Representations of Arbitrary Three-Manifolds

- **Goal:** Develop numerical methods that are easily adapted to solving elliptic PDEs on three-manifolds \( \Sigma \) with arbitrary topology, and parabolic or hyperbolic PDEs on manifolds \( R \times \Sigma \).

Every two- and three-manifold admits a triangulation (Radó 1925, Moire 1952), i.e. can be represented as a set of triangles (or tetrahedra), plus a list of rules for gluing their edges (or faces) together.

Cubes make more convenient computational domains for finite difference and spectral numerical methods.

Can arbitrary two- and three-manifolds be "cubed", i.e. represented as a set of squares or cubes plus a list of rules for gluing their edges or faces together?
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“Multi-Cube” Representations of Three-Manifolds

Every two- and three-dimensional triangulation can be refined to a “multi-cube” representation: For example, in three-dimensions divide each tetrahedron into four “distorted” cubes:
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![Diagram of multi-cube representation](image-url)
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Boundary Maps: Fixing the Topology

- Multi-cube representations of topological manifolds consist of a set of cubic regions, $B_A$, plus maps that identify the faces of neighboring regions, $\Psi_{B^\alpha}^{A\beta}(\partial B_B) = \partial A B_A$.

Choose cubic regions to have uniform size and orientation. Choose linear interface identification maps $\Psi: x_i^A = c_i^A + C_{A\alpha}^B x_k^B (x_k^B - c_k^B)$, where $C_{A\alpha}^B$ is a rotation-reflection matrix, and $c_i^A$ is the center of $\alpha$ face of region $A$. Examples:
Boundary Maps: Fixing the Topology

- Multi-cube representations of topological manifolds consist of a set of cubic regions, $\mathcal{B}_A$, plus maps that identify the faces of neighboring regions, $\Psi_{\beta}^{\alpha}(\partial_\beta \mathcal{B}_B) = \partial_\alpha \mathcal{B}_A$.
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- Choose cubic regions to have uniform size and orientation.
- Choose linear interface identification maps $\Psi_{B\beta}^{A\alpha}$:
  $$x^i_A = c^i_{A\alpha} + C_{A\alpha \beta k}^{B} (x^k_B - c^k_{B\beta}),$$
  where $C_{A\alpha \beta k}^{B}$ is a rotation-reflection matrix, and $c^i_{A\alpha}$ is the center of $\alpha$ face of region $A$. 

Examples:

Lee Lindblom (CASS UCSD)

Solving Einstein's Equation Numerically

2014/12/11–MSC Tsinghua U
Boundary Maps: Fixing the Topology

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  $$x_A^i = c_{A\alpha}^i + C_{B\beta k}^{A\alpha} (x_B^k - c_{B\beta}^k),$$
  
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- Examples:
  - $S_2 \times S_1$
  - $S_3$
Fixing the Differential Structure

- The boundary identification maps, $\psi^A_{B\beta}$, used to construct multi-cube topological manifolds are continuous, but typically are not differentiable at the interfaces.

- Smooth tensor fields expressed in multi-cube Cartesian coordinates are not (in general) even continuous at the interfaces.
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- Differential structure provides the framework in which smooth functions and tensors are defined on a manifold.

- The standard construction assumes the existence of overlapping coordinate domains having smooth transition maps.

- Multi-cube manifolds need an additional layer of infrastructure: e.g., overlapping domains $D_A \supset B_A$ with transition maps that are smooth in the overlap regions.
Fixing the Differential Structure II

All that is needed to define continuous tensor fields at interface boundaries is the Jacobian $J^{A\alpha i}_{B\beta k}$ and its dual $J^{*B\beta k}_{A\alpha i}$ that transform tensors from one multi-cube coordinate region to another.
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Define the transformed tensors across interface boundaries:

$$\langle v_i^i \rangle_A = J_{B\beta k}^{A\alpha i} v_B^k, \quad \langle w_{Bi} \rangle_A = J^{*B_{\beta k}}_{A\alpha i} w_{Bk}.$$ 

Tensor fields are continuous across interface boundaries if they are equal to their transformed neighbors:

$$v_A^i = \langle v_B^i \rangle_A, \quad w_{Ai} = \langle w_{Bi} \rangle_A$$

If there exists a covariant derivative $\tilde{\nabla}_i$ determined by a smooth connection, then differentiability across interface boundaries can be defined as continuity of the covariant derivatives:

$$\tilde{\nabla}_A v_{Bi} = \langle \tilde{\nabla}_B v_{Bi} \rangle_A, \quad \tilde{\nabla}_A w_{Bi} = \langle \tilde{\nabla}_B w_{Bi} \rangle_A.$$
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$$\tilde{\nabla}_{Aj} v_A^i = \langle \tilde{\nabla}_{Bj} v_B^i \rangle_A, \quad \tilde{\nabla}_{Aj} w_{Ai} = \langle \tilde{\nabla}_{Bj} w_{Bi} \rangle_A$$
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- A smooth reference metric $\tilde{g}_{ij}$ determines both the needed Jacobians and the smooth connection.
Let \( \tilde{g}_{Aij} \) and \( \tilde{g}_{Bij} \) be the components of a smooth reference metric in the multi-cube coordinates of regions \( B_A \) and \( B_B \) that are identified at the faces \( \partial \alpha B_A \leftrightarrow \partial \beta B_B \).

Use the reference metric to define the outward directed unit normals:

- \( \tilde{n}_{A \alpha i} \), \( \tilde{n}_{A \alpha i} \),
- \( \tilde{n}_{B \beta i} \), and \( \tilde{n}_{B \beta i} \).

The needed Jacobians are given by:

\[
J_{A \alpha i B \beta k} = C_{A \alpha i B \beta \ell} (\delta_{\ell k} - \tilde{n}_{\ell B \beta} \tilde{n}_{B \beta k}) - \tilde{n}_{A \alpha i} \tilde{n}_{B \beta k},
\]

\[
J^*_{B \beta k A \alpha i} = (\delta_{\ell i} - \tilde{n}_{A \alpha i} \tilde{n}_{\ell A \alpha}) C_{B \beta k A \alpha \ell} - \tilde{n}_{A \alpha i} \tilde{n}_{k B \beta}.
\]

These Jacobians satisfy:

\[
\tilde{n}_{A \alpha i} = - J_{A \alpha i B \beta k} \tilde{n}_{k B \beta},
\]

\[
\tilde{n}_{B \beta i} = - J^*_{B \beta k A \alpha i} \tilde{n}_{B \beta k}.
\]

Require that a smooth reference metric \( \tilde{g}_{ab} \) be provided as part of the multi-cube representation of any manifold.
Let $\tilde{g}_{Aij}$ and $\tilde{g}_{Bij}$ be the components of a smooth reference metric in the multi-cube coordinates of regions $\mathcal{B}_A$ and $\mathcal{B}_B$ that are identified at the faces $\partial_\alpha \mathcal{B}_A \leftrightarrow \partial_\beta \mathcal{B}_B$.

Use the reference metric to define the outward directed unit normals: $\tilde{n}_{A\alpha i}$, $\tilde{n}^i_{A\alpha}$, $\tilde{n}_{B\beta i}$, and $\tilde{n}^i_{B\beta}$.
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The needed Jacobians are given by

$$J_{B\beta k}^{A\alpha i} = C_{B\beta k}^{\alpha i} \left( \delta_{i}^{\ell} - \tilde{n}_{B\beta}^{\ell} \tilde{n}_{B\beta k} \right) - \tilde{n}_{A\alpha}^{i} \tilde{n}_{B\beta k},$$

$$J_{A\alpha i}^{*B\beta k} = \left( \delta_{i}^{\ell} - \tilde{n}_{A\alpha}^{i} \tilde{n}_{A\alpha}^{\ell} \right) C_{A\alpha k}^{\beta} - \tilde{n}_{A\alpha i} \tilde{n}_{B\beta}^{k}.$$
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The needed Jacobians are given by

$$J_{B\beta k}^{A\alpha i} = C_{B\beta \ell}^{A\alpha i} \left( \delta_{k \ell} - \tilde{n}_{B\beta}^\ell \tilde{n}_{B\beta^k} \right) - \tilde{n}_{A\alpha}^i \tilde{n}_{B\beta^k},$$

$$J^*_{A\alpha i}^{B\beta^k} = \left( \delta_{i \ell} - \tilde{n}_{A\alpha i} \tilde{n}_{A\alpha}^\ell \right) C_{A\alpha \ell}^{B\beta^k} - \tilde{n}_{A\alpha i} \tilde{n}_{B\beta^k}.$$

These Jacobians satisfy:

$$\tilde{n}_{A\alpha}^i = -J_{B\beta k}^{A\alpha i} \tilde{n}_{B\beta}^k,$$

$$\tilde{n}^i_{A\alpha} = -J^*_{A\alpha i}^{B\beta^k} \tilde{n}_{B\beta^k},$$

$$u^i_{A\alpha} = J_{B\beta k}^{A\alpha i} u^k_{B\beta} = C_{B\beta k}^{A\alpha i} u^k_{B\beta},$$

$$\delta_{A\alpha}^i = J_{B\beta \ell}^{A\alpha i} J^*_{A\alpha k}^{B\beta^\ell}.$$

Require that a smooth reference metric $\tilde{g}_{ab}$ be provided as part of the multi-cube representation of any manifold.
Solving PDEs on Multi-Cube Manifolds

- Solve PDEs in each cubic region separately.
- Use boundary conditions on cube faces to select the correct smooth global solution.
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- Solve PDEs in each cubic region separately.
- Use boundary conditions on cube faces to select the correct smooth global solution.
- For first-order symmetric hyperbolic systems whose dynamical fields are tensors: set incoming characteristic fields, $\hat{u}^-$, with outgoing characteristics, $\hat{u}^+$, from neighbor,

$$\hat{u}_A^- = \langle \hat{u}_B^+ \rangle_A \quad \hat{u}_B^- = \langle \hat{u}_A^+ \rangle_B.$$
Solving Einstein’s Equation on Multi-Cube Manifolds

Multi-cube methods were designed to solve first-order hyperbolic systems, $\partial_t u^\alpha + A^k{}_{\alpha\beta}(u) \tilde{\nabla}_k u^\beta = F^\alpha(u)$, where the dynamical fields $u^\alpha$ are tensors that can be transformed across interface boundaries using the Jacobians $J^A_{B\beta k}$, etc.
Solving Einstein’s Equation on Multi-Cube Manifolds

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- The usual first-order representations of Einstein’s equation fail to meet these conditions in two important ways:
  - The usual choice of dynamical fields, \( u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c \partial_c \psi_{ab}, \Phi_{iab} = \partial_i \psi_{ab}\} \) are not tensor fields.
  - The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of \( \Pi_{ab} \), \( \Pi_{ab} = -t^c \partial_c \psi_{ab} \), and the one that comes from preserving the constraint \( C_{iab} = \Phi_{iab} - \partial_i \psi_{ab}, t^c \partial_c C_{iab} = -\gamma_2 C_{iab} \).
Multi-cube methods were designed to solve first-order hyperbolic systems, \( \partial_t u^\alpha + A^k_{\alpha \beta}(u) \tilde{\nabla}_k u^\beta = F^\alpha(u) \), where the dynamical fields \( u^\alpha \) are tensors that can be transformed across interface boundaries using the Jacobians \( J^{A\alpha i}_{B\beta k} \), etc.

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Our attempts to construct the transformations for non-tensor quantities like \( \partial_i \psi_{ab} \) and \( \Phi_{iab} \) across the non-smooth multi-cube interface boundaries failed to result in stable numerical evolutions.

A spatially covariant first-order representation of the Einstein evolution system seems to be needed.
Let \( \tilde{\psi}_{ab} \) denote a smooth reference metric on the manifold \( R \times \Sigma \). For convenience we choose \( ds^2 = \tilde{\psi}_{ab} dx^a dx^b = -dt^2 + \tilde{g}_{ij} dx^i dx^j \), where \( \tilde{g}_{ij} \) is the smooth multi-cube reference three-metric on \( \Sigma \).
Covariant Representations of Einstein’s Equation

- Let $\tilde{\psi}_{ab}$ denote a smooth reference metric on the manifold $R \times \Sigma$. For convenience we choose $ds^2 = \tilde{\psi}_{ab}dx^a dx^b = -dt^2 + \tilde{g}_{ij} dx^i dx^j$, where $\tilde{g}_{ij}$ is the smooth multi-cube reference three-metric on $\Sigma$.

- A fully covariant expression for the Ricci tensor can be obtained using the reference covariant derivative $\tilde{\nabla}_a$:

  $$ R_{ab} = -\frac{1}{2} \psi^{cd} \tilde{\nabla}_c \tilde{\nabla}_d \psi_{ab} + \nabla (a \Delta_b) - \psi^{cd} \tilde{R}^e_{cd(a \psi_b)e} $$

  $$ + \psi^{cd} \psi^{ef} \left( \tilde{\nabla}_e \psi_{ca} \tilde{\nabla}_f \psi_{ab} - \Delta_{ace} \Delta_{bdf} \right), $$

where $\Delta_{abc} = \psi_{ad} \left( \Gamma^d_{bc} - \tilde{\Gamma}^d_{bc} \right)$, and $\Delta_a = \psi^{bc} \Delta_{abc}$.
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  \]
  where \( \Delta_{abc} = \psi_{ad} \left( \Gamma_{bc}^d - \tilde{\Gamma}_{bc}^d \right) \), and \( \Delta_a = \psi^{bc} \Delta_{abc} \).

- A fully-covariant manifestly hyperbolic representation of the Einstein equations can be obtained by fixing the gauge with a covariant generalized harmonic condition: \( \Delta_a = -H_a(\psi_{cd}) \).

- The vacuum Einstein equations then become:
  \[
  \psi^{cd} \tilde{\nabla}_c \tilde{\nabla}_d \psi_{ab} = -2 \nabla (a H_b) + 2 \psi^{cd} \psi^{ef} (\tilde{\nabla}_e \psi_{ca} \tilde{\nabla}_f \psi_{ab} - \Delta_{ace} \Delta_{bdf}) \\
  - 2 \psi^{cd} \tilde{R}_{cd(a} \psi_{b)e} + \gamma_0 \left[ 2 \delta^c_{(a} t_{b)} - \psi_{ab} t^c \right] (H_c + \Delta_c) .
  \]
A first-order representation of the Einstein equations can be obtained from this covariant generalized harmonic representation by choosing dynamical fields:

\[ u^\alpha = \{ \psi_{ab}, \Pi_{ab} = -t^c \nabla_c \psi_{ab}, \Phi_{iab} = \nabla_i \psi_{ab} \} , \]

which are tensors with respect to spatial coordinate transformations.

The resulting first-order Einstein evolution system,

\[ \partial_t u^\alpha + A^k_{\alpha \beta} (u^\beta) \tilde{\nabla}^k u^\beta = F^\alpha (u^\beta) , \]

is symmetric-hyperbolic and covariant with respect to spatial coordinate transformations. The expressions for the characteristic speeds and fields of this covariant system have the same forms as the standard ones in terms of \( u^\alpha = \{ \psi_{ab}, \Pi_{ab}, \Phi_{iab} \} \). These fields are now tensors, so the values of the characteristic fields are somewhat different.
A first-order representation of the Einstein equations can be obtained from this covariant generalized harmonic representation by choosing dynamical fields:

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The first order equation that arises from the definition of \( \Pi_{ab} \), \( t^c \tilde{\nabla}_c \psi_{ab} = -\Pi_{ab} \) is now covariant, as is the equation for \( t^c \tilde{\nabla}_c \Phi_{iab} \) that follows from the covariant constraint evolution equation, \( t^c \tilde{\nabla}_c C_{iab} = -\gamma_2 C_{iab} \), where \( C_{iab} = \Phi_{iab} - \tilde{\nabla}_i \psi_{ab} \).
Covariant Representations of Einstein’s Equation II

- A first-order representation of the Einstein equations can be obtained from this covariant generalized harmonic representation by choosing dynamical fields:
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The simplest solution to Einstein’s equation on $S^3$ is the “Einstein Static Universe”.

The geometry of this spacetime is described by the standard round metric on $S^3$:

\[
d s^2 = -d t^2 + R_3^2 \left[ d \chi^2 + \sin^2 \chi \left( d \theta^2 + \sin^2 \theta \, d \varphi^2 \right) \right],
\]

This metric solves Einstein’s equation with cosmological constant $\Lambda = 1/R_3^2$ and pressure-less matter with density $\rho = 1/(4\pi R_3^2)$ on a manifold with spatial topology $S^3$.

Dynamical evolutions of “dust” generically develop shell crossing singularities, making it a poor choice to use in tests of our spectral evolution code.
Testing the Einstein Solver: Static Universe on $S^3$

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$$= -dt^2 + \left( \frac{\pi R_3}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)(1 + Z_A^2)}{(1 + X_A^2 + Y_A^2 + Z_A^2)^2} \left[ \frac{(1 + X_A^2)(1 + Y_A^2 + Z_A^2)}{(1 + Y_A^2)(1 + Z_A^2)} \, dx^2 + \frac{(1 + Y_A^2)(1 + X_A^2 + Z_A^2)}{(1 + X_A^2)(1 + Z_A^2)} \, dy^2 + \frac{(1 + Z_A^2)(1 + X_A^2 + Y_A^2)}{(1 + X_A^2)(1 + Y_A^2)} \, dz^2 - \frac{2X_A Y_A}{1 + Z_A^2} \, dx \, dy - \frac{2X_A Z_A}{1 + Y_A^2} \, dx \, dz - \frac{2Y_A Z_A}{1 + X_A^2} \, dy \, dz \right].$$
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\[
\begin{align*}
    ds^2 &= -dt^2 + R_3^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \ d\varphi^2 \right) \right], \\
    &= -dt^2 + \left( \frac{\pi R_3}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)(1 + Z_A^2)}{(1 + X_A^2 + Y_A^2 + Z_A^2)^2} \left[ \frac{(1 + X_A^2)(1 + Y_A^2 + Z_A^2)}{(1 + Y_A^2)(1 + Z_A^2)} \ dx^2 + \frac{(1 + Y_A^2)(1 + X_A^2 + Z_A^2)}{(1 + X_A^2)(1 + Z_A^2)} \ dy^2 \\
    &\quad + \frac{(1 + Z_A^2)(1 + X_A^2 + Y_A^2)}{(1 + X_A^2)(1 + Y_A^2)} \ dz^2 - \frac{2X_A Y_A}{1 + Z_A^2} \ dx \ dy - \frac{2X_A Z_A}{1 + Y_A^2} \ dx \ dz - \frac{2Y_A Z_A}{1 + X_A^2} \ dy \ dz \right].
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The simplest solution to Einstein’s equation on $S^3$ is the “Einstein Static Universe”.

The geometry of this spacetime is described by the standard round metric on $S^3$:

$$ds^2 = -dt^2 + R_3^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right],$$

This metric solves Einstein’s equation with cosmological constant $\Lambda = 1/R_3^2$ and pressure-less matter with density $\rho = 1/(4\pi R_3^2)$ on a manifold with spatial topology $S^3$.

Dynamical evolutions of “dust” generically develop shell crossing singularities, making it a poor choice to use in tests of our spectral evolution code.
The scalar field $\phi = \phi_0 e^{i\mu t}$ satisfies the Klein-Gordon equation
$\nabla^a \nabla_a \phi = \mu^2 \phi$ on the Einstein static universe geometry.

This solution has energy density $\rho = \mu^2 |\phi_0|^2$ and no pressure. This could be used as the matter in the Einstein static universe by requiring $\rho = \mu^2 |\phi_0|^2 = 1/4\pi R_3^2$. 
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Couple Einstein’s equation to a complex scalar field with stress energy tensor:

$$T_{ab} = \frac{1}{2} (\nabla_a \phi \nabla_b \phi^* + \nabla_b \phi \nabla_a \phi^*) - \frac{1}{2} \psi_{ab} \left( \psi^{cd} \nabla_c \phi \nabla_d \phi^* + \mu^2 |\phi|^2 \right).$$
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Choose initial data corresponding to the Einstein-Klein-Gordon static universe solution:

$$\psi_{ab} = \psi^0_{ab}, \quad \Pi_{ab} = 0, \quad \Phi_{iab} = 0,$$
$$\phi = \phi_0, \quad \Pi^\phi = -i \mu \phi_0, \quad \Phi^\phi_i = 0.$$
Testing the Einstein Solver: Static Universe on $S^3$ II

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  $$\phi = \phi_0, \quad \Pi^\phi = -i\mu \phi_0, \quad \Phi^\phi_i = 0.$$
- Choose the scalar field amplitude and cosmological constant to have the Einstein Static universe values: $\Lambda = 1/R^2_3$ and $\mu^2 |\phi_0|^2 = 1/4\pi R^2_3$. 
Testing the Einstein Solver: Static Universe on $S^3$ III

- Monitor the accuracy of numerical metric solution by evaluating the norm of its error, $\Delta \psi_{ab} = \psi_{Nab} - \psi_{Aab}$:
  \[ E_{\psi} = \sqrt{\int \sum_{ab} |\Delta \psi_{ab}|^2 \sqrt{g} d^3x} / \sqrt{\int \sum_{ab} |\psi_{ab}|^2 \sqrt{g} d^3x}. \]

- Monitor the accuracy of numerical scalar field solution by evaluating the norm of its error, $\Delta \phi = \phi_{N} - \phi_{A}$:
  \[ E_{\phi} = \sqrt{\int |\Delta \phi|^2 \sqrt{g} d^3x} / \sqrt{\int |\phi|^2 \sqrt{g} d^3x}. \]

- Monitor how well the numerical solutions satisfy the Einstein system by evaluating the norm of the various constraints:
  \[ E_{C} = \sqrt{\int \sum |C|^2 \sqrt{g} d^3x} / \sqrt{\int \sum |\partial_i u|^2 \sqrt{g} d^3x}. \]
Testing the Einstein Solver: Static Universe on $S^3$ III

- Monitor the accuracy of numerical metric solution by evaluating the norm of its error, $\Delta \psi_{ab} = \psi_{Nab} - \psi_{Aab}$:

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The constraints are well satisfied for $t \lesssim 25$, so these are good (approximate) solutions for early times.
Testing the Einstein Solver: Static Universe on $S^3$ IV

- The constraints are well satisfied for $t \lesssim 25$, so these are good (approximate) solutions for early times.
- The instability seen in the static Einstein-Klein-Gordon evolutions is caused by two unstable modes: one $k = 0$ and one $k = 1$ mode.
- The numerically determined growth rates of these modes are $1/\tau^0_N = 1.100501(1)$ and $1/\tau^1_N = 0.6180(1)$ respectively.
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\[
1/\tau_N^0 = 1.100501(1) \quad \text{and} \quad 1/\tau_N^1 = 0.6180(1)
\]

Eddington (1930) predicted the \( k = 0 \) instability.

Analytical perturbation theory reveals exactly two unstable modes. One \( k = 0 \) and one \( k = 1 \) mode with frequencies

\[
\begin{align*}
\omega_0^2 R_3^2 &= 2(\mu^2 R_3^2 - 1) - 2\sqrt{(\mu^2 R_3^2 - 1)^2 + \mu^2 R_3^2} \\
\omega_1^2 R_3^2 &= -\frac{1}{4} \left( \mu_G R_3 - \sqrt{4 + \mu_G^2 R_3^2} \right)^2
\end{align*}
\]

The values of these growth rates for the parameters used in the numerical evolutions are: 

\[
1/\tau_0 = 1.1005010 \quad \text{and} \quad 1/\tau_1 = 0.618034.
\]
Mode Damping

- Can we test long term numerical stability by damping out the unstable modes while leaving the other dynamics untouched?
- Add small unphysical mode damping forces to the Einstein and Klein Gordon evolution systems:

\[
\partial_t \psi_{ab} = f_{ab} + Df_{ab}, \\
\partial_t \Pi_{ab} = F_{ab} + DF_{ab}, \\
\partial_t \phi = f^\phi + Df^\phi, \\
\partial_t \Pi^\phi = F^\phi + DF^\phi.
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All modes of the system (including the unstable modes) have specific spatial structures, best expressed in terms of tensor harmonics on \( S^3 \).

Define the spherical harmonic projection \( \bar{Q}^{k\ell m}(t) \) of a scalar quantity \( Q(t, x) \) by

\[ \bar{Q}^{k\ell m}(t) = \int Y^{*k\ell m} Q(t, x) \sqrt{\bar{g}} \, d^3 x. \]
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  \[
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  \]

- Construct the unphysical damping forces, e.g. $Df_{ab}$, to suppress any growth in those structures corresponding to the unstable modes of the system, i.e. the $k = 0$ and $k = 1$ harmonics.
Example mode damping force for the $\psi_{tt}$ evolution equation:

$$\partial_t \psi_{tt} = f_{tt} - \left[ \bar{f}^{k\ell m}_{tt} + \eta \bar{\psi}^{k\ell m}_{tt} \right] Y^{k\ell m} / R^3_3.$$ 

Multiply the modified evolution equations by $Y^{*k\ell m}$ and integrate, to obtain the modified evolution of the damped mode. For the $\psi_{tt}$ equation shown above you get:

$$\partial_t \bar{\psi}^{k\ell m}_{tt} = \bar{f}^{k\ell m}_{tt} - \bar{f}^{k\ell m}_{tt} - \eta \bar{\psi}^{k\ell m}_{tt} = -\eta \bar{\psi}^{k\ell m}_{tt}.$$
Mode Damping II

- Example mode damping force for the $\psi_{tt}$ evolution equation:

$$\partial_t \psi_{tt} = f_{tt} - \left[ f_{tt}^{kelm} + \eta \bar{\psi}_{tt}^{kelm} \right] Y^{kelm}/R_3^3.$$  

- Multiply the modified evolution equations by $Y^{*kelm}$ and integrate, to obtain the modified evolution of the damped mode. For the $\psi_{tt}$ equation shown above you get:

$$\partial_t \bar{\psi}_{tt}^{kelm} = f_{tt}^{kelm} - f_{tt}^{kelm} - \eta \bar{\psi}_{tt}^{kelm} = -\eta \bar{\psi}_{tt}^{kelm}.$$  

- Apply mode damping to the Einstein-Klein-Gordon static evolution:
Mode Damping III

- Measure the constraint norm $C$, and the norm of the unphysical mode damping forces $\mathcal{E}_D$:

![Graphs showing the convergence of constraints and mode damping forces](image)
Mode Damping III

- Measure the constraint norm $C$, and the norm of the unphysical mode damping forces $\mathcal{E}_D$:

![Graph showing $C$ and $\mathcal{E}_D$ over time](image)

- Both the constraints and the unphysical mode damping forces converge to zero. These numerical solutions therefore converge to solutions of the physical Einstein-Klein-Gordon system!
Testing the Einstein Solver: Perturbed Static $S^3$

- Construct a more interesting and challenging test problem by examining the perturbed Einstein-Klein-Gordon static universe solution. First, find the normal modes of the perturbed system analytically, e.g.,

$$\delta \psi_{tt} = R \left( A_{tt} Y^{k\ell m} e^{i\omega t} \right), \ldots$$

- The frequencies of the “scalar” modes of this system for $k \geq 2$ are given by

$$\omega_0^2 R_3^2 = k(k + 2)$$

and

$$\omega_{\pm}^2 R_3^2 = k(k + 2) + 2(\mu^2 R_3^2 - 1) \pm \sqrt{(\mu^2 R_3^2 - 1)^2 + [k(k + 2) + 1] \mu^2 R_3^2}.$$

- Use the solutions of the perturbation equations to construct analytical metric and scalar fields:

$$\psi^A_{ab} = \psi^0_{ab} + \delta \psi_{ab}$$

and

$$\phi^A = \phi_0 e^{i\mu t} + \delta \phi.$$

- Evolve initial data constructed from fifteen superimposed normal modes for $2 \leq k \leq 6$ with one mode for each value of $k$ from each frequency class $\omega_0$ and $\omega_{\pm}$. 

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Testing the Einstein Solver: Perturbed Static $S^3$ II

- Create initial data with mode amplitudes smaller than $10^{-6}$ to insure non-linear terms will be of order $10^{-12}$.

- Visualize the perturbations in $\delta \psi_{tt}$ on the equatorial $\chi = \pi/2$ two-sphere.
Testing the Einstein Solver: Perturbed Static $S^3$ II

- Create initial data with mode amplitudes smaller than $10^{-6}$ to insure non-linear terms will be of order $10^{-12}$.
- Visualize the perturbations in $\delta \psi_{tt}$ on the equatorial $\chi = \pi/2$ two-sphere.
- Compare non-linear evolution with analytical perturbation solution. Measure field error norms: $\Delta \psi_{ab} = \psi^A_{ab} - \psi^N_{ab}$ and $\Delta \phi = \phi^A - \phi^N$.
Testing the Einstein Solver: Perturbed Static $S^3$ III

- Norms of the constraints, $C$, and the unphysical mode damping forces, $\mathcal{E}_D$, for the perturbed Einstein-Klein-Gordon evolution:

Both the constraints and the unphysical mode damping forces converge to zero. These numerical solutions also converge to solutions of the physical Einstein-Klein-Gordon system!
Next examine the perturbed Einstein-Klein-Gordon static universe solution numerically with perturbations in the "tensor" modes of the system that represent the gravitational wave degrees of freedom:

\[
\delta \psi_{ab} = \Re \left( A_{T(4)}^{k \ell m} Y_{(4)ab}^{k \ell m} e^{i \omega_T t} + A_{T(5)}^{k \ell m} Y_{(5)ab}^{k \ell m} e^{i \omega_T t} \right),
\]

with frequencies \( \omega_T^2 = k(k + 2)/R_3^2 \), where \( R_3 \) is the radius of the Einstein-Klein-Gordon static solution.

Use the solutions of the perturbation equations to construct analytical metric and scalar field solutions: \( \psi_{ab}^A = \psi_{ab}^0 + \delta \psi_{ab} \) and \( \phi^A = \phi^0 e^{i \mu t} \).

Evolve initial data constructed from the analytical solutions for ten superimposed normal modes with \( 2 \leq k \leq 6 \), with one mode for each value of \( k \) from each of the transverse-traceless tensor harmonics \( Y_{(4)ab}^{k \ell m} \) and \( Y_{(5)ab}^{k \ell m} \).
Testing the Einstein Solver: Tensor Modes on $S^3$ II

- Create tensor mode initial data with amplitudes smaller than $10^{-6}$ to insure non-linear terms will be of order $10^{-12}$.
- Visualize $\sqrt{\delta\psi_{ab}\delta\psi^{ab}}$ on the equatorial $\chi = \pi/2$ two-sphere.
Testing the Einstein Solver: Tensor Modes on $S^3$ II

- Create tensor mode initial data with amplitudes smaller than $10^{-6}$ to insure non-linear terms will be of order $10^{-12}$.
- Visualize $\sqrt{\delta \psi_{ab} \delta \psi^{ab}}$ on the equatorial $\chi = \pi/2$ two-sphere.
- Compare non-linear evolution with analytical perturbation solution. Measure field error norms: $\Delta \psi_{ab} = \psi^A_{ab} - \psi^N_{ab}$ and $\Delta \phi = \phi^A - \phi^N$. 

![Graphs showing error norms over time](image.png)
Testing the Einstein Solver: Tensor Modes on $S^3$ III

- Norms of the constraints, $C$, and the unphysical mode damping forces, $\mathcal{E}_D$, for the tensor mode Einstein-Klein-Gordon evolution:

![Graph showing norms of constraints and damping forces over time](image)

- Both the constraints and the unphysical mode damping forces converge to zero. These numerical solutions also converge to solutions of the physical Einstein-Klein-Gordon system!
Summary

- We have developed a simple and flexible multi-cube numerical method for solving partial differential equations on manifolds with arbitrary spatial topologies.
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- Each new topology requires:
  - A multi-cube representation of the topology, i.e. a list of cubic regions and a list of boundary identification maps.
  - A smooth reference metric $\tilde{g}_{ab}$ to define the global differential structure on this multi-cube representation of the manifold.

These methods have been tested by solving simple elliptic and hyperbolic equations on several compact manifolds. A first-order symmetric-hyperbolic representation of the generalized harmonic Einstein evolution equations has been constructed that is covariant with respect to general spatial coordinate transformations. These methods have been tested successfully for Einstein evolutions by finding simple solutions numerically on compact manifolds using our new covariant Einstein evolution system.
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