A ‘New’ Generalized-Harmonic Evolution System

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- Generalized Harmonic (GH) coordinates have two interesting properties exploited recently by Frans Pretorius to perform some very impressive numerical evolutions of binary black-hole spacetimes.
- Outline of this talk:
  - How these special properties of GH coordinates make stable numerical evolutions possible.
  - Extending the GH system in a way that makes the formulation of appropriate boundary conditions easier, etc.
  - Numerical tests of the new extended GH evolution system.

Methods of Specifying Spacetime Coordinates

- The lapse N and shift N^i are generally used to specify how coordinates are layed out on a spacetime manifold: $\partial_t = Nt^a + N^k\partial_k$.
- An alternate way to specify how the coordinates are layed out on spacetime is through the generalized harmonic gauge source function $H_a(x)$:
  - Let $H_a(x)$ denote the function obtained by the action of the scalar wave operator on the coordinates x^b:

$$H_a(x) \equiv \psi_{ab}\nabla^c\nabla_c x^b = -\Gamma_a,$$

    where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$.
  - Specifying *generalized harmonic* (GH) coordinates is accomplished by choosing a gauge-source function $H_a(x)$, and requiring that $H_a(x) = -\Gamma_a$.

Important Properties of GH Coordinates

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function: The Ricci tensor may be written as

$$R_{ab} \quad = \quad -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + F_{ab}(\psi, \partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$. The vacuum Einstein equation, $R_{ab} = 0$, has the same principal part as the scalar wave equation when the GH gauge condition, $H_a(x) = -\Gamma_a$, is imposed.

- Imposing coordinates using a GH gauge function changes the nature of the constraints of the Einstein system in a profound way. The GH constraint equation, $\mathscr{C}_a = 0$, where

$$\mathscr{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathscr{M}_a = 0$, are determined by the derivatives of the gauge constraint $\mathscr{C}_a$:

$$\mathscr{M}_a \equiv G_{ab}t^b = t^b\left(\nabla_{(a}\mathscr{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathscr{C}_c\right).$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\, t_{(a}\mathcal{C}_{b)} - \frac{1}{2}\gamma_0\, \psi_{ab}\, t^c\,\mathcal{C}_c,$$

  where t^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

- The evolution of the constraints $\mathcal{C}_a$ for this system can be deduced from the Bianci identities:

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[t_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \frac{1}{2}\gamma_0\, t_a\mathcal{C}^c\mathcal{C}_c.$$

- This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

First Order Generalized Harmonic Evolution System

- We prefer to solve first-order equations numerically. (More is known about first-order systems: how to formulate well-posed boundary conditions, when shocks form, etc.)
- Kashif Alvi (2002) derived a nice (symmetric hyperbolic) first-order form for the generalized-harmonic evolution system:

$$\begin{aligned}
\partial_t\psi_{ab} - N^k\partial_k\psi_{ab} &= -N\,\Pi_{ab},\\
\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} &\simeq 0,\\
\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} &\simeq 0,
\end{aligned}$$

  where $\Phi_{kab} = \partial_k\psi_{ab}$.

- This system has two immediate problems:
  - This system has new constraints, $\mathscr{C}_{iab} = \partial_i\psi_{ab} - \Phi_{iab}$, that tend to grow exponentially during numerical evolutions.
  - This system is not linearly degenerate, so it is possible (likely) that shocks will develop (e.g. the shift evolution equation is of the form $\partial_t N^i - N^k\partial_k N^i \simeq 0$).

A 'New' Generalized Harmonic Evolution System

- We can correct these problems by adding additional multiples of the constraints to the evolution system:

$$\begin{aligned}
\partial_t \psi_{ab} - (1+\gamma_1) N^k \partial_k \psi_{ab} &= -N\Pi_{ab} - \gamma_1 N^k \Phi_{kab}, \\
\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + N g^{ki} \partial_k \Phi_{iab} - \gamma_1 \gamma_2 N^k \partial_k \psi_{ab} &\simeq -\gamma_1 \gamma_2 N^k \Phi_{kab}, \\
\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} - \gamma_2 N \partial_i \psi_{ab} &\simeq -\gamma_2 N \Phi_{iab},
\end{aligned}$$

- This 'new' generalized-harmonic evolution system has several nice properties:
  - This system is linearly degenerate for $\gamma_1 = -1$ (and so shocks should not form from smooth initial data).
  - The Φ_{iab} evolution equation can be written in the form, $\partial_t \mathscr{C}_{iab} - N^k \partial_k \mathscr{C}_{iab} \simeq -\gamma_2 N \mathscr{C}_{iab}$, so the new constraints are damped when $\gamma_2 > 0$.
  - This system is symmetric hyperbolic for all values of the parameters γ_1 and γ_2.

Constraint Evolution for the New GH System

- The evolution of the constraints, $c^A = \{\mathscr{C}_a, \mathscr{C}_{kab}, \mathscr{F}_a \approx t^c\partial_c\mathscr{C}_a, \mathscr{C}_{ka} \approx \partial_k\mathscr{C}_a, \mathscr{C}_{klab} = \partial_{[k}\mathscr{C}_{l]ab}\}$ are determined by the evolution of the dynamical fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$:

$$\partial_t c^A + A^{kA}{}_B(u)\partial_k c^B = F^A{}_B(u,\partial u)\, c^B.$$

- This constraint evolution system is symmetric hyperbolic with principal part:

$$\begin{aligned}
\partial_t\mathscr{C}_a &\simeq 0,\\
\partial_t\mathscr{F}_a - N^k\partial_k\mathscr{F}_a - Ng^{ij}\partial_i\mathscr{C}_{ja} &\simeq 0,\\
\partial_t\mathscr{C}_{ia} - N^k\partial_k\mathscr{C}_{ia} - N\partial_i\mathscr{F}_a &\simeq 0,\\
\partial_t\mathscr{C}_{iab} - (1+\gamma_1)N^k\partial_k\mathscr{C}_{iab} &\simeq 0,\\
\partial_t\mathscr{C}_{ijab} - N^k\partial_k\mathscr{C}_{ijab} &\simeq 0.
\end{aligned}$$

- An analysis of this system shows that all of the constraints are damped in the WKB limit when $\gamma_0 > 0$ and $\gamma_2 > 0$. So, this system has constraint suppression properties that are similar to those of the Pretorius (and Gundlach, et al.) system.

Boundary Condition Basics

- Boundary conditions are imposed on first-order hyperbolic evolutions systems,

$$\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\partial_k u^\beta \quad = \quad F^\alpha(u),$$

  in the following way:

- Find the eigenvectors of the characteristic matrix $n_k A^{k\,\alpha}{}_\beta$ at each boundary point:

$$e^{\hat\alpha}{}_\alpha\, n_k A^{k\,\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta$$

  where n_k is the outward directed unit normal.

- Define the characteristic fields:

$$u^{\hat\alpha} = e^{\hat\alpha}{}_\alpha u^\alpha.$$

  (For hyperbolic evolution systems the eigenvectors $e^{\hat\alpha}{}_\alpha$ are complete so $\det e^{\hat\alpha}{}_\alpha \neq 0$.)

- A boundary condition must be imposed on every incoming characteristic field (*i.e.* every field with $v_{(\hat\alpha)} < 0$), and must not be imposed on any outgoing field (*i.e.* any field with $v_{(\hat\alpha)} > 0$).

Boundary Condition Basics II

- Why are these incoming characteristic fields the right ones to impose boundary conditions on?
- Consider the propagation of short wavelength perturbations (the WKB limit) normal to the boundary. Multiply the evolution equations,

$$\partial_t \delta u^\alpha + n_k A^{k\,\alpha}{}_\beta(u)\partial_\perp \delta u^\beta \quad \approx \quad 0,$$

  by the characteristic eigenvector matrix $e^{\hat\alpha}{}_\alpha$, to obtain a decoupled set of evolution equations for the perturbed characteristic fields:

$$\partial_t \delta u^{\hat\alpha} + v_{(\hat\alpha)}\partial_\perp \delta u^{\hat\alpha} \quad \approx \quad 0.$$

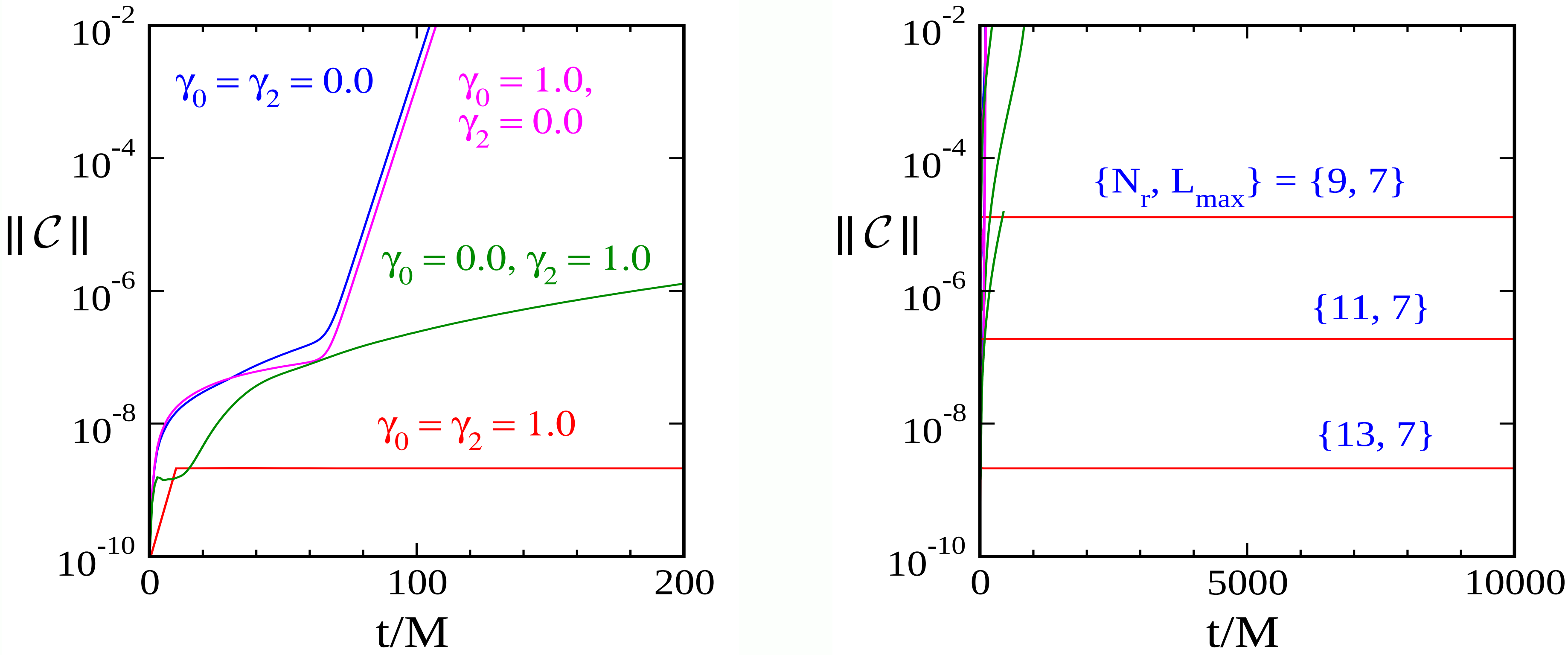
- The solutions to these perturbed characteristic evolution equations (in the WKB limit) are simple traveling waves of the form,

$$\delta u^{\hat\alpha} = \delta u^{\hat\alpha}(x_\perp - v_{(\hat\alpha)}t).$$

- The incoming characteristic waves, those with $v_{(\hat\alpha)} < 0$, must have boundary conditions imposed on them. It is inconsistent to impose any boundary condition on the outgoing characteristic waves, those with $v_{(\hat\alpha)} > 0$.

Numerical Tests of the ‘New’ GH System

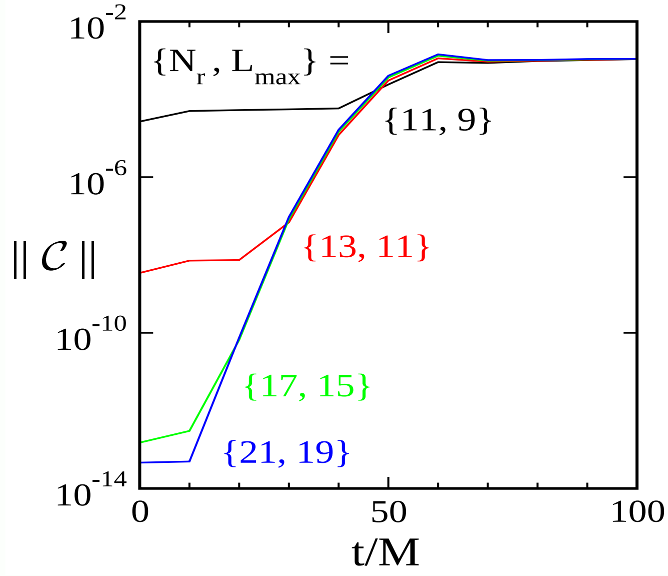
- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of this new GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.



- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

Evolutions of a Perturbed Schwarzschild Black Hole

- A more stringent numerical test uses Schwarzschild initil data and a boundary condition on the incoming physical GW characteristic projection of $\psi_{ab} = f(t)(\hat{x}^a\hat{x}^b + \hat{y}^a\hat{y}^b - 2\hat{z}^a\hat{z}^b)$ having a time profile $f(t) = \mathscr{A}\, e^{-(t-t_p)^2/w^2}$ with $\mathscr{A} = 10^{-3}$, $t_p = 60M$, and $w = 10M$.
- Evolutions using simple "freezing" boundary conditions on the remaining incoming characteristic fields are stable and convergent. But these solutions have non-negligable constraint violations (which do not converge away).



- Better boundary conditions are needed to prevent the influx of constraint violations.

Constraint Preserving Boundary Conditions

- The evolution of the GH constraints are described by a hyperbolic evolution system:

$$\partial_t c^A + A^{kA}{}_B(u)\partial_k c^B = F^A{}_B(u,\partial u)\, c^B.$$

- Find the characteristic constraint fields $c^{\hat{A}} = e^{\hat{A}}{}_A c^A$.
- Divide the constraints into incoming and outgoing fields:

$$\mathbf{c} = \{\mathbf{c}^-, \mathbf{c}^+\}.$$

- Use the definitions of the constraints to re-express them in terms of the principal characteristic fields:

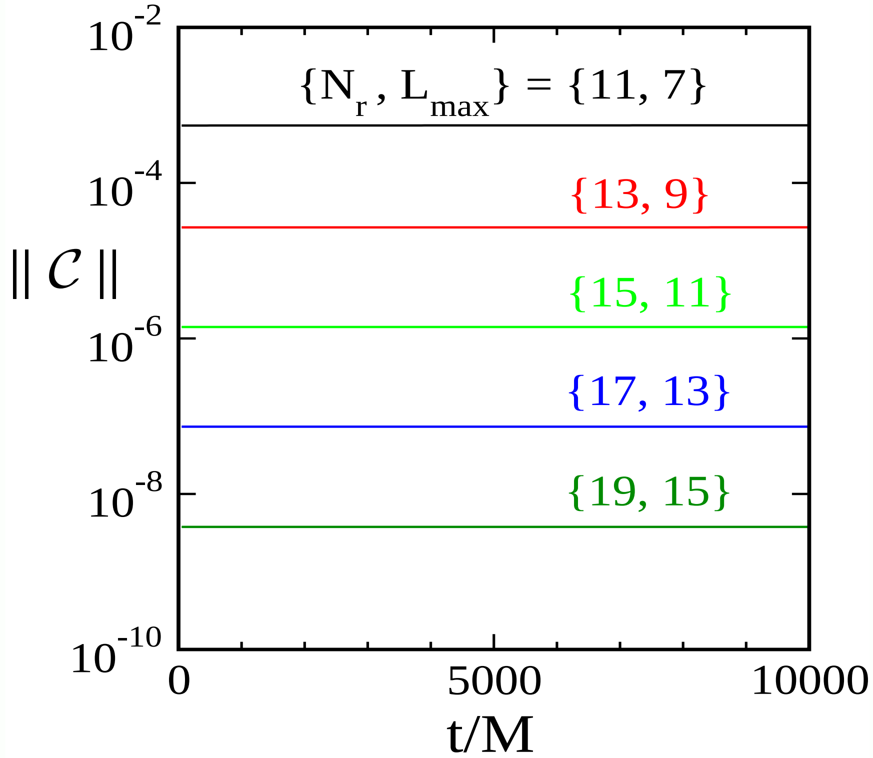
$$\mathbf{c}^- = d_\perp \mathbf{u}^- + \mathbf{f}^-(\partial_\| u, u),$$

  where $\mathbf{f}^-(\partial_\| u, u)$ indicates terms with derivatives of the fields tangent to the boundary, and terms algebraic in the fields.
- Set boundary conditions on those incoming dynamical fields that will ensure the incoming constraint fields are zero at the boundaries:

$$d_\perp \mathbf{u}^- = -\mathbf{f}^-(\partial_\| u, u).$$

Numerical Tests of Constraint Preserving Boundary Conditions

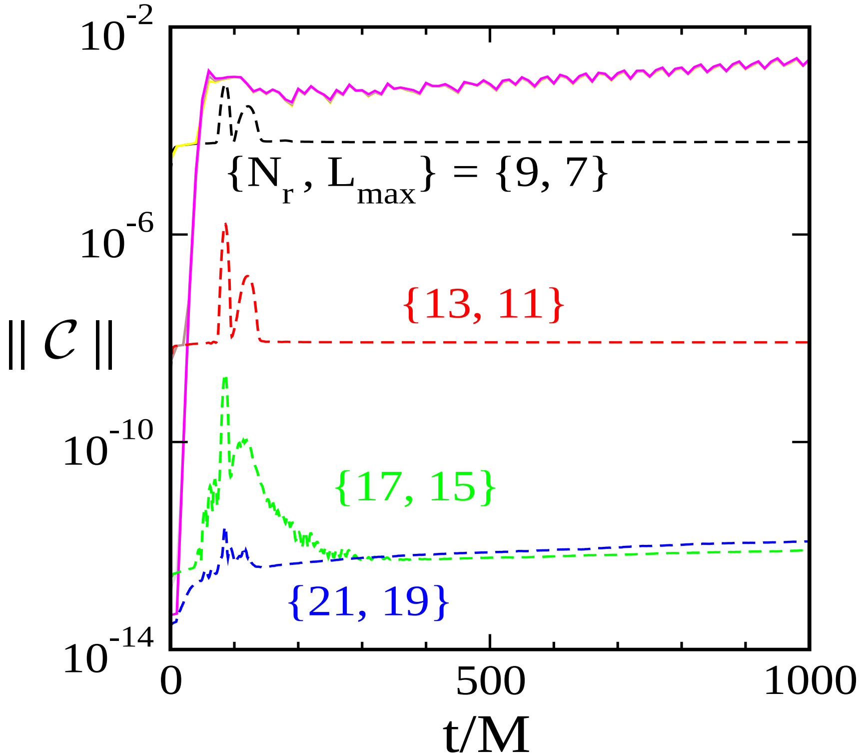
- 3D numerical evolutions of Kerr with $\vec{s} = (0.1, 0.2, 0.3)$ using $\gamma_0 = \gamma_2 = 1$, constraint preserving boundary conditions, and physical boundary conditions that set $\partial_t \Psi_0 = 0$:

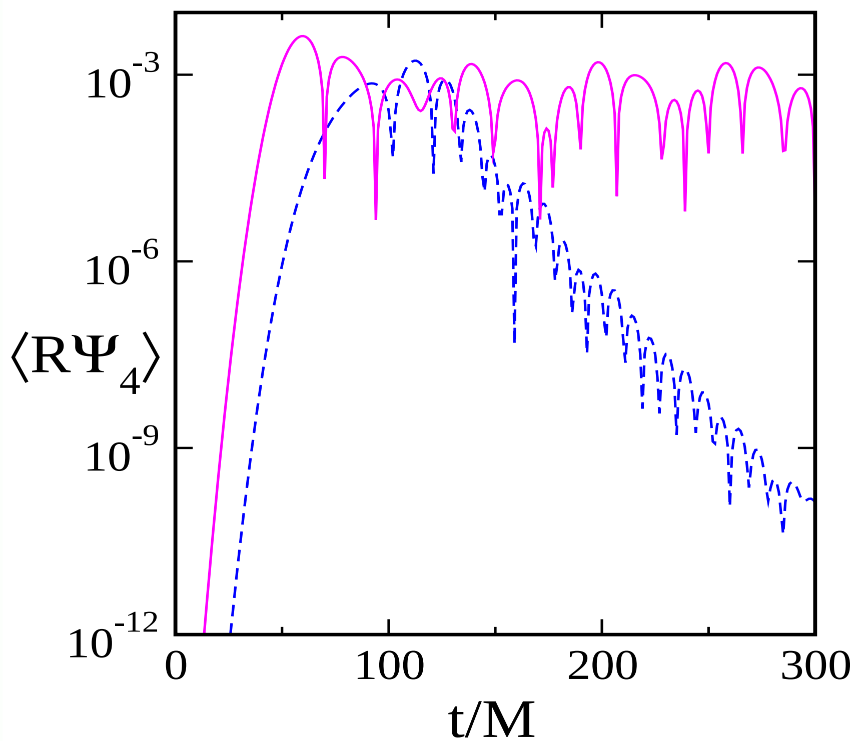


- These evolutions are also stable and convergent when $\gamma_0 = \gamma_2 = 1$, even though no rigorous well-posedness theorem currently exists for these constraint-preserving and physical boundary conditions.

Perturbed Schwarzschild Black Hole with Constaint Preserving BC

- Evolve a Schwarzschild black hole perturbed by the incoming gravitational waves as before. Impose constraint preserving boundary conditions on the remaining incoming characteristic fields.

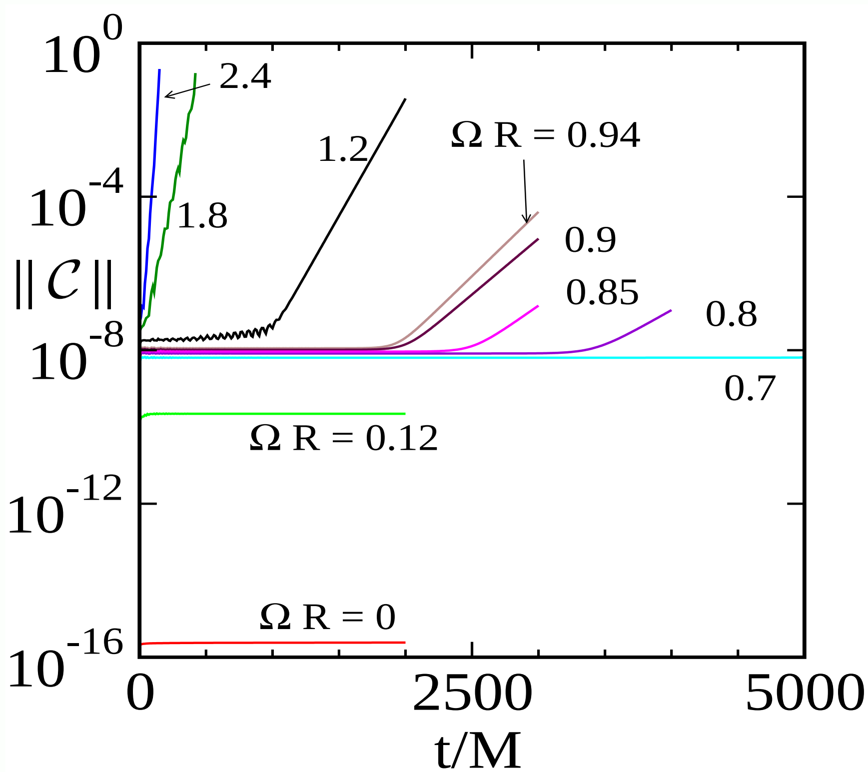




- Evolutions using these new constraint-preserving boundary conditions are still stable and convergent.
- The Weyl curvature component Ψ_4 shows clear quasi-normal mode oscillations in the outgoing gravitational wave flux when constraint-preserving boundary conditions are used.

Trouble in Paradise

- 3D numerical evolutions in spacetimes with large tangential shift components on the boundary of the compuational domain (e.g. any spacetime in rotating coordinates) all appear to be unstable.
- Even evolutions of flat space in coordinates that rotate with angular velocity $\Omega > 0$ are unstable in the new GH system:



- This may be a problem with our boundary conditions, or ... ?