Solving Einstein's Equations for Binary Black Hole Spacetimes

Lee Lindblom

Theoretical Astrophysics, Caltech

University of Wisconsin at Milwaukee
Department of Physics Colloquium
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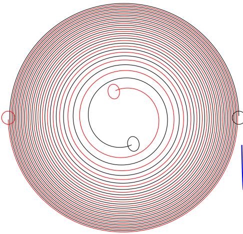

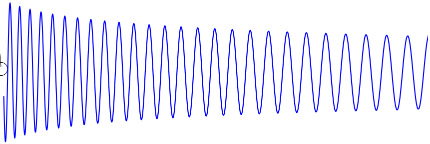

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- Unequal masses – Goddard + Penn State groups (2006).
- Non-zero spins – Brownsville + AEI (2006-07).
- Post merger recoils (up to ~ 4000 km/s) – Jena + AEI + Rochester (2007).
- Large mass ratios (1:10) – Jena (2009).
- Generic spins with precession – Rochester (2009).
- High precision inspiral + merger + ringdown waveforms – AEI + Caltech/Cornell (2009).
- Very large mass ratios (1:100) – Rochester (2010).
- Very high spins ($\chi \approx 0.95$) – Caltech/Cornell (2010).

Outline of Talk:

- Brief History of the BBH Problem.
- Fundamental Einstein Equation Issues.
- Numerical Method Issues.
- Sample of Interesting BBH Evolution Results.

General Relativity Theory

- Einstein's theory of gravitation, general relativity theory, is a geometrical theory in which gravitational effects are described as geometrical structures on spacetime.
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- Coordinates x^a are used to label points in spacetime.
- The spacetime metric determines the physical lengths of curves $x^a(\lambda)$ in spacetime, $L^2 = \pm\int \psi_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}d\lambda$.

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- In these special coordinates the second-order differential operator $\psi^{ab}\partial_a\partial_b$ reduces to the standard wave operator:

$$\psi^{ab}\partial_a\partial_b = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$$

.

General Relativity Theory II

- The spacetime metric ψ_{ab} is determined by Einstein's equation:
  $$R_{ab} - \tfrac{1}{2} R \psi_{ab} = 8\pi T_{ab},$$
  where R_{ab} is the Ricci curvature tensor associated with ψ_{ab}, $R = \psi^{ab} R_{ab}$ is the scalar curvature, and T_{ab} is the stress-energy tensor of the matter present in spacetime.

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- For "vacuum" spacetimes (like binary black hole systems) $T_{ab} = 0$, so Einstein's equations can be reduced to $R_{ab} = 0$.
- The Ricci curvature R_{ab} is determined by derivatives of the metric:
$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{bc} + \Gamma^c{}_{cd} \Gamma^d{}_{ab} - \Gamma^c{}_{ad} \Gamma^d{}_{bc},$$
where $\Gamma^c{}_{ab} = \frac{1}{2} \psi^{cd} (\partial_a \psi_{db} + \partial_b \psi_{da} - \partial_d \psi_{ab})$.

General Relativity Theory III

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- How do we go about solving them?
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- What are the appropriate boundary and/or initial data needed to determine unique solutions to these equations?
- The important fundamental ideas needed to understand and then answer these questions are:
  - gauge freedom,
  - constraints.
- Maxwell's equations are a simpler (and more familiar) system in which these same fundamental issues play analogous roles.

Gauge and Hyperbolicity in Electromagnetism

- Maxwell's equations split into evolution equations and constraints:

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0,$$
$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0.$$

- Introduce Maxwell tensor, F_{ab} with components are $\vec{E}$ and $\vec{B}$. The Maxwell equations then reduce to $\nabla^a F_{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$.

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- Maxwell's equations can be solved in part by introducing a vector potential $F_{ab} = \nabla_a A_b - \nabla_b A_a$. This reduces the system to the single equation: $\nabla^a \nabla_a A_b - \nabla_b \nabla^a A_a = 0$.

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- This form of the equations can be made hyperbolic by choosing the gauge correctly, e.g., let $\nabla^a A_a = H(x, t, A)$, giving:

$$\nabla^a \nabla_a A_b = \left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right) A_b = \nabla_b H.$$

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- Hyperbolic equations (like Maxwell's) can be solved by giving values of the fields, A_a and $\partial_t A_a$, at an initial time $t = 0$, and integrating to determine the fields A_a for future times $t > 0$.

Gauge and Hyperbolicity in General Relativity

- The spacetime Ricci curvature tensor can be written as:

$$R_{ab} = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi, \partial\psi),$$

where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi_{ad}\psi^{bc}\Gamma^d{}_{bc}$.

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- Like Maxwell's equations, these equations can not be solved without specifying suitable gauge conditions.
- The gauge freedom in general relativity theory is the freedom to choose any coordinates x^a on spacetime. Solving the equations requires some specific choice of coordinates be made.
- One way to impose the needed gauge conditions is to specify H^a, the source term for a wave equation for each coordinate x^a:

$$H^a = \nabla^c\nabla_c x^a = \psi^{bc}(\partial_b\partial_c x^a - \Gamma^e{}_{bc}\partial_e x^a) = -\Gamma^a,$$

where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$.

Einstein's Equation with the GH Method

- The spacetime Ricci tensor can be written as:

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  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$.

- The Generalized Harmonic Einstein equation is obtained by replacing Γ_a with $-H_a(x,\psi) = -\psi_{ab}H^b(x,\psi)$:

$$R_{ab} - \nabla_{(a}\left[\Gamma_{b)} + H_{b)}\right] = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(\psi,\partial\psi).$$

- The vacuum GH Einstein equation, $R_{ab} = 0$ with $\Gamma_a + H_a = 0$, is therefore manifestly hyperbolic, having the same principal part as Maxwell's equation:

$$\nabla^c\nabla_c A_a = \psi^{cd}\partial_c\partial_d A_a + Q_a(A,\partial A) = \nabla_b H.$$

- Einstein's equations can be solved by specifying initial values of ψ_{ab} and $\partial_t\psi_{ab}$ (subject to the constraints $\mathcal{C}_a = \Gamma_a + H_a = 0$ and $\partial_t\mathcal{C}_a = \partial_t\Gamma_a + \partial_t H_a = 0$) at $t = 0$, and then integrating to find the solutions for $t > 0$.

The Constraint Problem

- Fixing the gauge in an appropriate way makes the Einstein equations hyperbolic, so the initial value problem becomes well-posed mathematically.
- In a well-posed representation, the constraints, $\mathcal{C} = 0$, remain satisfied for all time if they are satisfied initially.

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- There is no guarantee, however, that constraints that are "small" initially will remain "small".
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- Constraint violating instabilities were one of the major problems that made progress on binary black hole solutions so slow.
- Special representations of the Einstein equations are needed that control the growth of any constraint violations.
- Constraints play a similar role in Maxwell’s equation, so we will discuss that simpler case first.

Constraint Damping in Electromagnetism

- Electromagnetism is described as the hyperbolic evolution equation $\nabla^a\nabla_a A_b = \nabla_b H$.
  Where have the usual $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ constraints gone?

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- Gauge condition becomes a constraint: $0 = \mathcal{C} \equiv \nabla^a A_a - H$.
- Maxwell's equations imply that this constraint is preserved:

$$\nabla^a\nabla_a \left(\nabla^b A_b - H\right) = \nabla^a\nabla_a \mathcal{C} = 0.$$

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- Modify evolution equations by adding multiples of the constraints:

$$\nabla^a\nabla_a A_b = \nabla_b H + \gamma_0 t_b\,\mathcal{C} = \nabla_b H + \gamma_0 t_b\left(\nabla^a A_a - H\right).$$

- These changes effect the constraint evolution equation,

$$\nabla^a\nabla_a\mathcal{C} - \gamma_0 t^b\nabla_b\mathcal{C} = 0,$$

  so constraint violations are damped when $\gamma_0 > 0$.

Generalized Harmonic Evolution System

- A similar constraint damping mechanism exists for the GH evolution system:

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

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- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to $\mathcal{C}_a$:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0 \left[t_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\, t^c\, \mathcal{C}_c \right],$$

  where t^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the basic hyperbolic structure of the system.

Constraint Damping Generalized Harmonic System

- Evolution of constraints $\mathcal{C}_a$ follow from the Bianchi identities.
- Apply them to the (trace reversed) Pretorius evolution system,

$$0 = R_{ab} - \tfrac{1}{2}\psi_{ab}R - \nabla_{(a}\mathcal{C}_{b)} + \tfrac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c + \gamma_0 t_{(a}\mathcal{C}_{b)},$$

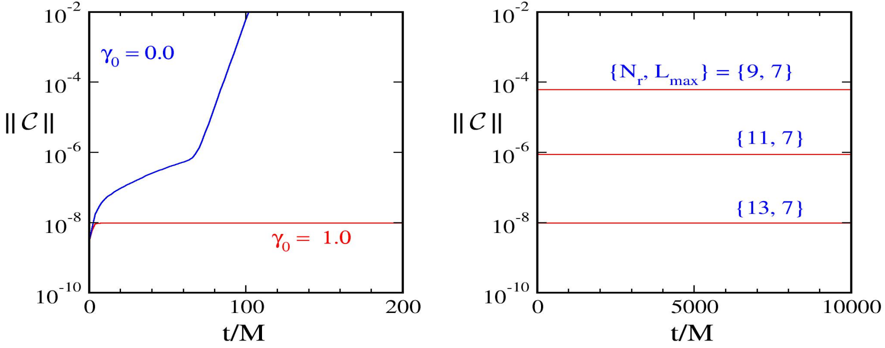
to obtain

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[t_{(c}\,\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \tfrac{1}{2}\gamma_0\; t_a\,\mathcal{C}^c\mathcal{C}_c.$$

- This damped wave equation for $\mathcal{C}_a$ drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

Numerical Tests of the GH Evolution System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = 1$.



Bad Old BBH Movie

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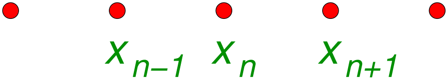
$$\partial_t u = Q(u, \partial_x u, x, t).$$

- Choose a grid of spatial points, x_n.



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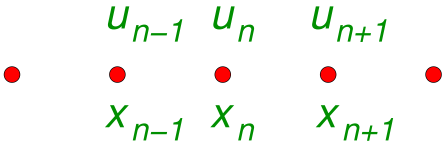

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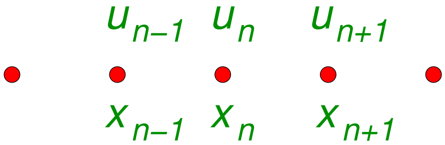

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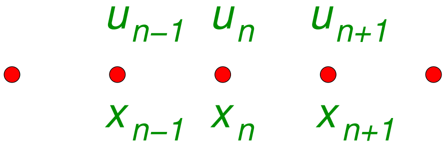

- Approximate the spatial derivatives at the grid points $\partial_x u(x_n) = \sum_k D_{n\,k} u_k$.
- Evaluate Q at the grid points x_n in terms of the u_k: $Q(u_k, x_n, t)$.

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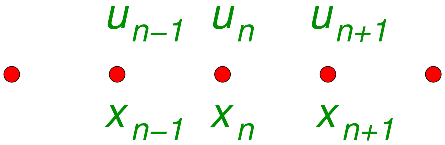

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- Evaluate Q at the grid points x_n in terms of the u_k: $Q(u_k, x_n, t)$.
- Solve the coupled system of ordinary differential equations,

$$\frac{du_n(t)}{dt} = Q[u_k(t), x_n, t],$$

using standard numerical methods (e.g. Runge-Kutta).

Basic Numerical Methods

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- Grid spacing decreases as the number of grid points N increases, $\Delta x \sim 1/N$. Errors in finite difference methods scale as N^{-p}.
- Most groups now use finite difference codes with $p = 6$ or $p = 8$.

Basic Numerical Methods II

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- Estimate the errors (e.g. for Fourier series of *smooth* functions):

$$\tilde{u}_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-iNx} dx \leq \frac{1}{N^p} \max \left| \frac{d^p u(x)}{dx^p} \right|.$$

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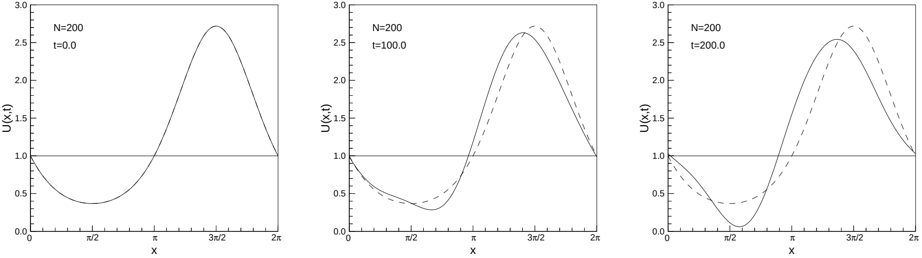
- Errors in spectral methods decrease faster than any power N^p.
- This means that a given level of accuracy can be achieved using many fewer grid points with spectral methods.

Comparing Different Numerical Methods

- Wave propagation with second-order finite difference method:



Figures from Hesthaven, Gottlieb, & Gottlieb (2007).

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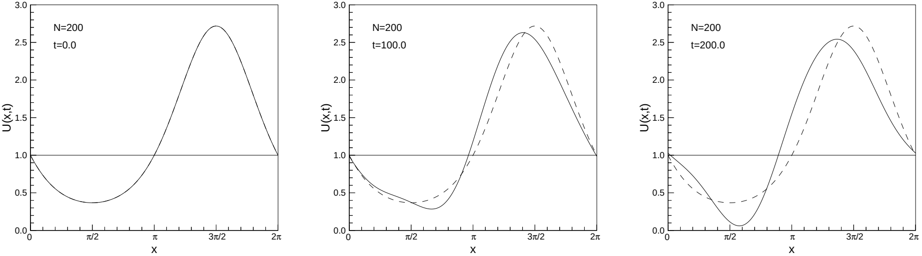
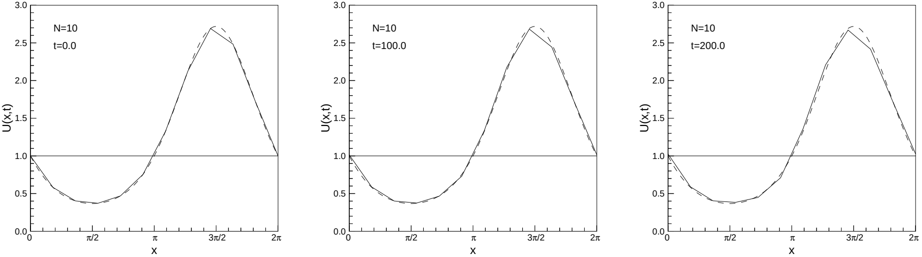


- Wave propagation with spectral method:



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Moving Black Holes in a Spectral Code

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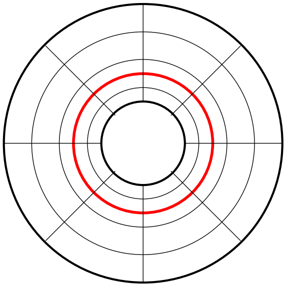

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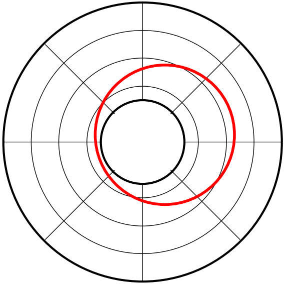

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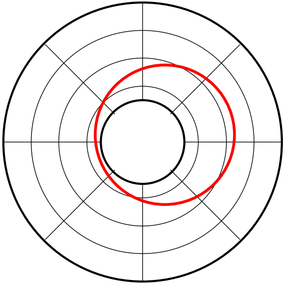

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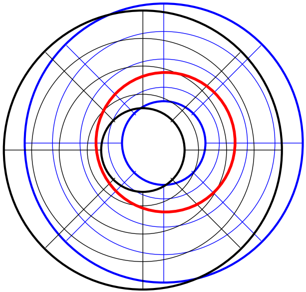

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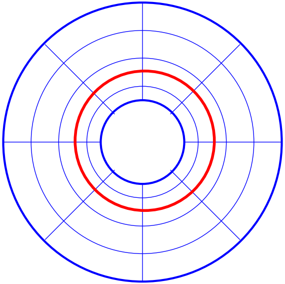

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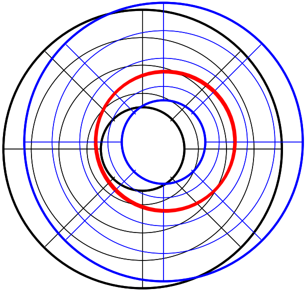

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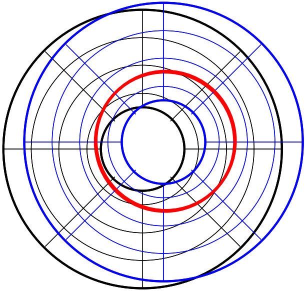

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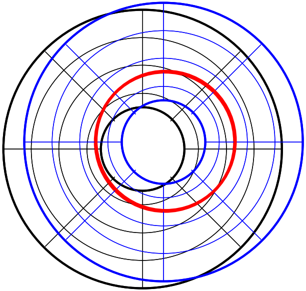

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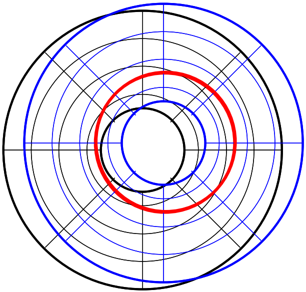

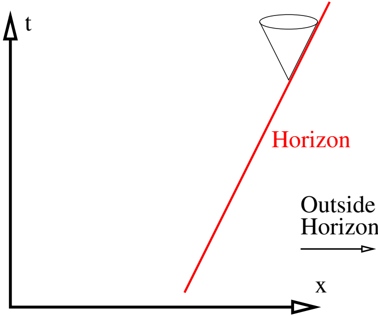

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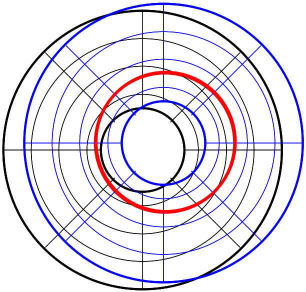

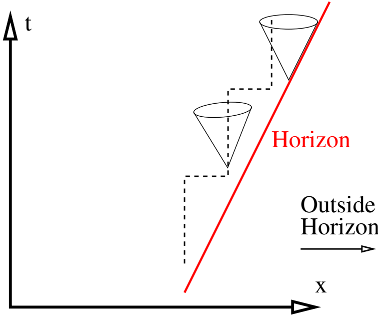

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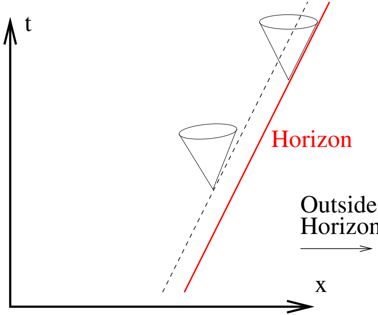
- Solution:

  Choose coordinates that smoothly track the location of the black hole.

  For a black hole binary this means using coordinates that rotate with respect to inertial frames at infinity.



Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- This can be implemented by using a coordinate transformation from inertial coordinates, $\bar{x}^i$, to co-moving coordinates x^i, consisting of a translation followed by a rotation followed by an expansion:

$$\begin{aligned} x^i &= a(\bar{t})\, R^{(z)\,i}{}_j[\varphi(\bar{t})]\, R^{(y)\,j}{}_k[\xi(\bar{t})]\, \left[\bar{x}^k - c^k(\bar{t})\right], \\ t &= \bar{t}. \end{aligned}$$

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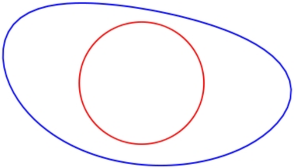
- This transformation keeps the holes fixed in co-moving coordinates for suitably chosen $a(\bar{t})$, $\varphi(\bar{t})$, $\xi(\bar{t})$, and $c^k(\bar{t})$.
- Motions of the holes are not known *a priori*, so $a(\bar{t})$, $\varphi(\bar{t})$, $\xi(\bar{t})$, and $c^k(\bar{t})$ must be chosen dynamically and adaptively.
- A simple feedback-control system has been used to choose $a(\bar{t})$, $\varphi(\bar{t})$, $\xi(\bar{t})$, and $c^k(\bar{t})$ by fixing the black-hole positions, even in evolutions with precession and “kicks”.

Horizon Distortion Maps

- Tidal deformation, along with kinematic and gauge effects cause the shapes of the black holes to deform:



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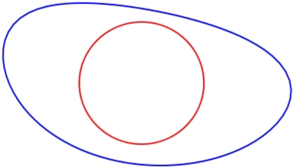

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  - Some points on the excision boundary are much deeper inside the singular black hole interior. Numerical errors and constraint violations are largest there, sometimes leading to instabilities.

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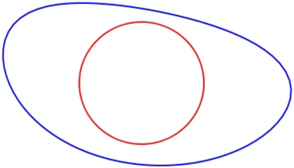

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  - Some points on the excision boundary are much deeper inside the singular black hole interior. Numerical errors and constraint violations are largest there, sometimes leading to instabilities.
  - When the horizons move relative to the excision boundary points, the excision boundary can become timelike, and boundary conditions are then needed there.

Horizon Distortion Maps II

- Adjust the placement of grid points near each black hole using a horizon distortion map that connects grid coordinates x^i to points in the black-hole rest frame $\tilde{x}^i$:

$$\begin{aligned}\tilde{\theta}_A &= \theta_A, \qquad \tilde{\varphi}_A = \varphi_A,\\ \tilde{r}_A &= r_A - f_A(r_A, \theta_A, \varphi_A) \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \lambda_A^{\ell m}(t) Y_{\ell m}(\theta_A, \varphi_A).\end{aligned}$$

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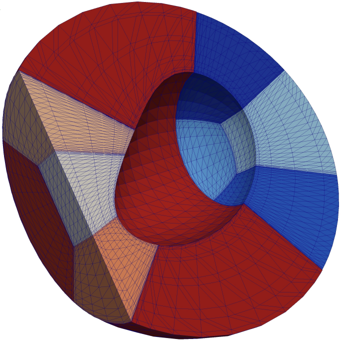
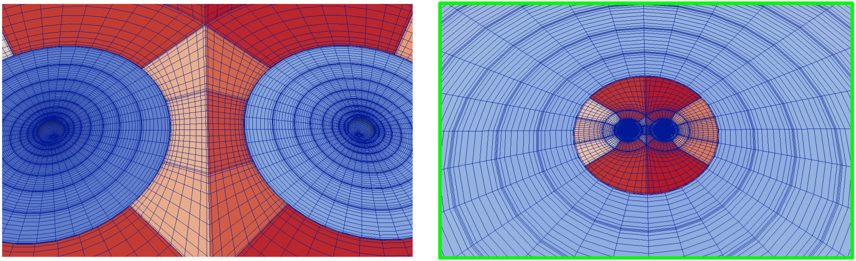
- Adjust the coefficients $\lambda_A^{\ell m}(t)$ using a feedback-control system to keep the excision surface the same shape and slightly smaller than the horizon, and to keep the boundary spacelike.
- Choose f_A to scale linearly from $f_A = 1$ on the excision boundary, to $f_A = 0$ on cut sphere.



Caltech/Cornell Spectral Einstein Code (SpEC):

- Multi-domain pseudo-spectral evolution code.



Lovelace, Scheel, & Szilágyi (2010) high spin evolution grids.

- Constraint damped "generalized harmonic" Einstein equations:

$$\psi^{cd}\partial_c\partial_d\psi_{ab} = Q_{ab}(\psi, \partial\psi).$$

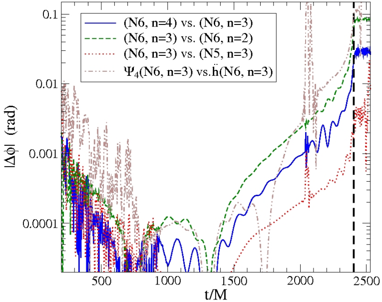
- Dual frame evolutions with horizon tracking and distortion maps.
- Constraint-preserving, physical and gauge boundary conditions.
- Spectral AMR.   Event Horizon Movie

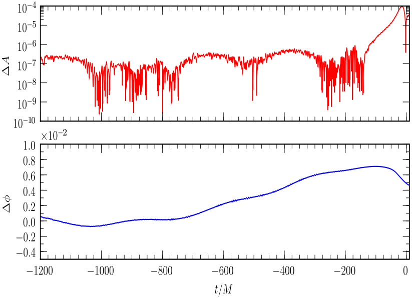
Outline of Talk:

- Brief History of the BBH Problem.
- Fundamental Einstein Equation Issues.
- Numerical Method Issues.
- Sample of Interesting BBH Evolution Results.

Accurate Long Waveform Simulations

- Numerical waveforms must be accurate enough to satisfy LIGO's data analysis requiements.
- Numerical waveforms must be long enough to allow matching onto PN or EOB waveforms without loss of accuracy.

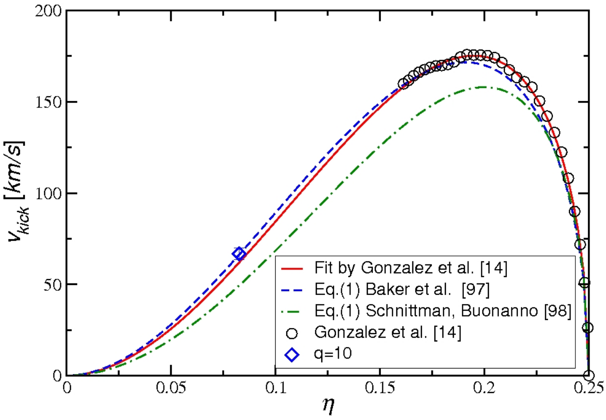




- Recent Caltech/Cornell: accurate aligned-spin waveforms, Pan, et al. (2010).
- AEI/LSU: accurate non-spinning waveforms, Pollney, et al. (2010).

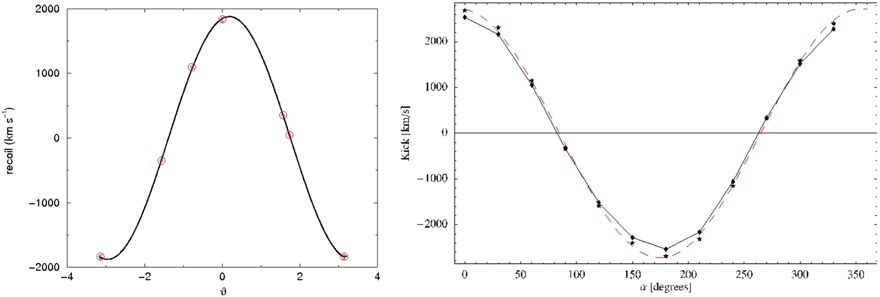
Post-Merger Recoils

- Mergers of asymmetric binaries (unequal masses and/or unequal or nonaligned spins) emit gravitational waves asymmetrically.
- Resulting single black hole has a “kick” velocity relative to the pre-merger center of mass.
- Kicks in asymmetric non-spinning binaries first studied by the Penn State and Jena groups (2006-07).
- Figure from González, Sperhake, and Brügmann (2009).



Post-Merger Recoils with Spin

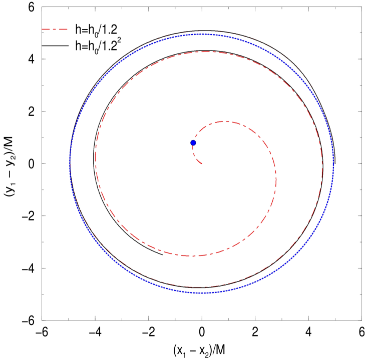
- Mergers of spinning black-hole binaries can result in large recoils.
- Maximum kicks are produced by mergers with anti-parallel spins tangent to the orbital plane.

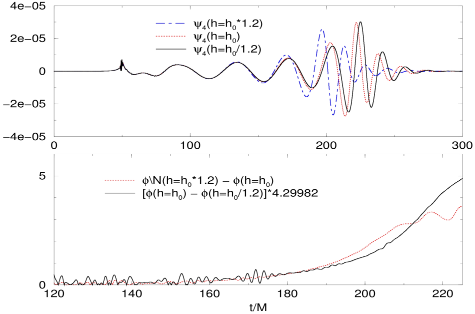


- Campanelli, et al. (2007): Kick velocities as function of orbital phase for black holes with spin $\chi \approx 0.5$.
- Brügmann, et al. (2007): Analogous results for black holes with spin $\chi \approx 0.72$.
- Maximum kick velocity $v_{\text{max}} \approx 4000$ km/s predicted for maximum spin, $\chi_1 = -\chi_2 = 1$, equal-mass black-hole mergers.

Very High Mass Ratios

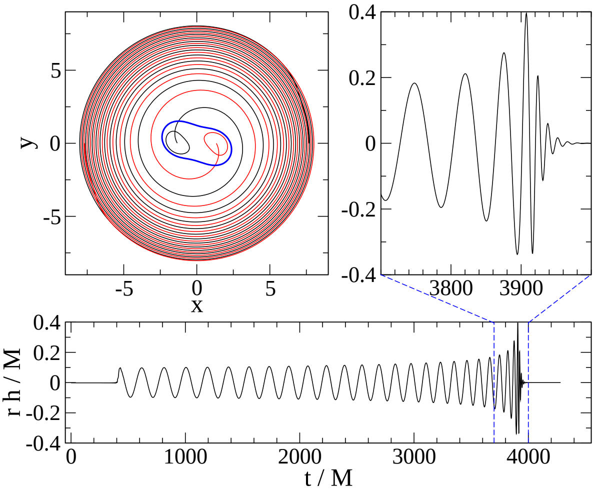
- Numerical simulation of high mass-ratio binaries is very difficult:
  - Very high spatial resolution needed near the smaller black hole.
  - Time steps set by the smallest spatial resolution (explicit schemes).
  - Radiation reaction timescale proportional to mass ratio $M/m \gg 1$, so many orbits required to achieve merger.
- Jena group performed $M/m \approx 10$ simulations (2009), RIT group later performed $M/m \approx 100$ simulations (2010).





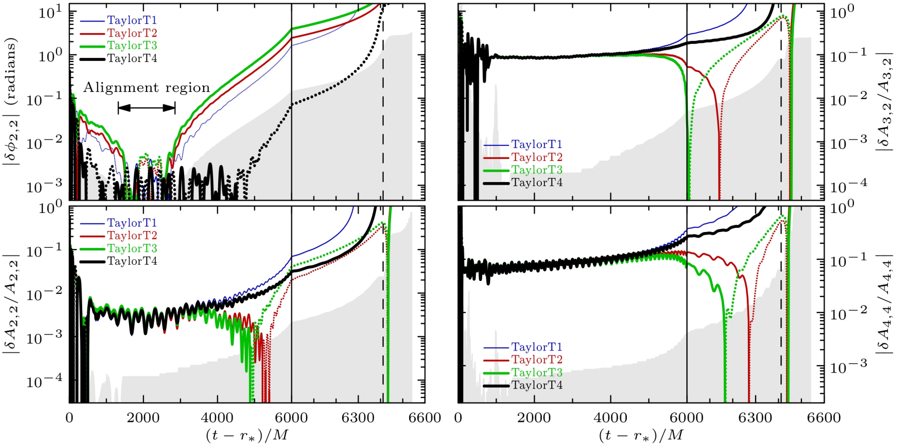
High Spin Evolutions

- Lovelace, Scheel, & Szilágyi (2010) use high spin conformal initial data from superimposed boosted Kerr-Schild black holes.
- Spins $\chi \approx 0.95$ anti-aligned with orbital angular momentum.
- Evolve through 12.5 orbits, merger, and ringdown.
- High accuracy gravitational waveform extracted.



High Spin Evolutions II

- Lovelace, Boyle, Scheel, & Szilágyi (2011) evolve initial data with high spins, $\chi \approx 0.97$, aligned with orbital angular momentum, for about 25.5 orbits followed by merger and ringdown.
- Comparisons of numerical waveforms with PN waveforms:



Summary

- The NR community has made great progress on a number of fundamental problems:
  - Hyperbolic representations of GR: GH and BSSN and ...
  - Constraint violating instabilities controlled.
  - Plus several others (that there wasn't time to talk about), e.g. good gauge conditions, outer boundary conditions, ...
- Great progress on numerical and code development issues:
  - Higher order FD and spectral numerical methods.
  - Methods of controlling boundaries inside black holes: Excision plus dual-frame dynamical horizon-tracking coordinates using feedback-control.
  - Plus several others, e.g. Moving Puncture method, Adaptive Mesh Refinement (AMR), ...
- Interesting physical results:
  - Long acccurate waveforms for GW data analysis.
  - Large astrophysically interesting post-merger kicks.