Solving Einstein's Equation Numerically VI

Lee Lindblom

Center for Astrophysics and Space Sciences
University of California at San Diego

Mathematical Sciences Center Lecture Series
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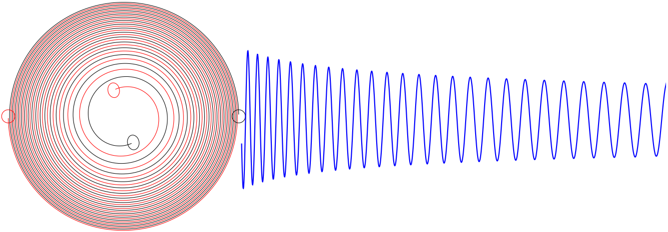

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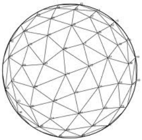

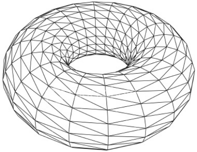

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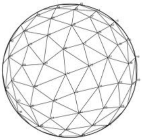
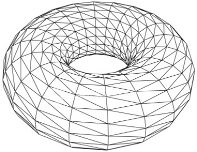
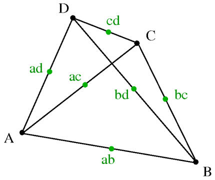




- Cubes make more convenient computational domains for finite difference and spectral numerical methods.
- Can arbitrary two- and three-manifolds be "cubed", i.e. represented as a set of squares or cubes plus a list of rules for gluing their edges or faces together?

“Multi-Cube” Representations of Three-Manifolds

- Every two- and three-dimensional triangulation can be refined to a “multi-cube” representation: For example, in three-dimensions divide each tetrahedron into four “distorted” cubes:



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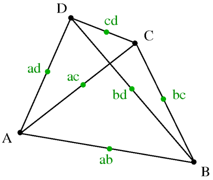

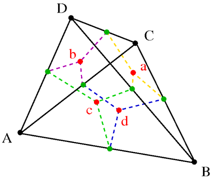

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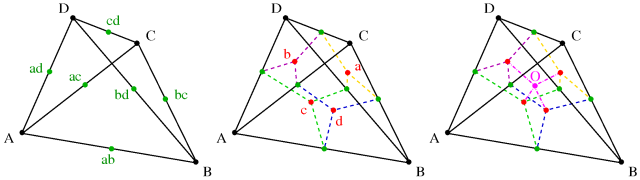

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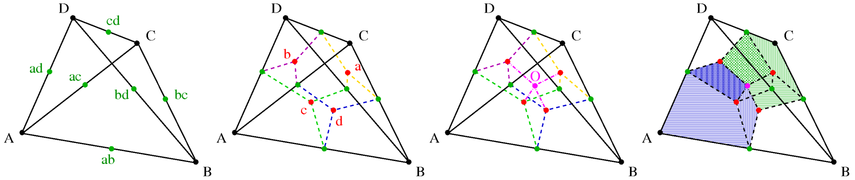

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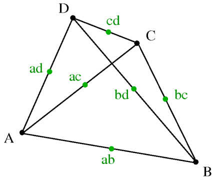
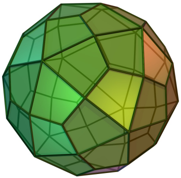
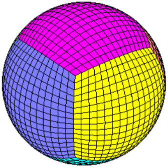
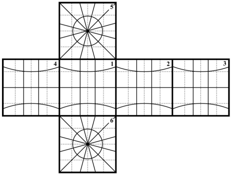


- Every two- or three-manifold can be represented as a set of squares or cubes, plus maps that identify their edges or faces.







Boundary Maps: Fixing the Topology

- Multi-cube representations of topological manifolds consist of a set of cubic regions, $\mathcal{B}_A$, plus maps that identify the faces of neighboring regions, $\Psi^{A\alpha}_{B\beta}(\partial_\beta \mathcal{B}_B) = \partial_\alpha \mathcal{B}_A$.

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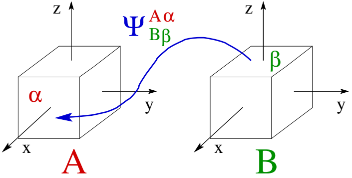
- Choose cubic regions to have uniform size and orientation.
- Choose linear interface identification maps $\Psi^{A\alpha}_{B\beta}$: $x^i_A = c^i_{A\,\alpha} + C^{A\,\alpha\,i}_{B\,\beta\,k}(x^k_B - c^k_{B\,\beta})$, where $C^{A\,\alpha\,i}_{B\,\beta\,k}$ is a rotation-reflection matrix, and $c^i_{A\,\alpha}$ is center of α face of region A.



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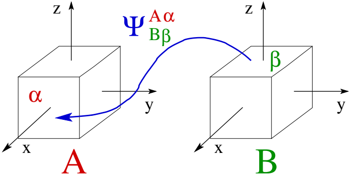
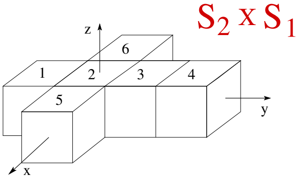
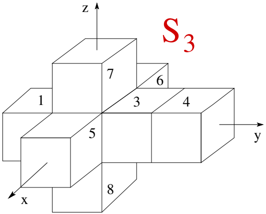


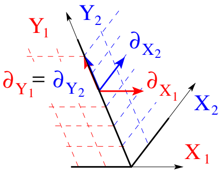
- Examples:





Fixing the Differential Structure

- The boundary identification maps, $\Psi^{A\alpha}_{B\beta}$, used to construct multi-cube topological manifolds are continuous, but typically are not differentiable at the interfaces.



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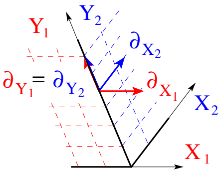

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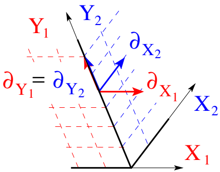
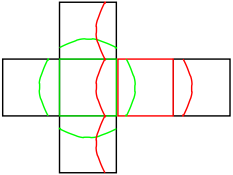


- Smooth tensor fields expressed in multi-cube Cartesian coordinates are not (in general) even continuous at the interfaces.
- Differential structure provides the framework in which smooth functions and tensors are defined on a manifold.
- The standard construction assumes the existence of overlapping coordinate domains having smooth transition maps.
- Multi-cube manifolds need an additional layer of infrastructure: e.g., overlapping domains $\mathcal{D}_A \supset \mathcal{B}_A$ with transition maps that are smooth in the overlap regions.



Fixing the Differential Structure II

- All that is needed to define continuous tensor fields at interface boundaries is the Jacobian $J^{A\alpha i}_{B\beta k}$ and its dual $J^{*B\beta k}_{A\alpha i}$ that transform tensors from one multi-cube coordinate region to another.

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- Define the transformed tensors across interface boundaries:

$$\langle v^i_B \rangle_A = J^{A\alpha i}_{B\beta k} v^k_B, \qquad \langle w_{Bi} \rangle_A = J^{*B\beta k}_{A\alpha i} w_{Bk}.$$

- Tensor fields are continuous across interface boundaries if they are equal to their transformed neighbors:

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- If there exists a covariant derivative $\tilde{\nabla}_i$ determined by a smooth connection, then differentiability across interface boundaries can be defined as continuity of the covariant derivatives:

$$\tilde{\nabla}_{Aj} v^i_A = \langle \tilde{\nabla}_{Bj} v^i_B \rangle_A, \qquad \tilde{\nabla}_{Aj} w_{Ai} = \langle \tilde{\nabla}_{Bj} w_{Bi} \rangle_A$$

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- A smooth reference metric $\tilde g_{ij}$ determines both the needed Jacobians and the smooth connection.

Fixing the Differential Structure III

- Let $\tilde{g}_{Aij}$ and $\tilde{g}_{Bij}$ be the components of a smooth reference metric in the multi-cube coordinates of regions $\mathcal{B}_A$ and $\mathcal{B}_B$ that are identified at the faces $\partial_\alpha\mathcal{B}_A \leftrightarrow \partial_\beta\mathcal{B}_B$.

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- Use the reference metric to define the outward directed unit normals: $\tilde{n}_{A\alpha i}$, $\tilde{n}^i_{A\alpha}$, $\tilde{n}_{B\beta i}$, and $\tilde{n}^i_{B\beta}$.

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- Use the reference metric to define the outward directed unit normals: $\tilde{n}_{A\alpha i}$, $\tilde{n}^i_{A\alpha}$, $\tilde{n}_{B\beta i}$, and $\tilde{n}^i_{B\beta}$.
- The needed Jacobians are given by
$J^{A\alpha i}_{B\beta k} = C^{A\alpha i}_{B\beta\ell}\left(\delta^\ell_k - \tilde{n}^\ell_{B\beta}\tilde{n}_{B\beta k}\right) - \tilde{n}^i_{A\alpha}\tilde{n}_{B\beta k}$,
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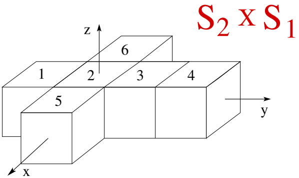
- These Jacobians satisfy:

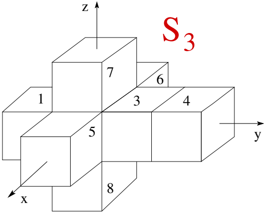
$$\tilde{n}^i_{A\alpha} = -J^{A\alpha i}_{B\beta k}\tilde{n}^k_{B\beta}, \qquad \tilde{n}_{A\alpha i} = -J^{*B\beta k}_{A\alpha i}\tilde{n}_{B\beta k}$$

$$u^i_{A\alpha} = J^{A\alpha i}_{B\beta k}u^k_{B\beta} = C^{A\alpha i}_{B\beta k}u^k_{B\beta}, \qquad \delta^{Ai}_{Ak} = J^{A\alpha i}_{B\beta\ell}J^{*B\beta\ell}_{A\alpha k}.$$

- Require that a smooth reference metric $\tilde{g}_{ab}$ be provided as part of the multi-cube representation of any manifold.

Solving PDEs on Multi-Cube Manifolds





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- Use boundary conditions on cube faces to select the correct smooth global solution.

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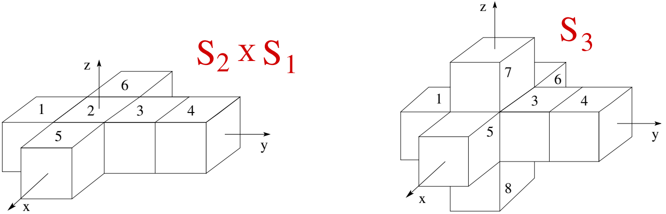

- Solve PDEs in each cubic region separately.
- Use boundary conditions on cube faces to select the correct smooth global solution.
- For first-order symmetric hyperbolic systems whose dynamical fields are tensors: set incoming characteristic fields, $\hat{u}^-$, with outgoing characteristics, $\hat{u}^+$, from neighbor,

$$\hat{u}^-_A = \langle \hat{u}^+_B \rangle_A \qquad \hat{u}^-_B = \langle \hat{u}^+_A \rangle_B.$$

Solving Einstein's Equation on Multi-Cube Manifolds

- Multi-cube methods were designed to solve first-order hyperbolic systems, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\tilde\nabla_k u^\beta = F^\alpha(u)$, where the dynamical fields u^α are tensors that can be transformed across interface boundaries using the Jacobians $J^{A\alpha i}_{B\beta k}$, etc.

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- The usual first-order representations of Einstein's equation fail to meet these conditions in two important ways:
  - The usual choice of dynamical fields, $u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c\partial_c\psi_{ab}, \Phi_{iab} = \partial_i\psi_{ab}\}$ are not tensor fields.
  - The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of Π_{ab}, $\Pi_{ab} = -t^c\partial_c\psi_{ab}$, and the one that comes from preserving the constraint $C_{iab} = \Phi_{iab} - \partial_i\psi_{ab}$, $t^c\partial_c C_{iab} = -\gamma_2 C_{iab}$.

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- Our attempts to construct the transformations for non-tensor quantities like $\partial_i\psi_{ab}$ and Φ_{iab} across the non-smooth multi-cube interface boundaries failed to result in stable numerical evolutions.
- A spatially covariant first-order representation of the Einstein evolution system seems to be needed.

Covariant Representations of Einstein's Equation

- Let $\tilde{\psi}_{ab}$ denote a smooth reference metric on the manifold $R \times \Sigma$. For convenience we choose $ds^2 = \tilde{\psi}_{ab} dx^a dx^b = -dt^2 + \tilde{g}_{ij} dx^i dx^j$, where $\tilde{g}_{ij}$ is the smooth multi-cube reference three-metric on Σ.

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- A fully covariant expression for the Ricci tensor can be obtained using the reference covariant derivative $\tilde{\nabla}_a$:

$$\begin{aligned} R_{ab} &= -\tfrac{1}{2}\psi^{cd}\tilde{\nabla}_c\tilde{\nabla}_d\psi_{ab} + \nabla_{(a}\Delta_{b)} - \psi^{cd}\tilde{R}^e{}_{cd(a}\psi_{b)e} \\ &\quad + \psi^{cd}\psi^{ef}\left(\tilde{\nabla}_e\psi_{ca}\tilde{\nabla}_f\psi_{ab} - \Delta_{ace}\Delta_{bdf}\right), \end{aligned}$$

where $\Delta_{abc} = \psi_{ad}\left(\Gamma^d_{bc} - \tilde{\Gamma}^d_{bc}\right)$, and $\Delta_a = \psi^{bc}\Delta_{abc}$.

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- A fully-covariant manifestly hyperbolic representation of the Einstein equations can be obtained by fixing the gauge with a covariant generalized harmonic condition: $\Delta_a = -H_a(\psi_{cd})$.
- The vacuum Einstein equations then become:

$$
\begin{aligned}
\psi^{cd}\tilde{\nabla}_c\tilde{\nabla}_d\psi_{ab} &= -2\nabla_{(a}H_{b)} + 2\psi^{cd}\psi^{ef}\left(\tilde{\nabla}_e\psi_{ca}\tilde{\nabla}_f\psi_{ab} - \Delta_{ace}\Delta_{bdf}\right) \\
&\quad -2\psi^{cd}\tilde{R}^e{}_{cd(a}\psi_{b)e} + \gamma_0\left[2\delta^c_{(a}t_{b)} - \psi_{ab}t^c\right](H_c + \Delta_c).
\end{aligned}
$$

Covariant Representations of Einstein's Equation II

- A first-order representation of the Einstein equations can be obtained from this covariant generalized harmonic representation by choosing dynamical fields:

$$u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c\tilde\nabla_c\psi_{ab}, \Phi_{iab} = \tilde\nabla_i\psi_{ab}\},$$

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- The resulting first-order Einstein evolution system, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\tilde\nabla_k u^\beta = F^\alpha(u)$, is symmetric-hyperbolic and covariant with respect to spatial coordinate transformations.
- The expressions for the characteristic speeds and fields of this covariant system have the same forms as the standard ones in terms of $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{iab}\}$. These fields are now tensors, so the values of the characteristic fields are somewhat different.

Testing the Einstein Solver: Static Universe on S^3

- The simplest solution to Einstein's equation on S^3 is the "Einstein Static Universe".
- The geometry of this spacetime is described by the standard round metric on S^3:

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- Dynamical evolutions of "dust" generically develop shell crossing singularities, making it a poor choice to use in tests of our spectral evolution code.

Testing the Einstein Solver: Static Universe on S^3 II

- The scalar field $\phi = \phi_0 e^{i\mu t}$ satisfies the Klein-Gordon equation $\nabla^a \nabla_a \phi = \mu^2 \phi$ on the Einstein static universe geometry.
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  $$T_{ab} = \tfrac{1}{2}\left(\nabla_a\phi\nabla_b\phi^* + \nabla_b\phi\nabla_a\phi^*\right) - \tfrac{1}{2}\psi_{ab}\left(\psi^{cd}\nabla_c\phi\nabla_d\phi^* + \mu^2|\phi|^2\right).$$

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- Choose the scalar field amplitude and cosmological constant to have the Einstein Static universe values: $\Lambda = 1/R_3^2$ and $\mu^2|\phi_0|^2 = 1/4\pi R_3^2$.

Testing the Einstein Solver: Static Universe on S^3 III

- Monitor the accuracy of numerical metric solution by evaluating the norm of its error, $\Delta\psi_{ab} = \psi_{Nab} - \psi_{Aab}$:

$$\mathcal{E}_\psi = \sqrt{\frac{\int \sum_{ab} |\Delta\psi_{ab}|^2 \sqrt{g} d^3x}{\int \sum_{ab} |\psi_{ab}|^2 \sqrt{g} d^3x}}.$$

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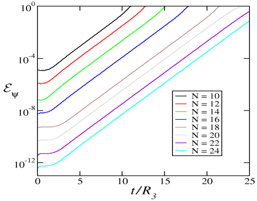

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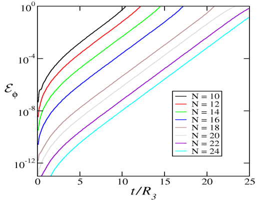

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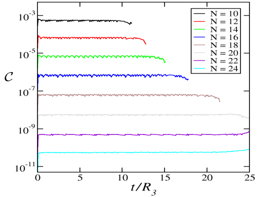

Testing the Einstein Solver: Static Universe on S^3 IV

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- Eddington (1930) predicted the $k = 0$ instability.
- Analytical perturbation theory reveals exactly two unstable modes. One $k = 0$ and one $k = 1$ mode with frequencies

$$\omega_0^2 R_3^2 = 2(\mu^2 R_3^2 - 1) - 2\sqrt{(\mu^2 R_3^3 - 1)^2 + \mu^2 R_3^2}$$

$$\omega_1^2 R_3^2 = -\frac{1}{4}\left(\mu_G R_3 - \sqrt{4 + \mu_G^2 R_3^2}\right)^2$$

- The values of these growth rates for the parameters used in the numerical evolutions are: $1/\tau_0 = 1.1005010$ and $1/\tau_1 = 0.618034$.

Mode Damping

- Can we test long term numerical stability by damping out the unstable modes while leaving the other dynamics untouched?
- Add small unphysical mode damping forces to the Einstein and Klein Gordon evolution systems:

$$\partial_t \psi_{ab} = f_{ab} + \mathcal{D} f_{ab}, \qquad \partial_t \Pi_{ab} = F_{ab} + \mathcal{D} F_{ab},$$
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- All modes of the system (including the unstable modes) have specific spatial structures, best expressed in terms of tensor harmonics on S^3.
- Define the spherical harmonic projection $\bar{Q}^{k\ell m}(t)$ of a scalar quantity $Q(t, \mathbf{x})$ by

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- Construct the unphysical damping forces, e.g. $\mathcal{D}f_{ab}$, to suppress any growth in those structures corresponding to the unstable modes of the system, i.e. the $k = 0$ and $k = 1$ harmonics.

Mode Damping II

- Example mode damping force for the ψ_{tt} evolution equation:

$$\partial_t \psi_{tt} \;=\; f_{tt} \;-\; \left[\bar{f}^{k\ell m}_{tt} + \eta \bar{\psi}^{k\ell m}_{tt}\right] Y^{k\ell m}/R_3^3.$$

- Multiply the modified evolution equations by $Y^{*k\ell m}$ and integrate, to obtain the modified evolution of the damped mode. For the ψ_{tt} equation shown above you get:

$$\partial_t \bar{\psi}^{k\ell m}_{tt} = \bar{f}^{k\ell m}_{tt} - \bar{f}^{k\ell m}_{tt} - \eta \bar{\psi}^{k\ell m}_{tt} = -\eta \bar{\psi}^{k\ell m}_{tt}.$$

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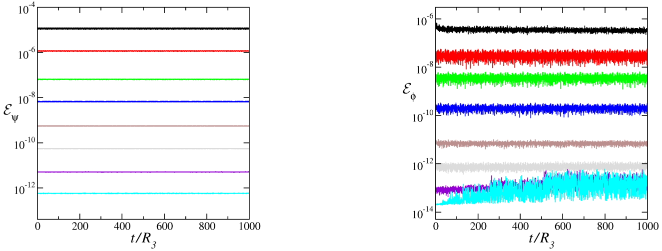
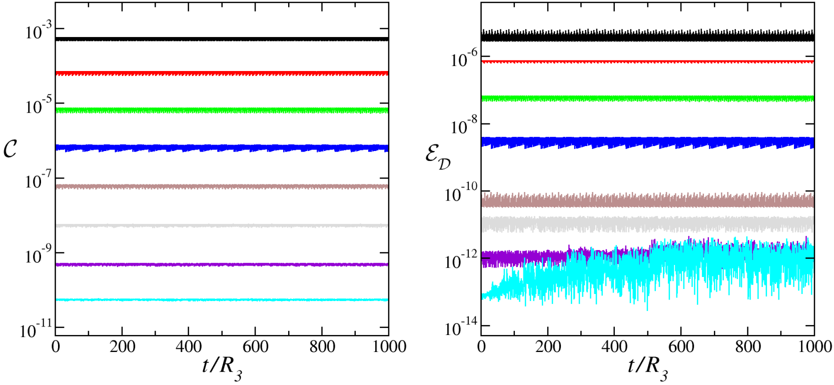
- Apply mode damping to the Einstein-Klein-Gordon static evolution:



Mode Damping III

- Measure the constraint norm $\mathcal{C}$, and the norm of the unphysical mode damping forces $\mathcal{E}_{\mathcal{D}}$:



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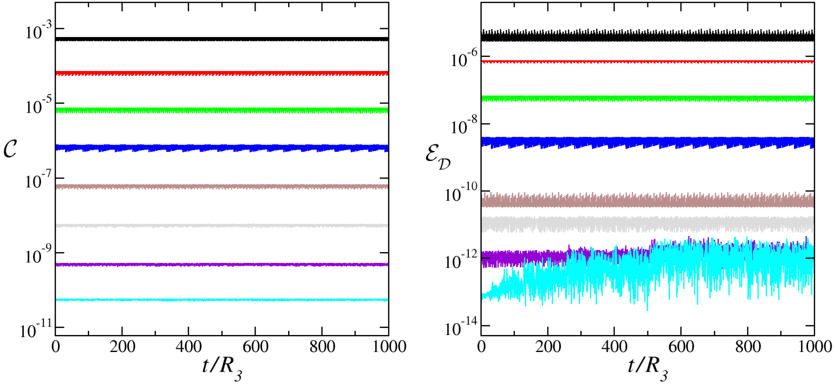


- Both the constraints and the unphysical mode damping forces converge to zero. These numerical solutions therefore converge to solutions of the physical Einstein-Klein-Gordon system!

Testing the Einstein Solver: Perturbed Static S^3

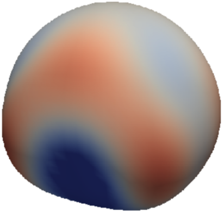
- Construct a more interesting and challenging test problem by examining the perturbed Einstein-Klein-Gordon static universe solution. First, find the normal modes of the perturbed system analytically, e.g., $\delta\psi_{tt} = \Re\left(A_{tt} Y^{k\ell m} e^{i\omega t}\right)$, ...
- The frequencies of the "scalar" modes of this system for $k \geq 2$ are given by $\omega_0^2 R_3^2 = k(k+2)$ and

$$\begin{aligned}\omega_\pm^2 R_3^2 &= k(k+2) + 2(\mu^2 R_3^2 - 1) \\ &\qquad \pm\sqrt{(\mu^2 R_3^2 - 1)^2 + [k(k+2) + 1]\mu^2 R_3^2}.\end{aligned}$$

- Use the solutions of the perturbation equations to construct analytical metric and scalar fields: $\psi^A_{ab} = \psi^0_{ab} + \delta\psi_{ab}$ and $\phi^A = \phi_0 e^{i\mu t} + \delta\phi$.
- Evolve initial data constructed from fifteen superimposed normal modes for $2 \leq k \leq 6$ with one mode for each value of k from each frequency class ω_0 and $\omega_\pm$.

Testing the Einstein Solver: Perturbed Static S^3 II

- Create initial data with mode amplitudes smaller than 10^{-6} to insure non-linear terms will be of order 10^{-12}.
- Visualize the perturbations in $\delta\psi_{tt}$ on the equatorial $\chi = \pi/2$ two-sphere.



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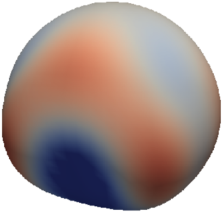
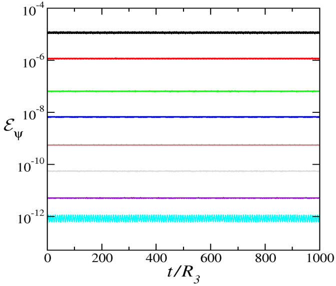
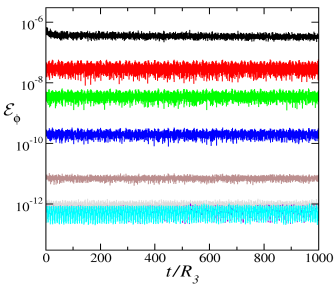


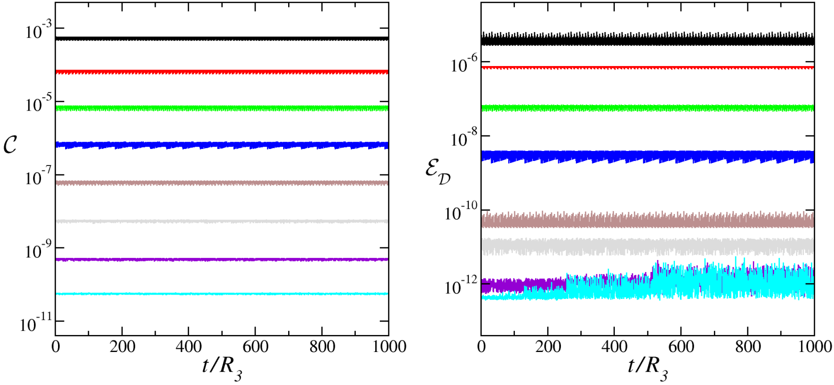
- Compare non-linear evolution with analytical perturbation solution. Measure field error norms: $\Delta\psi_{ab} = \psi^A_{ab} - \psi^N_{ab}$ and $\Delta\phi = \phi^A - \phi^N$.





Testing the Einstein Solver: Perturbed Static S^3 III

- Norms of the constraints, $\mathcal{C}$, and the unphysical mode damping forces, $\mathcal{E}_{\mathcal{D}}$, for the perturbed Einstein-Klein-Gordon evolution:



- Both the constraints and the unphysical mode damping forces converge to zero. These numerical solutions also converge to solutions of the physical Einstein-Klein-Gordon system!

Testing the Einstein Solver: Tensor Modes on S^3

- Next examine the perturbed Einstein-Klein-Gordon static universe solution numerically with perturbations in the ''tensor" modes of the system that represent the gravitational wave degrees of freedom:

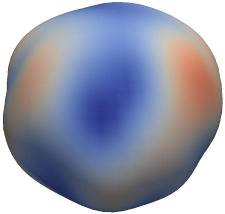
$$\delta\psi_{ab} = \Re\left(A^{k\ell m}_{T(4)}\,Y^{k\ell m}_{(4)ab}e^{i\omega_T t} + A^{k\ell m}_{T(5)}\,Y^{k\ell m}_{(5)\,ab}e^{i\omega_T t}\right),$$

  with frequencies $\omega_T^2 = k(k+2)/R_3^2$, where R_3 is the radius of the Einstein-Klein-Gordon static solution.

- Use the solutions of the perturbation equations to construct analytical metric and scalar field solutions: $\psi^A_{ab} = \psi^0_{ab} + \delta\psi_{ab}$ and $\phi^A = \phi_0 e^{i\mu t}$.
- Evolve initial data constructed from the analytical solutions for ten superimposed normal modes with $2 \le k \le 6$, with one mode for each value of k from each of the transverse-traceless tensor harmonics $Y^{k\ell m}_{(4)\,ab}$ and $Y^{k\ell m}_{(5)\,ab}$.

Testing the Einstein Solver: Tensor Modes on S^3 II

- Create tensor mode initial data with amplitudes smaller than 10^{-6} to insure non-linear terms will be of order 10^{-12}.
- Visualize $\sqrt{\delta\psi_{ab}\delta\psi^{ab}}$ on the equatorial $\chi = \pi/2$ two-sphere.



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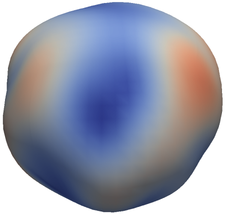
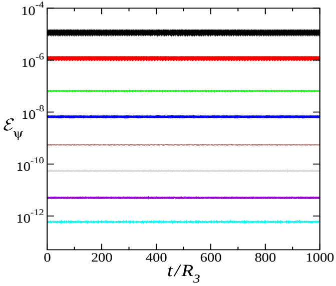
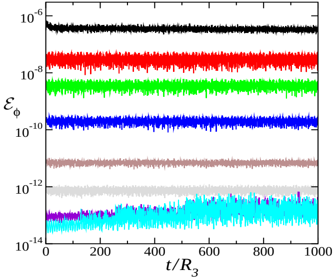


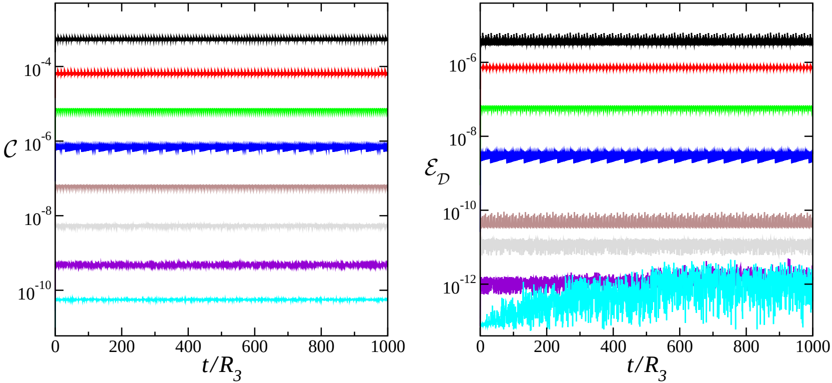
- Compare non-linear evolution with analytical perturbation solution. Measure field error norms: $\Delta\psi_{ab} = \psi^A_{ab} - \psi^N_{ab}$ and $\Delta\phi = \phi^A - \phi^N$.





Testing the Einstein Solver: Tensor Modes on S^3 III

- Norms of the constraints, $\mathcal{C}$, and the unphysical mode damping forces, $\mathcal{E}_{\mathcal{D}}$, for the tensor mode Einstein-Klein-Gordon evolution:



- Both the constraints and the unphysical mode damping forces converge to zero. These numerical solutions also converge to solutions of the physical Einstein-Klein-Gordon system!

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- These methods have been tested successfully for Einstein evolutions by finding simple solutions numerically on compact manifolds using our new covariant Einstein evolution system.