Solving Einstein's Equation Numerically V

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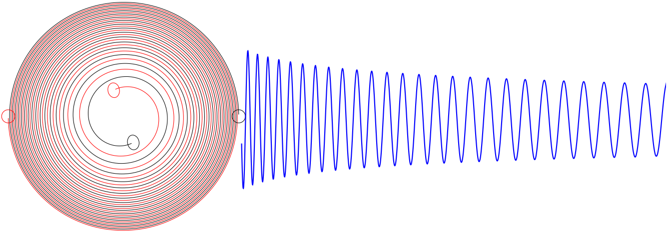
Mathematical Sciences Center Lecture Series
Tsinghua University – 5 December 2014



Characteristic Fields for the Einstein System

- The characteristic fields $u^{\hat\alpha} = e^{\hat\alpha}_{\beta} u^\beta$ for the generalized harmonic version of the Einstein evolution equations look very much like their scalar field counterparts: $u^{\hat\alpha} = \{u^{\hat 0}_{ab}, u^{\hat 1 \pm}_{ab}, u^{\hat 2}_{iab}\}$, given by

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u^{\hat 0}_{ab} &= \psi_{ab}, \\
u^{\hat 1 \pm}_{ab} &= \Pi_{ab} \pm n^k \Phi_{kab} - \gamma_2 \psi_{ab}, \\
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- The coordinate characteristic speeds associated with these fields also have the same forms as those for the scalar field system: $v_{(\hat 0)} = -(1+\gamma_1) n_k N^k$ for the fields $u^{\hat 0}_{ab}$, $v_{(\hat 1\pm)} = -n^k N_k \pm N$ for the fields $u^{\hat 1\pm}_{ab}$, and $v_{(\hat 2)} = -n_k N^k$ for the fields $u^{\hat 2}_{iab}$.

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- A boundary condition must be imposed on each characteristic field whose characteristic speed is negative on that boundary.
- A boundary condition may not be imposed on any characteristic field whose characteristic speed is non-negative on that boundary.

Constraint Preserving Boundary Conditions

- Construct the characteristic fields, $\hat{c}^{\hat{A}} = e^{\hat{A}}{}_{A} c^{A}$, associated with the constraint evolution system, $\partial_t c^A + A^{k\,A}{}_B \partial_k c^B = F^A{}_B c^B$.

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$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$

- Set boundary conditions on the fields $\hat{u}^-$ by requiring

$$d_\perp \hat{u}^- = -\hat{F}(u, d_\parallel u).$$

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- The characteristic fields associated with the constraint evolution system, and their associated characteristic speeds for the first-order Einstein system are:

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c_a^{\hat{0}\pm} &= \mathcal{F}_a \mp n^k \mathcal{C}_{ka} \approx t^c \partial_c \mathcal{C}_a \mp n^k \partial_k \mathcal{C}_a, & v_{(\hat{0}\pm)} &= -n_k N^k \pm N,\\
c_a^{\hat{1}} &= \mathcal{C}_a, & v_{(\hat{1})} &= 0,\\
c_{ia}^{\hat{2}} &= P^k{}_i \mathcal{C}_{ka} \approx (\delta^k{}_i - n^k n_i)\partial_k \mathcal{C}_a, & v_{(\hat{2})} &= -n_k N^k,\\
c_{iab}^{\hat{3}} &= \mathcal{C}_{iab}, & v_{(\hat{3})} &= -(1+\gamma_1) n_k N^k,\\
c_{ijab}^{\hat{4}} &= \mathcal{C}_{ijab} = 2\partial_{[j}\mathcal{C}_{i]ab}, & v_{(\hat{4})} &= -n_k N^k.
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- The constraint characteristic fields $c_a^{\hat{0}-}$, $c_{iab}^{\hat{3}}$ and $c_{ijab}^{\hat{4}}$ have the same characteristic speeds as the principal dynamical fields $u_{ab}^{\hat{1}-}$, $u_{ab}^{\hat{0}}$ and $u_{iab}^{\hat{2}}$ respectively. These constraint fields will be incoming under the same conditions as these dynamical fiels.

Constraint Characteristic Fields II

- Fortunately, the incoming constraint characteristic fields, $c^{\hat{0}-}_a$, $c^{\hat{3}}_{iab}$ and $c^{\hat{4}}_{ikab}$, can be expressed in terms of the corresponding principal dynamical characteristic fields:

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\begin{aligned}
c^{\hat{0}-}_a &\approx \sqrt{2}\left[k^{(c}\psi^{d)}{}_a - \tfrac{1}{2}k_a\psi^{cd}\right] d_\perp u^{\hat{1}-}_{cd}\,,\\
n^k c^{\hat{3}}_{kab} &\approx d_\perp u^{\hat{0}}_{ab},\\
n^k c^{\hat{4}}_{kiab} &\approx d_\perp u^{\hat{2}}_{iab},
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- Setting these incoming characteristic constraint fields to zero therefore provides boundary conditions on the normal derivatives $d_\perp u^{\hat{\alpha}} = e^{\hat{\alpha}}{}_\beta n^k \partial_k u^\beta$ of some of the primary dynamical characteristic fields.

Physical Boundary Conditions

- The Weyl curvature tensor C_{abcd} satisfies a system of evolution equations from the Bianchi identities: $\nabla_{[a}C_{bc]de} = 0$.
- The characteristic fields of this system corresponding to physical gravitational waves are the quantities:

$$\hat{w}^{\pm}_{ab} = (P_a{}^c P_b{}^d - \tfrac{1}{2} P_{ab} P^{cd})(t^e \mp n^e)(t^f \mp n^f) C_{cedf},$$

where t^a is a unit timelike vector, n^a a unit spacelike vector (with $t^a n_a = 0$), and $P_{ab} = \psi_{ab} + t_a t_b - n_a n_b$.

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  where t^a is a unit timelike vector, n^a a unit spacelike vector (with $t^a n_a = 0$), and $P_{ab} = \psi_{ab} + t_a t_b - n_a n_b$.
- The incoming field $\hat{w}^{-}_{ab}$ can be expressed in terms of the characteristic fields of the primary evolution system:

$$\hat{w}^{-}_{ab} = d_\perp u^{\hat{1}-}_{ab} + \hat{F}_{ab}(u, d_\parallel u).$$

- We impose boundary conditions on the physical graviational wave degrees of freedom then by setting:

$$d_\perp u^{\hat{1}-}_{ab} = -\hat{F}_{ab}(u, d_\parallel u) + \hat{w}^{-}_{ab}|_{t=0}.$$

Imposing Neumann-like Boundary Conditions

- Consider Neumann-like boundary conditions of the form
$$e^{\hat\alpha}{}_{\beta} n^k \partial_k u^\beta \equiv d_\perp u^{\hat\alpha} = d_\perp u^{\hat\alpha}|_{\mathrm{BC}}.$$

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$$d_t u^{\hat\alpha} \equiv e^{\hat\alpha}{}_\beta \partial_t u^\beta = e^{\hat\alpha}{}_\beta\left(-A^{k\beta}{}_\gamma \partial_k u^\gamma + F^\beta\right) \equiv D_t u^{\hat\alpha}.$$

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- The spatial derivatives of u^γ in this expression can be re-written:
$$e^{\hat\alpha}{}_\beta A^{k\,\beta}{}_\gamma\partial_k u^\gamma = v_{(\hat\alpha)} e^{\hat\alpha}{}_\gamma n^k\partial_k u^\gamma + e^{\hat\alpha}{}_\beta A^{\ell\,\beta}{}_\gamma(\delta^k{}_\ell - n^k n_\ell)\partial_k u^\gamma.$$

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- We impose these Neumann-like boundary conditions by changing the appropriate components of the evolution equations at the boundary to:
$$d_t u^{\hat\alpha} = D_t u^{\hat\alpha} + v_{(\hat\alpha)}\left(d_\perp u^{\hat\alpha} - d_\perp u^{\hat\alpha}|_{\rm BC}\right).$$

Gauge Boundary Conditions

- Constraint preserving and physical boundary conditions discussed above place conditions on some (but not all) of the components of the incoming characteristic field $u^{\hat{1}-}_{ab}$:
- Constraint preserving boundary conditions place conditions on

$$P^{C\,cd}_{ab} d_\perp u^{\hat{1}-}_{cd} \equiv \left(\tfrac{1}{2} P_{ab} P^{cd} - 2 l_{(a} P_{b)}{}^{(c} k^{d)} + l_a l_b k^c k^d\right) d_\perp u^{\hat{1}-}_{cd}.$$

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- Additional "gauge" boundary conditions are needed for the remaining components of $u^{\hat{1}-}_{ab}$:

$$P^{G\,cd}_{ab} u^{\hat{1}-}_{cd} \equiv \left(\delta_a{}^c \delta_b{}^d - P^{C\,cd}_{ab} - P^{P\,cd}_{ab}\right) u^{\hat{1}-}_{cd}.$$

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- We first tried just freezing these boundary degrees of freedom:

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These conditions are mathematically well posed, but they tend to generate a lot of boundary reflections of the gauge degrees of freedom. These conditions do not produce solutions that converge rapidly with increasing numerical resolution.

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  These conditions are mathematically well posed, but they tend to generate a lot of boundary reflections of the gauge degrees of freedom. These conditions do not produce solutions that converge rapidly with increasing numerical resolution.
- Better, less reflective, gauge boundary conditions can be obtained by imposing what amount to Sommerfeld conditions on these gauge degrees of freedom:

$$P^{G\,cd}_{ab} d_t \left[u^{\hat{1}-}_{cd} + (\gamma_2 - r^{-1}) u^{\hat{0}}_{cd} \right] = 0.$$

Outer Boundary Conditions for the Einstein System

- The combined constraint preserving boundary conditions, plus the simple "no incoming Ψ_0" physical gravitational wave boundary conditions for the first-order generalized harmonic Einstein evolution system are given by:

$$
\begin{aligned}
d_t u^{\hat{0}}_{ab} &= D_t u^{\hat{0}}_{ab} - (1+\gamma_1) n_j N^j n^k c^{\hat{3}}_{kab}, \\
d_t u^{\hat{1}-}_{ab} &= P_{ab}^{P\,cd}\left[D_t u^{\hat{1}-}_{cd} - (N + n_j N^j)(\hat{w}^-_{cd} - \gamma_2 n^i c^{\hat{3}}_{icd})\right] \\
&\quad -(\gamma_2 - r^{-1}) P_{ab}^{G\,cd} d_t u^{\hat{0}}_{cd} + P_{ab}^{C\,cd} D_t u^{\hat{1}-}_{cd} \\
&\quad +\sqrt{2}(N + n_j N^j)\left[l_{(a}P_{b)}{}^c - \tfrac{1}{2} P_{ab} l^c - \tfrac{1}{2} l_a l_b k^c\right] c^{\hat{0}-}_c, \\
d_t u^{\hat{2}}_{kab} &= D_t u^{\hat{2}}_{kab} - n_l N^l n^i P^j{}_k c^{\hat{4}}_{ijab}.
\end{aligned}
$$

The quantity P_{ab} in these expressions is the projection tensor, $P_{ab} = \psi_{ab} + t_a t_b - n_a n_b$, the incoming null vector k^a is defined by $k^a = (t^a - n^a)/\sqrt{2}$, and the outgoing null vector l^a is defined by $l^a = (t^a + n^a)/\sqrt{2}$.

Tests of Constraint Preserving and Physical BC

- Evolve the perturbed black-hole spacetime using the resulting constraint preserving boundary conditions for the generalized harmonic evolution systems.
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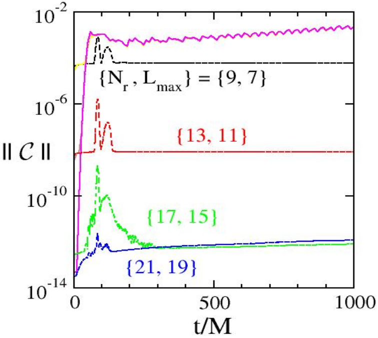
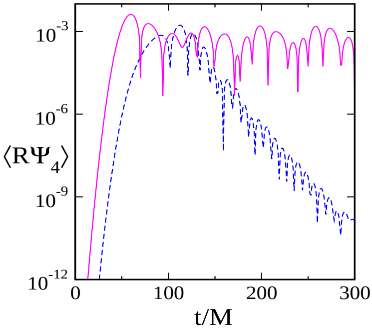
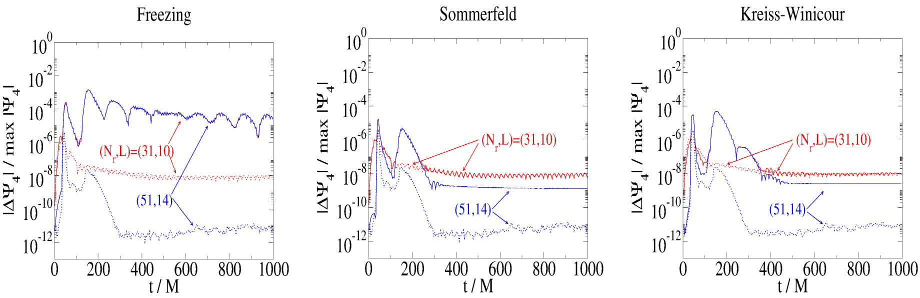




- Evolutions using new BC are stable and convergent.
- The Weyl curvature Ψ_4 shows quasi-normal mode oscillations when new BC are used.

Tests of Constraint Preserving and Physical BC II

- Evolve a perturbed black-hole spacetime using the resulting constraint preserving boundary conditions for the generalized harmonic evolution systems. Compare results of evolutions on a small domain with $R = 41.9M$ with results from a large domain with $R = 961.9M$. Compare results using constraint preserving BC (dotted lines), with other possible outer boundary treatments (solid lines), see Rinne, et. al (2007) for details.



Moving Black Holes Using Spectral Methods

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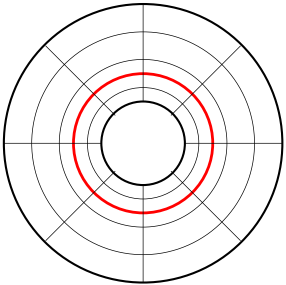

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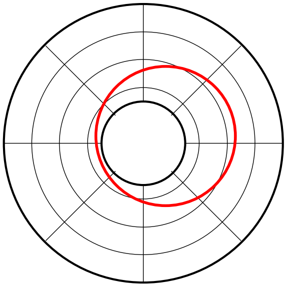

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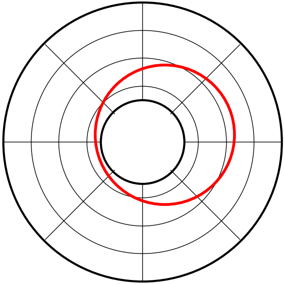

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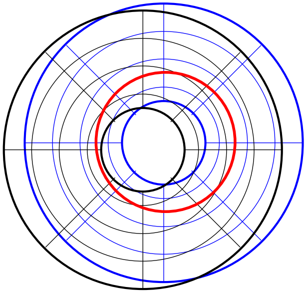

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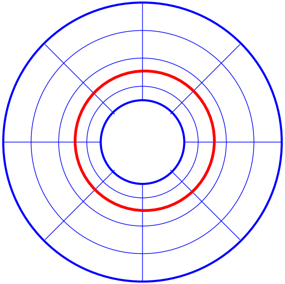

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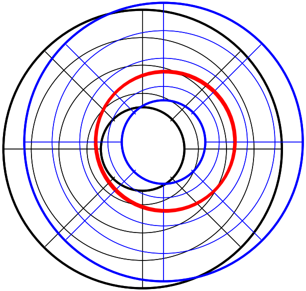

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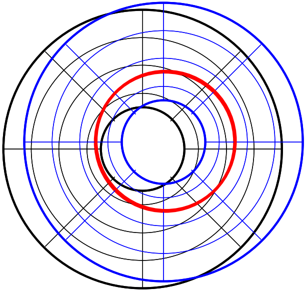

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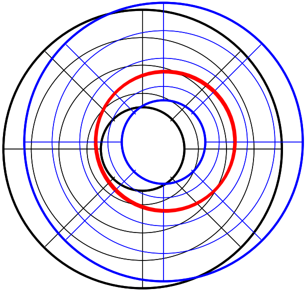

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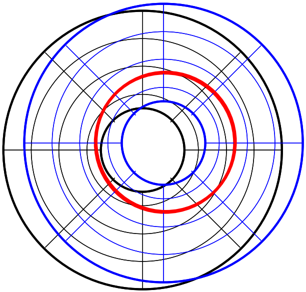

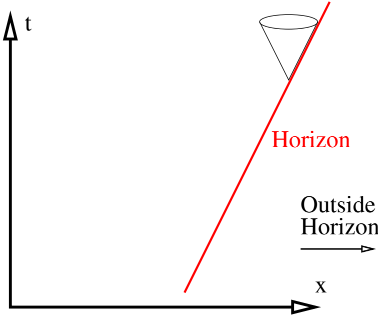

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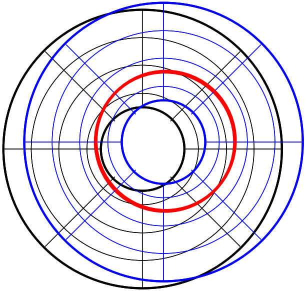
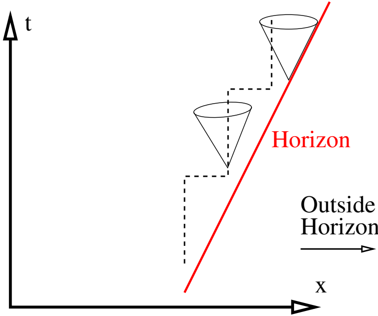
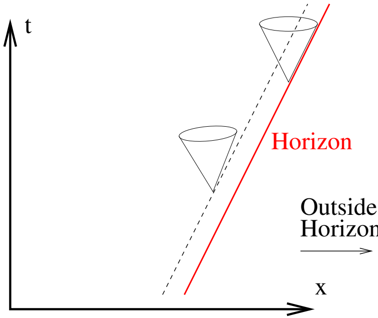




Moving Black Holes Using Spectral Methods

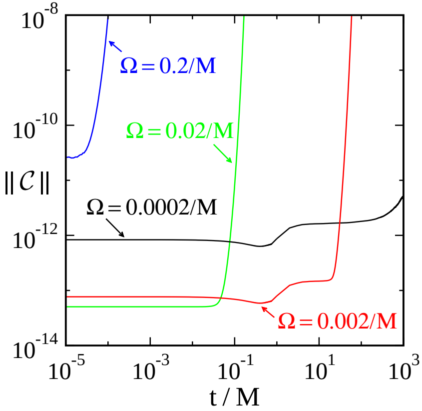
- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: re-grid when holes move too far.
- Problems:
  - Re-gridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.
- Solution:
  Choose coordinates that smoothly track the location of the black hole.

  For a black hole binary this means using coordinates that rotate with respect to inertial frames at infinity.



Evolving Black Holes in Rotating Frames

- Coordinates that rotate with respect to the inertial frames at infinity are needed to track the horizons of orbiting black holes.
- Evolutions of Schwarzschild in rotating coordinates are unstable.



- Evolutions shown use a computational domain that extends to $r = 1000M$.
- Angular velocity needed to track the horizons of an equal mass binary at merger is about $\Omega \approx 0.2/M$.
- Problem caused by asymptotic behavior of metric in rotating coordinates: $\psi_{tt} \sim \varpi^2\Omega^2$, $\psi_{ti} \sim \varpi\Omega$, $\psi_{ij} \sim 1$.

Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar{a}} = \{\bar{t}, x^{\bar{\imath}}\}$, to define field components, $u^{\bar{\alpha}} = \{\psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{\imath}\bar{a}\bar{b}}\}$, and the same coordinates to determine these components by solving Einstein's equation for $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^{\bar{a}})$:

$$\partial_{\bar{t}} u^{\bar{\alpha}} + A^{\bar{k}\,\bar{\alpha}}{}_{\bar{\beta}} \partial_{\bar{k}} u^{\bar{\beta}} = F^{\bar{\alpha}}.$$

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- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^{\bar{a}})$, to represent these components as functions, $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^a)$.

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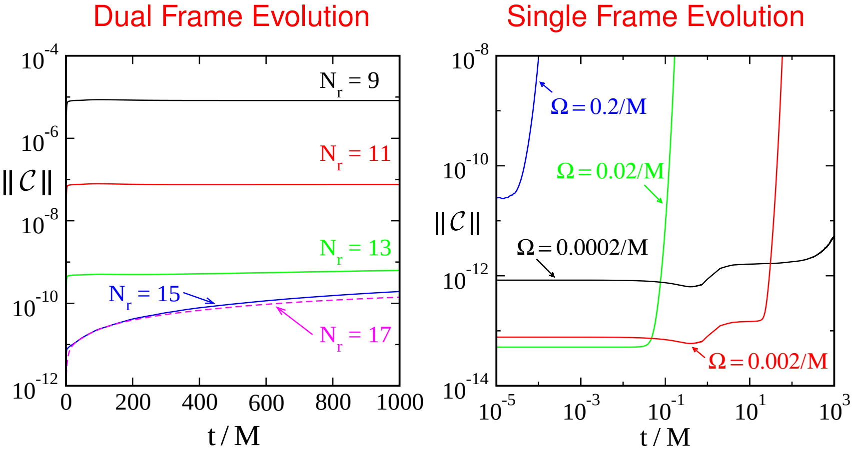
- These functions are determined by solving the transformed Einstein equation:

$$\partial_t u^{\bar{\alpha}} + \left[\frac{\partial x^i}{\partial \bar{t}} \delta^{\bar{\alpha}}{}_{\bar{\beta}} + \frac{\partial x^i}{\partial x^{\bar{k}}} A^{\bar{k}\,\bar{\alpha}}{}_{\bar{\beta}}\right] \partial_i u^{\bar{\beta}} = F^{\bar{\alpha}}.$$

Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:



- Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/M$ on a domain with outer radius $r = 1000M$.

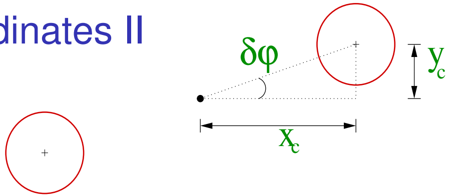
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- The coordinate transformation from inertial coordinates, $(\bar{x}, \bar{y}, \bar{z})$, to co-moving coordinates (x, y, z),

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{a(\bar{t})} \begin{pmatrix} \cos\varphi(\bar{t}) & -\sin\varphi(\bar{t}) & 0 \\ \sin\varphi(\bar{t}) & \cos\varphi(\bar{t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix},$$

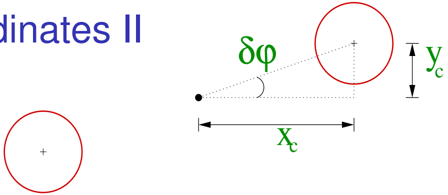
  with $t = \bar{t}$, is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions $a(\bar{t})$ and $\varphi(\bar{t})$.
- Since the motions of the holes are not known *a priori*, the functions $a(\bar{t})$ and $\varphi(\bar{t})$ must be chosen dynamically and adaptively as the system evolves.

Horizon Tracking Coordinates II



- Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.
- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.

Horizon Tracking Coordinates II



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- Changing the map parameters by the small amounts, δa and $\delta\varphi$, results in associated small changes in δQ^x and δQ^y:

$$\delta Q^x = -\delta a, \qquad \delta Q^y = -\delta\varphi.$$

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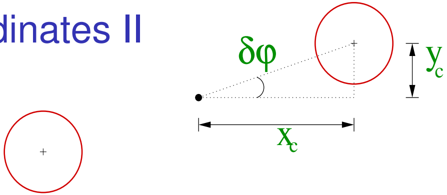

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- Measure the quantities $Q^y(t)$, $dQ^y(t)/dt$, $d^2Q^y(t)/dt^2$, and set

$$\frac{d^3\varphi}{dt^3} = \lambda^3 Q^y + 3\lambda^2 \frac{dQ^y}{dt} + 3\lambda \frac{d^2Q^y}{dt^2} = -\frac{d^3Q^y}{dt^3}.$$

The solutions to this "closed-loop" equation for Q^y have the form $Q^y(t) = (At^2 + Bt + C)e^{-\lambda t}$, so Q^y always decreases as $t \to \infty$.

Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times $t = t_i$.
- In the time interval $t_i < t < t_{i+1}$ we set:

$$\begin{aligned}\varphi(t) &= \varphi_i + (t - t_i)\frac{d\varphi_i}{dt} + \frac{(t-t_i)^2}{2}\frac{d^2\varphi_i}{dt^2}\\ &\quad + \frac{(t-t_i)^3}{2}\left(\lambda\frac{d^2Q^y_i}{dt^2} + \lambda^2\frac{dQ^y_i}{dt} + \lambda^3\frac{Q^y_i}{3}\right),\end{aligned}$$

where Q^x, Q^y, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop
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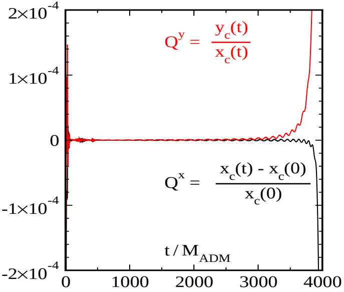
where Q^x, Q^y, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop equation at $t = t_i$.

- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.



Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- This can be implemented by using a coordinate transformation from inertial coordinates, $\bar{x}^i$, to co-moving coordinates x^i, consisting of a translation followed by a rotation followed by an expansion:

$$
\begin{aligned}
x^i &= e^{a(\bar{t})}\, R^{(z)\,i}{}_j[\varphi(\bar{t})]\, R^{(y)\,j}{}_k[\xi(\bar{t})]\, \left[\bar{x}^k - c^k(\bar{t})\right],\\
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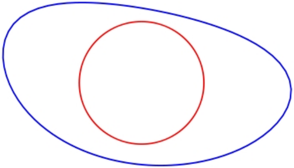
- This transformation keeps the holes fixed in co-moving coordinates for suitably chosen $a(\bar{t})$, $\varphi(\bar{t})$, $\xi(\bar{t})$, and $c^k(\bar{t})$.
- Motions of the holes are not known *a priori*, so $a(\bar{t})$, $\varphi(\bar{t})$, $\xi(\bar{t})$, and $c^k(\bar{t})$ must be chosen dynamically and adaptively.
- A simple feedback-control system has been used to choose $a(\bar{t})$, $\varphi(\bar{t})$, $\xi(\bar{t})$, and $c^k(\bar{t})$ by fixing the black-hole positions, even in evolutions with precession and "kicks".

Horizon Distortion Maps

- Tidal deformation, along with kinematic and gauge effects cause the shapes of the black holes to deform:



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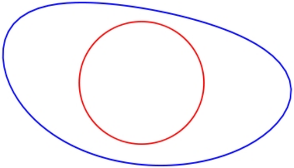

- If the holes become significantly distorted – relative to the spherical excision surface – bad things happen:
  - Some points on the excision boundary are much deeper inside the singular black hole interior. Numerical errors and constraint violations are largest there, sometimes leading to instabilities.

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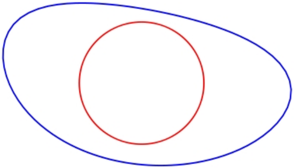

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  - Some points on the excision boundary are much deeper inside the singular black hole interior. Numerical errors and constraint violations are largest there, sometimes leading to instabilities.
  - When the horizons move relative to the excision boundary points, the excision boundary can become timelike, and boundary conditions are then needed there.

Horizon Distortion Maps II

- Adjust the placement of grid points near each black hole using a horizon distortion map that connects grid coordinates x^i to points in the black-hole rest frame $\tilde{x}^i$:

$$\begin{aligned}
\tilde{\theta}_A &= \theta_A, \qquad \tilde{\varphi}_A = \varphi_A, \\
\tilde{r}_A &= r_A - f_A(r_A, \theta_A, \varphi_A) \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \lambda_A^{\ell m}(t)\, Y_{\ell m}(\theta_A, \varphi_A).
\end{aligned}$$

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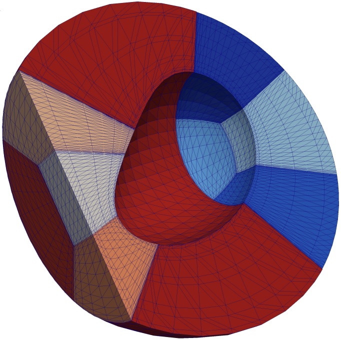
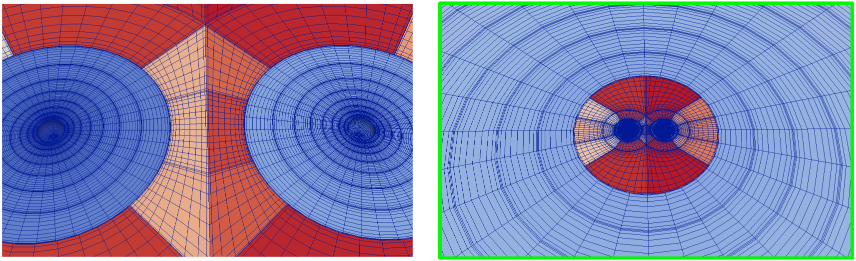
- Choose f_A to scale linearly from $f_A = 1$ on the excision boundary, to $f_A = 0$ on cut sphere.
- Adjust the coefficients $\lambda_A^{\ell m}(t)$ using a feedback-control system to keep the excision surface the same shape and slightly smaller than the horizon, and to keep the boundary spacelike.



Caltech/Cornell Spectral Einstein Code (SpEC):

- Multi-domain pseudo-spectral evolution code.



Lovelace, Scheel, & Szilágyi (2010) high spin evolution grids.

- Constraint damped "generalized harmonic" Einstein equations:
$$\psi^{cd}\partial_c\partial_d\psi_{ab} = Q_{ab}(\psi, \partial\psi).$$
- Dual frame evolutions with horizon tracking and distortion maps.
- Constraint-preserving, physical and gauge boundary conditions.
- Spectral AMR.