Gauge conditions are specified in the GH Einstein system by the gauge source function $H^a \equiv \nabla^c \nabla_c x^a$.

How do you choose $H^a$ corresponding to the familiar gauge conditions of numerical relativity — without destroying the hyperbolicity of the system?
The principal parts of the GH Einstein equations may be written as

$$\psi^{cd} \partial_c \partial_d \psi_{ab} = \nabla_a H_b + \nabla_b H_a + Q_{ab}(\psi, \partial \psi),$$

where $\psi_{ab}$ is the spacetime metric.

These equations are manifestly hyperbolic when $H^a$ is specified as a function of $x^a$ and $\psi_{ab}$: $H^a = H^a(x, \psi)$.

In this case the principal parts of the GH Einstein system are simple wave operators on each component of $\psi_{ab}$:

$$\psi^{cd} \partial_c \partial_d \psi_{ab} = \hat{Q}_{ab}(x, \psi, \partial \psi).$$
Imposing Useful Gauge Conditions

Specifying the gauge source function $H^a$ places constraints on the derivatives of the spacetime metric:

$$H^a = \nabla^c \nabla_c x^a = \psi^{bc} \Gamma^a_{bc} \equiv \Gamma^a.$$ 

where $\Gamma^a_{bc}$ is the Christoffel symbol.

The quantity $\Gamma^a$ depends on the time derivatives of the lapse $N$ and shift $N^i$. For example,

$$\Gamma^t = N^{-3} \left( \partial_t N - N^k \partial_k N + N^2 K \right),$$

where $K$ is the trace of the extrinsic curvature.

One could impose the slicing condition $\partial_t N - N^k \partial_k N = -2NK$, for example, by setting $H^t = N^{-2}(N - 2)K$.

Unfortunately this choice has the form $H^a = H^a(x, \psi, \partial \psi)$ which destroys the hyperbolicity of the GH Einstein system.
Elevate $H_a$ to the status of a dynamical field (Pretorius) and evolve it along with the spacetime metric $\psi_{ab}$:

$$\psi^{cd} \partial_c \partial_d H_a = Q_a(x, H, \partial H, \psi, \partial \psi),$$

$$\psi^{cd} \partial_c \partial_d \psi_{ab} = Q_{ab}(x, H, \partial H, \psi, \partial \psi),$$

Any gauge driver of this form produces a symmetric hyperbolic combined evolution system.

Choose $Q_a$ so that $H_a$ evolves toward the desired gauge target $F_a$ as the system evolves: $H_a \rightarrow F_a$.

For example, consider the simple gauge driver:

$$\psi^{cd} \partial_c \partial_d H_a = Q_a = \mu^2 (H_a - F_a) + 2\mu N^{-1} \partial_t H_a.$$ 

For constant $F_a$, this gauge driver causes $H_a \rightarrow F_a + O(e^{-\mu t})$. 

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Improved Gauge Driver Equations

- In the time independent (but spatially inhomogeneous) limit, the simple gauge driver evolution equation reduces to,

\[ \psi^{jk} \partial_j \partial_k H_a = \mu^2 (H_a - F_a) \]

so this equation does not achieve \( H_a \rightarrow F_a \) unless \( \partial_k H_a = 0 \).

- The system can be improved by adding a time averaging field,

\[ \partial_t \theta_a + \mu \theta_a = \psi^{jk} \partial_j \partial_k H_a, \]

which is used to modify the gauge driver:

\[ \psi^{cd} \partial_c \partial_d H_a = Q_a = \mu^2 (H_a - F_a) + 2 \mu N^{-1} \partial_t H_a + \mu \theta_a. \]

- The resulting gauge driver system is symmetric hyperbolic for all \( F_a = F_a(x, \psi, \partial \psi) \), has solutions \( H_a \) that exponentially approach any time independent \( F_a \) when the background geometry is fixed, and reduces to \( H_a = F_a \) in any time independent state.
Decoupled Gauge Driver Test:

- Test the gauge driver equation on a fixed (flat) background spacetime. Use a time and space dependent gauge target:

\[ F = \left[ 3 + e^{-(t-10)^2/9} \right] \sin(10x). \]
Fully Coupled Gauge Driver Test:

- Test the gauge driver equation (with a conformal gamma driver target $F_a$) coupled to the GH Einstein equations for a perturbed Schwarzschild spacetime: $\delta N^i = 0.01 \hat{r}^i Y_{30} e^{-(r-15)^2/9}$.