inside the singular black hole interior. Numerical errors and constraint violations are largest there, sometimes leading to instabilities.
- When the horizons move relative to the excision boundary points, the excision boundary can become timelike, and boundary conditions are then needed there.
Horizon Distortion Maps II

- Adjust the placement of grid points near each black hole using a horizon distortion map that connects grid coordinates $x^i$ to points in the black-hole rest frame $\tilde{x}^i$:

  \[
  \tilde{\theta}_A = \theta_A, \quad \tilde{\varphi}_A = \varphi_A, \\
  \tilde{r}_A = r_A - f_A(r_A, \theta_A, \varphi_A) \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \lambda_A^{\ell m}(t) Y_{\ell m}(\theta_A, \varphi_A).
  \]

Choose $f_A$ to scale linearly from $f_A = 1$ on the excision boundary, to $f_A = 0$ on cut sphere. Adjust the coefficients $\lambda_A^{\ell m}(t)$ using a feedback-control system to keep the excision surface the same shape and slightly smaller than the horizon, and to keep the boundary spacelike.
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Caltech/Cornell Spectral Einstein Code (SpEC):

- Multi-domain pseudo-spectral evolution code.

Lovelace, Scheel, & Szilágyi (2010) high spin evolution grids.

- Constraint damped “generalized harmonic” Einstein equations:
  \[ \psi^{cd} \partial_c \partial_d \psi_{ab} = Q_{ab}(\psi, \partial \psi). \]

- Dual frame evolutions with horizon tracking and distortion maps.
- Constraint-preserving, physical and gauge boundary conditions.
- Spectral AMR.