Solving Einstein's Equation Numerically on Manifolds with Arbitrary Spatial Topology

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Numerical Relativity Beyond Astrophysics
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- Representations of arbitrary 3-manifolds.
- Boundary conditions for elliptic and hyperbolic PDEs.
- Numerical tests for solutions of simple PDEs.
- Boundary conditions for Einstein's equation.
- Simple numerical Einstein evolutions.

Representations of Arbitrary 3-Manifolds

- Goal: Develop numerical methods that are easily adapted to solving elliptic PDEs on 3-manifolds Σ with arbitrary topology, and hyperbolic PDEs on manifolds with topology $R \times \Sigma$.

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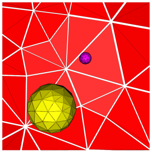

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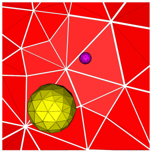
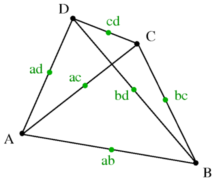


- Cubes make “better” computational domains than tetrahedrons.
- Can arbitrary 3-manifolds be “cubed”, i.e. represented as a set of cubes plus a list of rules for gluing their faces together?

“Cubed” Representations of Arbitrary 3-Manifolds

- Every triangulation can be refined to a ”cubed” representation: divide each tetrahedron into four “distorted” cubes.



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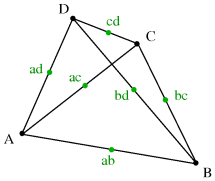

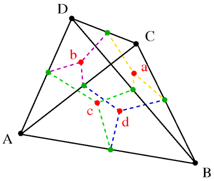

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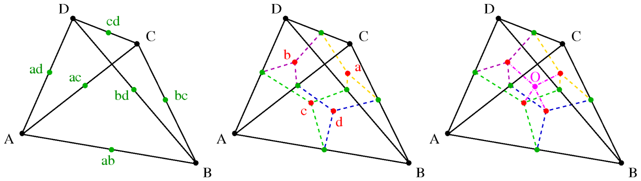

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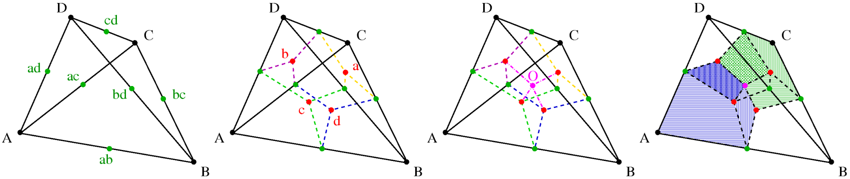

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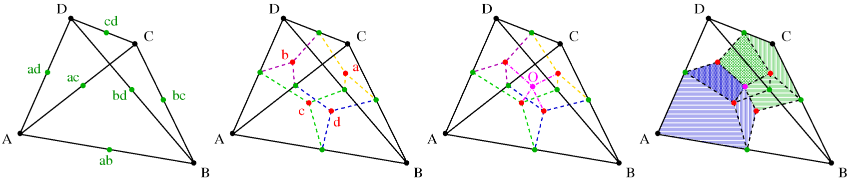
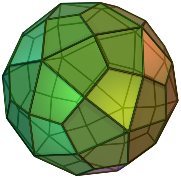
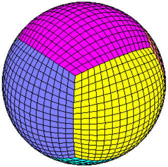
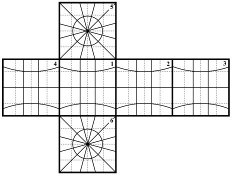


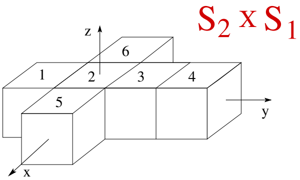
- Every 3-manifold can therefore be represented as a set of cubes, plus maps that identify their faces in the appropriate way.

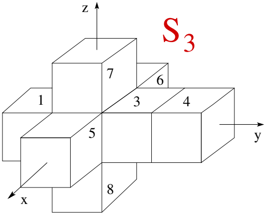






Solving PDEs on Cubed Manifolds





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- Use boundary conditions on cube faces to select the correct smooth global solution.

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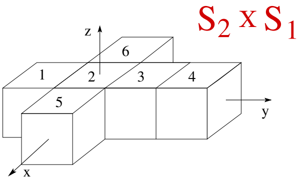

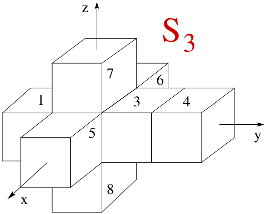

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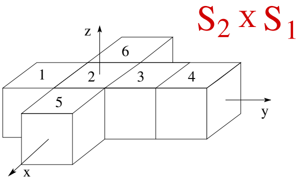

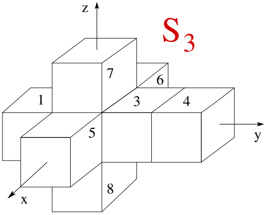

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- For first-order symmetric hyperbolic systems: set incoming characteristic fields with outgoing characteristics from neighbor,

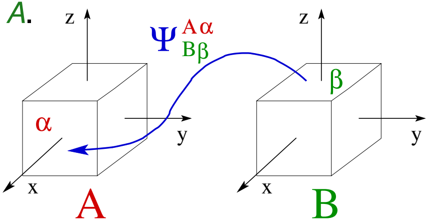
$$\tilde{u}_A^- = \tilde{u}_B^+ \qquad \tilde{u}_B^- = \tilde{u}_A^+.$$

Mapping Boundary Data: Scalars

- Choose the cubic-block coordinate patches to have uniform (coordinate) size and orientation.
- Maps $\Psi^{A\alpha}_{B\beta}$ between boundary faces are linear:

$$x^i_A = c^i_{A\,\alpha} + C^{A\,\alpha\,i}_{B\,\beta\,k}(x^k_B - c^k_{B\,\beta}),$$

where $C^{A\,\alpha\,i}_{B\,\beta\,k}$ is a rotation-reflection matrix, and $c^i_{A\,\alpha}$ is the center of the α face of block A.



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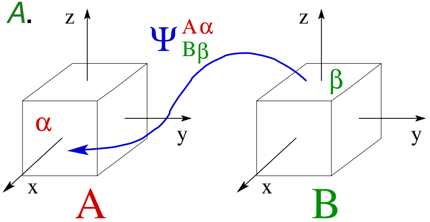

- This map provides the needed boundary transformation law for scalar fields: $\bar{u}_A(x^i_A) \equiv u_B(x^k_B)$, where x^i_A and x^k_B are related by the coordinate boundary map.

Mapping Boundary Data: Tensors

- Jacobian of the boundary coordinate map gives the appropriate transformation law for vectors tangent to the boundary surface:

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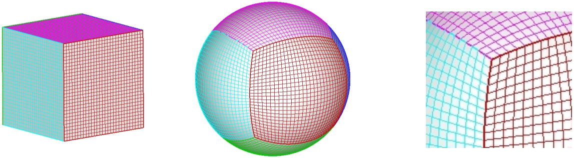
- In general the normal coordinate basis vector $\partial_{A\,\sigma}$ is not the smooth extension of $\partial_{B\,\sigma}$, so a more complicated transformation law is needed for generic vectors.



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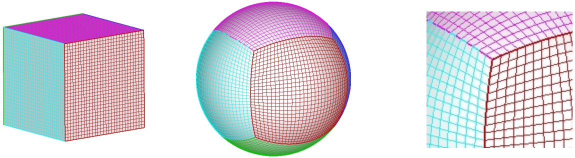

- The outward directed geometrical normals, n^a_A and n^b_B, can be used to define the natural transformation law for smooth vectors, $\bar{v}^a_A(x^i_A) \equiv J^{A\,\alpha\,a}_{B\,\beta\,b} v^b_B(x^k_B)$, with $J^{A\,\alpha\,a}_{B\,\beta\,b} = C^{A\,\alpha\,a}_{B\,\beta\,c}(\delta^c_b - n^c_B n_{Bb}) - n^a_A n_{Bb}$.

Testing the Elliptic PDE Solver

- Solve the elliptic PDE, $\nabla^i \nabla_i \psi - c^2 \psi = f$ where c^2 is a constant, and f is a given function.

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- Use the co-variant derivative ∇_i for the round metric on $S^2 \times S^1$:

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\begin{aligned}
ds^2 &= R_1^2 d\chi^2 + R_2^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),\\
&= \left(\frac{2\pi R_1}{L}\right)^2 dz^2 + \left(\frac{\pi R_2}{2L}\right)^2 \frac{(1+X_A^2)(1+Y_A^2)}{(1+X_A^2+Y_A^2)^2}\\
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where $X_A = \tan\left[\pi(x-c_A^x)/2L\right]$ and $Y_A = \tan\left[\pi(y-c_A^y)/2L\right]$ are “local” Cartesian coordinates in each cubic-block.

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- Let $f = -(\omega^2 + c^2)\psi_A$, where $\psi_A = \Re\left[e^{ik\chi}Y_{\ell m}(\theta,\varphi)\right]$. The angles χ, θ and φ are functions of the coordinates x, y and z.

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- The unique, exact, analytical solution to this problem is $\psi = \psi_A$, when $\omega^2 = \ell(\ell+1)/R_2^2 + k^2/R_1^2$.

Testing the Elliptic PDE Solver II

- Measure the accuracy of the numerical solution ψ_N as a function of numerical resolution N (grid points per dimension) in two ways:
  - First, with the residual $R_N \equiv \nabla^i \nabla_i \psi_N - c^2 \psi_N - f$, and its norm:
    $$\mathcal{E}_R = \sqrt{\frac{\int R_N^2 \sqrt{g} d^3x}{\int f^2 \sqrt{g} d^3x}}.$$
  - Second, with the solution error, $\Delta\psi = \psi_N - \psi_A$, and its norm:
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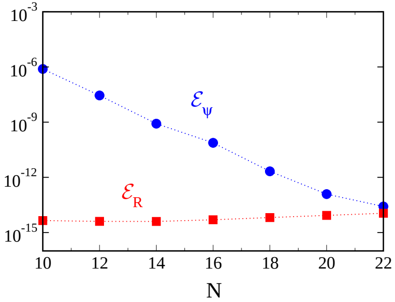

- All these numerical tests were performed by implementing the ideas described here into the Spectral Einstein Code (SpEC) developed originally by the Caltech/Cornell numerical relativity collaboration.

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- Solve the equation $\partial_t^2\psi = \nabla_i\nabla^i\psi$ with given initial data.
- Convert the second-order equation into an equivalent first-order system: $\partial_t\psi = -\Pi$, $\partial_t\Pi = -\nabla^i\Phi_i$ and $\partial_t\Phi_i = -\nabla_i\Pi$ with constraint $\mathcal{C}_i = \nabla_i\psi - \Phi_i$.

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- Choose initial data with $\psi_{t=0} = \Re[Y_{k\ell m}(\chi,\theta,\varphi)]$, $\Pi_{t=0} = -\Re[i\omega Y_{k\ell m}(\chi,\theta,\varphi)]$ and $\Phi_{i\,t=0} = \Re[\nabla_i Y_{k\ell m}(\chi,\theta,\varphi)]$ where $\omega^2 = k(k+2)/R_3^2$.

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  - Second, with the constraint error, $\mathcal{C}_i = \Phi_i - \nabla_i\psi$, and its norm:

$$\mathcal{E}_\mathcal{C} = \sqrt{\frac{\int g^{ij}\mathcal{C}_i\mathcal{C}_j \sqrt{g} d^3x}{\int g^{ij}(\Phi_i\Phi_j + \nabla_i\psi\nabla_j\psi)\sqrt{g} d^3x}}.$$

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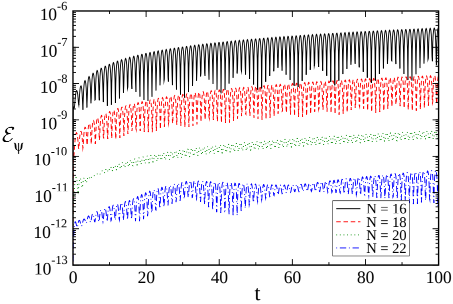

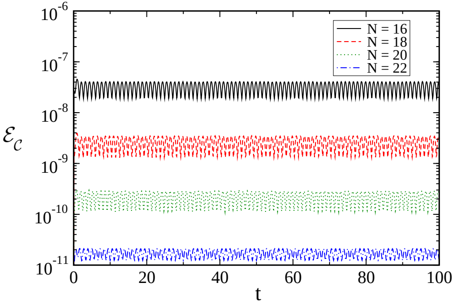

Boundary Conditions for Einstein's Equation

- Einstein's equation can be written as a first-order symmetric hyperbolic system: $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\partial_k u^\beta = F^\alpha(u)$, where u^α includes both spacetime metric ψ_{ab} and derivatives $\partial_c\psi_{ab}$.

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- Incoming characteristic fields must be specified on each boundary.
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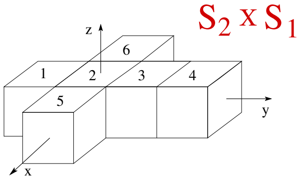

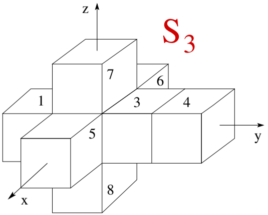

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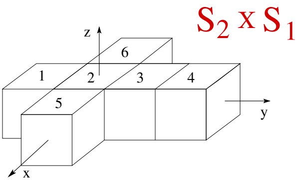

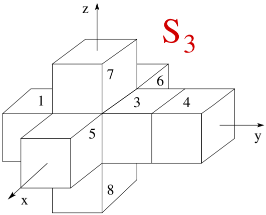

- For the Einstein system, characteristic fields depend on the spacetime metric ψ_{ab} and its derivatives $\partial_c\psi_{ab}$.
- ψ_{ab} and its derivatives $\partial_c\psi_{ab}$ must be mapped between cubic-block regions to construct the needed boundary conditions.

Mapping Boundary Data for Einstein's Equation

- The cubic-block boundary maps have the form

$$t_A = t_B, \qquad x^i_A = c^i_{A\,\alpha} + C^{A\,\alpha\,i}_{B\,\beta\,k}(x^k_B - c^k_{B\,\beta}),$$

where $C^{A\,\alpha\,i}_{B\,\beta\,k}$ is a rotation-reflection matrix.

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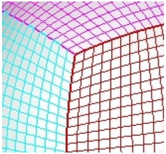
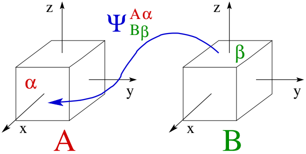
- The Jacobians needed to map tensor fields can be constructed using the outward directed normals, $\tilde{n}^a_A$ and $\tilde{n}^b_B$:

$$J^{A\,\alpha\,a}_{B\,\beta\,b} = C^{A\,\alpha\,a}_{B\,\beta\,c}(\delta^c_b - \tilde{n}^c_B \tilde{n}_{Bb}) - \tilde{n}^a_A \tilde{n}_{Bb}.$$





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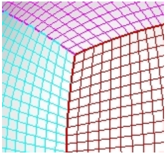

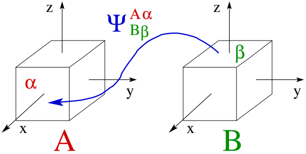

- Assume there exists a smooth (time independent) "reference" metric, whose representation $\tilde{g}_{ab}$ is known in terms of the global cubic-block Cartesian coordinates. Use this metric to construct the normals $\tilde{n}^a_A$, $\tilde{n}^b_B$ and $\tilde{n}_{Bb}$ needed for these boundary Jacobians.

Mapping Boundary Data for Einstein's Equation II

- The physical spacetime metric ψ_{ab} is a tensor mapped across region boundaries using the (inverse) boundary Jacobians:

$$\bar\psi_{Aab} = J^{B\,\beta\,c}_{A\,\alpha\,a} J^{B\,\beta\,d}_{A\,\alpha\,b} \psi_{Bcd}.$$

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- The derivatives of the physical spacetime metric $\partial_c \psi_{ab}$ are mapped across region boundaries using the covariant derivative $\tilde{\nabla}_c$ associated with the smooth reference metric $\tilde{g}_{ab}$.
- The covariant derivative of the physical spacetime metric $\tilde{\nabla}_c \psi_{ab}$ is a tensor mapped by the (inverse) boundary Jacobians:

$$\tilde{\nabla}_{Ac} \bar{\psi}_{Aab} = J^{B\,\beta\,d}_{A\,\alpha\,c} J^{B\,\beta\,e}_{A\,\alpha\,a} J^{B\,\beta\,f}_{A\,\alpha\,b} \tilde{\nabla}_{Bd} \psi_{Bef}.$$

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- The derivatives of the physical metric needed to construct the characteristic fields of the Einstein system are then determined from $\tilde\nabla_{Ac}\bar\psi_{Aab}$:

$$\partial_{Ac}\bar\psi_{Aab} = \tilde\nabla_{Ac}\bar\psi_{Aab} + \tilde\Gamma^d_{Aca}\bar\psi_{Adb} + \tilde\Gamma^d_{Acb}\bar\psi_{Aad}.$$

Testing the Einstein Solver: Non-Linear Gauge Wave

- This simple test evolves the non-linear gauge wave solution,

  $$ds^2 = \psi_{Aab} dx^a dx^b = -(1+F)dt^2 + (1+F)dx^2 + dy^2 + dz^2,$$

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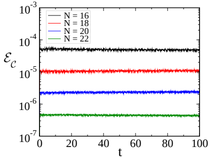

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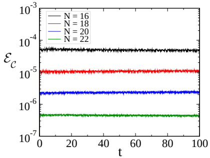

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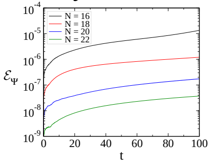

Testing the Einstein Solver: Static Universe on S^3

- Metric initial data is taken from the “Einstein Static Universe” geometry:

$$ds^2 = -dt^2 + R_3^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right],$$

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- This metric solves Einstein's equation with cosmological constant and complex scalar field source on a manifold with spatial topology S^3.
- Evolution of these initial data is the static universe geometry, if the cosmological constant is chosen to be $\Lambda = 1/R_3^2$, and the complex scalar field is $\varphi = \varphi_0 e^{i\mu t}$ with $\mu^2|\varphi_0|^2 = 1/4\pi R_3^2$.

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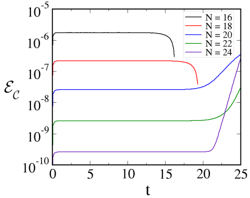

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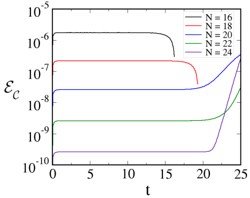

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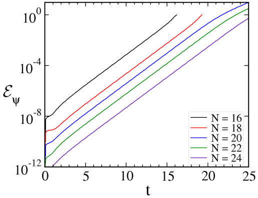

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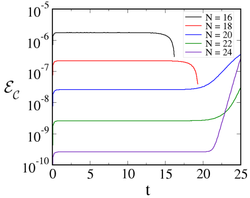

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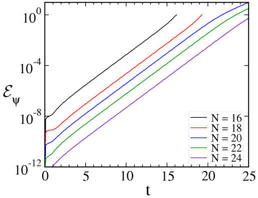

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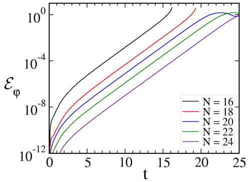

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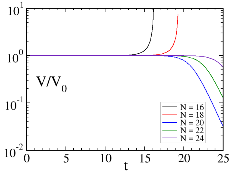

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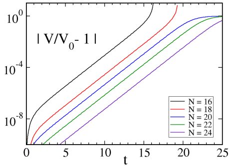

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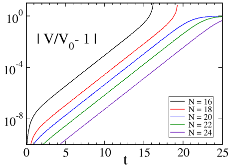

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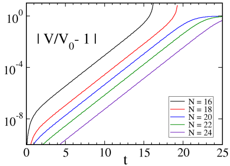

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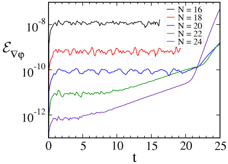

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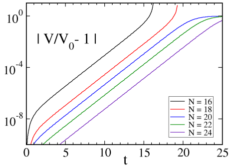

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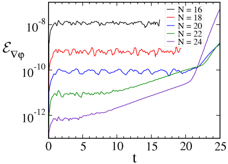

- These solutions appear to be unstable, spatially uniform ($k = 0$) modes of the static Einstein-Klein-Gordon system.

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  - A smooth reference metric $\tilde{g}_{ab}$ to define the global differential structure on this cubic-block representation of the manifold.
- These methods have been tested by solving simple elliptic and hyperbolic equations on several compact manifolds.
- These methods have also been tested by finding simple solutions to Einstein's equation on several compact manifolds.