Solving Einstein’s Equation Numerically on Manifolds with Arbitrary Spatial Topologies

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- Multi-cube representations of arbitrary three-manifolds.
- Boundary conditions for elliptic and hyperbolic PDEs.
- Numerical tests for solutions of simple PDEs.
- Covariant first-order representation of Einstein’s equation.
- Simple numerical Einstein evolutions.
Goal: Develop numerical methods that are easily adapted to solving elliptic PDEs on three-manifolds $\Sigma$ with arbitrary topology, and hyperbolic PDEs on manifolds with topology $R \times \Sigma$. 

Every two- and three-manifold admits a triangulation (Radó 1925, Moire 1952), i.e. can be represented as a set of triangles (or tetrahedra), plus a list of rules for gluing their edges (or faces) together.

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- Every two- or three-manifold can be represented as a set of squares or cubes, plus maps that identify their edges or faces.
Multi-cube representations of topological manifolds consist of a set of cubic regions, $\mathcal{B}_A$, plus maps that identify the faces of neighboring regions, $\Psi^{\alpha}_{\beta}(\partial_{\beta} \mathcal{B}_B) = \partial_{\alpha} \mathcal{B}_A$. Choose cubic regions to have uniform size and orientation. Choose linear interface identification maps $\Psi^{\alpha}_{\beta}$:

$$x_i^A = c_{i}^A + C^{\alpha}_{B \beta} (x_k^B - c_k^B),$$
where $C^{\alpha}_{B \beta}$ is a rotation-reflection matrix, and $c_i^A$ is the center of $\alpha$ face of region $A$. Examples:
Boundary Maps: Fixing the Topology

- Multi-cube representations of topological manifolds consist of a set of cubic regions, $\mathcal{B}_A$, plus maps that identify the faces of neighboring regions, $\Psi_{B\beta}^{A\alpha}(\partial_\beta \mathcal{B}_B) = \partial_\alpha \mathcal{B}_A$.
- Choose cubic regions to have uniform size and orientation.
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- Choose cubic regions to have uniform size and orientation.
- Choose linear interface identification maps $\psi_{B\beta}^{A\alpha}$:
  \[ x_A^i = c_{A\alpha}^i + C_{B\beta}^{A\alpha} i (x_B^k - c_{B\beta}^k), \]
  where $C_{B\beta}^{A\alpha} i$ is a rotation-reflection matrix, and $c_{A\alpha}^i$ is center of $\alpha$ face of region $A$. 

Examples:

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Numerical Methods for Arbitrary Topologies

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Boundary Maps: Fixing the Topology

- Multi-cube representations of topological manifolds consist of a set of cubic regions, \( \mathcal{B}_A \), plus maps that identify the faces of neighboring regions, \( \Psi^{A\alpha}_{B\beta}(\partial_{B}B) = \partial_{A}B_A \).

- Choose cubic regions to have uniform size and orientation.

- Choose linear interface identification maps \( \Psi^{A\alpha}_{B\beta} \):
  \[
  x^i_A = c^{i}_{A\alpha} + C^{A\alpha i}_{B\beta k}(x^k_B - c^k_{B\beta}),
  \]
  where \( C^{A\alpha i}_{B\beta k} \) is a rotation-reflection matrix, and \( c^{i}_{A\alpha} \) is center of \( \alpha \) face of region \( A \).

- Examples:
Fixing the Differential Structure

- The boundary identification maps, $\psi^{A\alpha}_{B\beta}$, used to construct multi-cube topological manifolds are continuous, but typically are not differentiable at the interfaces.

- Smooth tensor fields expressed in multi-cube coordinates are not (in general) even continuous at the interfaces.
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- Differential structure provides the framework in which smooth functions and tensors are defined on a manifold.

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- Multi-cube manifolds need an additional layer of infrastructure: e.g., overlapping domains $D_A \supset B_A$ with transition maps that are smooth in the overlap regions.
Fixing the Differential Structure II

All that is needed to define continuous tensor fields at interface boundaries is the Jacobian $J^{A\alpha i}_{B\beta k}$ and its dual $J^{*B\beta k}_{A\alpha i}$ that transform tensors from one multi-cube coordinate region to another: for example, $v^i_A = J^{A\alpha i}_{B\beta k} v^k_B$ and $w_{Ai} = J^{*B\beta k}_{A\alpha i} w_{Bk}$. 

A smooth reference metric $\tilde{g}_{ij}$ determines the needed Jacobians. Let $\tilde{g}^{Aij}$ and $\tilde{g}^{Bij}$ be the components of a smooth reference metric in the multi-cube coordinates of regions $B^A$ and $B^B$ that are identified at the faces $\partial^A \leftrightarrow \partial^B$. Use the reference metric to define the outward directed unit normals: $n^A_{\alpha i}$, $n^B_{\beta i}$, and $n_B^{i A_{\alpha}}$. The needed Jacobians are given by

$$
J^{A\alpha i}_{B\beta k} = C^{A\alpha i}_{B\beta \ell} (\delta^{\ell k} - n^\ell B_{\beta} n^{B\beta k}) - n^i A_{\alpha} n^{B\beta k},
$$

$$
J^{*B\beta k}_{A\alpha i} = (\delta^{i \ell} - n^A_{\alpha i} n^{i A_{\beta}}) C^{B\beta k}_{A\alpha \ell} - n^i A_{\alpha} n^{k B_{\beta}}.
$$

Use continuity of the covariant derivatives of tensors, e.g. $\tilde{\nabla}_A v^i_B v^k_A$, to define their differentiability. These Jacobians satisfy:

$$
n^i A_{\alpha} = - J^{A\alpha i}_{B\beta k} n^{k B_{\beta}},
$$

$$
n^A_{\alpha} = - J^{*B\beta k}_{A\alpha i} n_B^{i B_{\beta}},
$$

$$
t^i A_{\alpha} = J^{A\alpha i}_{B\beta k} t^k_B,
$$

$$
\delta^A_{\alpha} = J^{A\alpha i}_{B\beta k} J^{*B\beta k}_{A\alpha i}.
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- The needed Jacobians are given by

$$J_{B\beta k}^{A\alpha i} = C_{B\beta \ell}^{A\alpha i} \left( \delta_{k}^\ell - n_{B\beta}^\ell n_{B\beta k} \right) - n_{A\alpha}^i n_{B\beta k},$$

$$J_{A\alpha i}^{*B\beta k} = \left( \delta_{i}^\ell - n_{A\alpha i} n_{A\alpha}^\ell \right) C_{A\alpha \ell}^{B\beta k} - n_{A\alpha i} n_{B\beta}^k.$$
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Solving PDEs on Multi-Cube Manifolds

- Solve PDEs in each cubic region separately.
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- Use boundary conditions on cube faces to select the correct smooth global solution.
- For second-order strongly-elliptic systems: enforce continuity on one face and continuity of normal derivatives on neighboring face,
  \[ u_A \simeq u_B \quad \tilde{\nabla}_{n_B} u_B \simeq -\tilde{\nabla}_{n_A} u_A. \]
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- For first-order symmetric hyperbolic systems whose dynamical fields are tensors: set incoming characteristic fields with outgoing characteristics from neighbor,
  \[ \hat{u}^-_A \simeq \hat{u}^+_B \quad \hat{u}^-_B \simeq \hat{u}^+_A. \]
Numerical Methods

- Represent each component of each tensor function as a (finite) sum of spectral basis functions, \( u = \sum_{pqr} u_{pqr} T_p(x) T_q(y) T_r(z) \), in each cubic region.
Numerical Methods

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- Evaluate derivatives of the functions using the known derivatives of the basis functions: \( \partial_x u = \sum_{pqr} u_{pqr} \partial_x T_p(x) T_q(y) T_r(z) \).
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- Evaluate the PDEs and BCs on a set of collocation points, \( \{x_i, y_j, z_k\} \), chosen so that \( u(x_i, y_j, z_k) \) can be mapped efficiently onto the spectral coefficients \( u_{pqr} \). Derivatives become linear combinations of the fields: \( \partial_x u(x_i, y_j, z_k) = \sum_\ell D_i^\ell u(x_\ell, y_j, z_k) \).
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For elliptic systems, these pseudo-spectral equations become a system of algebraic equations for $$u(x_i, y_j, z_k)$$. Solve these algebraic equations using standard numerical methods.

For hyperbolic systems these equations become a system of ordinary differential equations for $$u(x_i, y_j, z_k, t)$$. Solve these equations by the method of lines using standard ode integrators.
Testing the Elliptic PDE Solver

Solve the elliptic PDE, $\nabla^i \nabla_i \psi - c^2 \psi = f$ where $c^2$ is a constant, and $f$ is a given function.
Testing the Elliptic PDE Solver

- Solve the elliptic PDE, $\nabla^i \nabla_i \psi - c^2 \psi = f$ where $c^2$ is a constant, and $f$ is a given function.
- Use the co-variant derivative $\nabla_i$ for the round metric on $S^2 \times S^1$:

$$ds^2 = R_1^2 d\chi^2 + R_2^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right),$$

$$= \left( \frac{2\pi R_1}{L} \right)^2 dz_A^2 + \left( \frac{\pi R_2}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)}{(1 + X_A^2 + Y_A^2)^2} \left( 1 + X_A^2 \right) dx_A^2 - 2X_A Y_A dx_A dy_A + (1 + Y_A^2) dy_A^2 \right].$$

where $X_A = \tan \left( \frac{\pi(x_A - c^X_A)/2L}{2L} \right)$ and $Y_A = \tan \left( \frac{\pi(y_A - c^Y_A)/2L}{2L} \right)$ are “local” Cartesian coordinates in each cubic region.
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  where $X_A = \tan \left[ \pi (x_A - c^x_A)/2L \right]$ and $Y_A = \tan \left[ \pi (y_A - c^y_A)/2L \right]$ are “local” Cartesian coordinates in each cubic region.
- Let $f = -(\omega^2 + c^2)\psi_E$, where $\psi_E = \Re \left[ e^{ik\chi} Y_{\ell m}(\theta, \varphi) \right]$. The angles $\chi$, $\theta$ and $\varphi$ are functions of the coordinates $x$, $y$ and $z$. 

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$$ds^2 = R_1^2 d\chi^2 + R_2^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right),$$

$$= \left( \frac{2\pi R_1}{L} \right)^2 dz_A^2 + \left( \frac{\pi R_2}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)}{(1 + X_A^2 + Y_A^2)^2} \left[ (1 + X_A^2) \, dx_A^2 - 2X_A Y_A \, dx_A \, dy_A + (1 + Y_A^2) \, dy_A^2 \right].$$

where $X_A = \tan \left[ \pi (x_A - c_A^X)/2L \right]$ and $Y_A = \tan \left[ \pi (y_A - c_A^Y)/2L \right]$ are “local” Cartesian coordinates in each cubic region.

- Let $f = -(\omega^2 + c^2)\psi_E$, where $\psi_E = \Re \left[ e^{ik\chi} Y_{\ell m}(\theta, \varphi) \right]$. The angles $\chi$, $\theta$ and $\varphi$ are functions of the coordinates $x$, $y$ and $z$.

- The unique, exact, analytical solution to this problem is $\psi = \psi_E$, when $\omega^2 = \ell(\ell + 1)/R_2^2 + k^2/R_1^2$. 
Testing the Elliptic PDE Solver II

Measure the accuracy of the numerical solution $\psi_N$ as a function of numerical resolution $N$ (grid points per dimension) in two ways:

- First, with the residual $R_N \equiv \nabla^i \nabla_i \psi_N - c^2 \psi_N - f$, and its norm:
  $$\mathcal{E}_R = \sqrt{\frac{\int R_N^2 \sqrt{g} d^3x}{\int f^2 \sqrt{g} d^3x}}.$$

- Second, with the solution error, $\Delta \psi = \psi_N - \psi_E$, and its norm:
  $$\mathcal{E}_\psi = \sqrt{\frac{\int \Delta \psi^2 \sqrt{g} d^3x}{\int \psi_E^2 \sqrt{g} d^3x}}.$$

All these numerical tests were performed by implementing the ideas described here into the Spectral Einstein Code (SpEC) developed originally by the Caltech/Cornell numerical relativity collaboration.
Testing the Elliptic PDE Solver II

- Measure the accuracy of the numerical solution $\psi_N$ as a function of numerical resolution $N$ (grid points per dimension) in two ways:
  - First, with the residual $R_N \equiv \nabla^i \nabla_i \psi_N - c^2 \psi_N - f$, and its norm:
    $$E_R = \sqrt{\frac{\int R_N^2 \sqrt{g} d^3x}{\int f^2 \sqrt{g} d^3x}}.$$
  - Second, with the solution error, $\Delta \psi = \psi_N - \psi_E$, and its norm:
    $$E_\psi = \sqrt{\frac{\int (\Delta \psi)^2 \sqrt{g} d^3x}{\int (\psi_E)^2 \sqrt{g} d^3x}}.$$

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Testing the Hyperbolic PDE Solver

- Solve the equation \( \partial_t^2 \psi = \nabla_i \nabla^i \psi \) with given initial data.
- Convert the second-order equation into an equivalent first-order system:
  \[
  \partial_t \psi = -\Pi, \quad \partial_t \Pi = -\nabla^i \Phi_i \quad \text{and} \quad \partial_t \Phi_i = -\nabla_i \Pi
  \]
  with constraint \( \mathcal{C}_i = \nabla_i \psi - \Phi_i \).
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- Solve the equation $\partial_t^2 \psi = \nabla_i \nabla^i \psi$ with given initial data.
- Convert the second-order equation into an equivalent first-order system: $\partial_t \psi = -\Pi$, $\partial_t \Pi = -\nabla^i \Phi_i$ and $\partial_t \Phi_i = -\nabla_i \Pi$ with constraint $C_i = \nabla_i \psi - \Phi_i$.
- Use the co-variant derivative $\nabla_i$ for the round metric on $S^3$:

$$ds^2 = R_3^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right],$$

$$= \left( \frac{\pi R_3}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)(1 + Z_A^2)}{(1 + X_A^2 + Y_A^2 + Z_A^2)^2} \left[ \frac{(1 + X_A^2)(1 + Y_A^2 + Z_A^2)}{(1 + Y_A^2)(1 + Z_A^2)} \, dx^2 + \frac{(1 + Y_A^2)(1 + X_A^2 + Z_A^2)}{(1 + X_A^2)(1 + Z_A^2)} \, dy^2 \right. \right.$$ 

$$+ \left. \frac{(1 + Z_A^2)(1 + X_A^2 + Y_A^2)}{(1 + X_A^2)(1 + Y_A^2)} \, dz^2 - \frac{2X_A Y_A}{1 + Z_A^2} \, dx \, dy - \frac{2X_A Z_A}{1 + Y_A^2} \, dx \, dz - \frac{2Y_A Z_A}{1 + X_A^2} \, dy \, dz \right].$$
Testing the Hyperbolic PDE Solver

- Solve the equation $\partial_t^2 \psi = \nabla_i \nabla^i \psi$ with given initial data.
- Convert the second-order equation into an equivalent first-order system: $\partial_t \psi = -\Pi$, $\partial_t \Pi = -\nabla^i \Phi_i$ and $\partial_t \Phi_i = -\nabla_i \Pi$ with constraint $C_i = \nabla_i \psi - \Phi_i$.
- Use the co-variant derivative $\nabla_i$ for the round metric on $S^3$:

$$ds^2 = R_3^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right],$$

$$= \left( \frac{\pi R_3}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)(1 + Z_A^2)}{(1 + X_A^2)^2 + Y_A^2 + Z_A^2)^2} \left[ \frac{(1 + X_A^2)(1 + Y_A^2 + Z_A^2)}{(1 + Y_A^2)(1 + Z_A^2)} \right] dx^2 + \frac{(1 + Y_A^2)(1 + X_A^2 + Z_A^2)}{(1 + X_A^2)(1 + Z_A^2)} dy^2$$

$$+ \frac{(1 + Z_A^2)(1 + X_A^2 + Y_A^2)}{(1 + X_A^2)(1 + Y_A^2)} dz^2 - \frac{2X_A Y_A}{1 + Z_A^2} dx dy - \frac{2X_A Z_A}{1 + Y_A^2} dx dz - \frac{2Y_A Z_A}{1 + X_A^2} dy dz.$$

- Choose initial data with $\psi_{t=0} = \Re[Y_{k\ell m}(\chi, \theta, \varphi)]$, $\Pi_{t=0} = -\Re[i\omega Y_{k\ell m}(\chi, \theta, \varphi)]$ and $\Phi_{i\, t=0} = \Re[\nabla_i Y_{k\ell m}(\chi, \theta, \varphi)]$ where $\omega^2 = k(k + 2)/R_3^2$. 
Testing the Hyperbolic PDE Solver

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$$ds^2 = R_3^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right],$$

$$= \left( \frac{\pi R_3}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)(1 + Z_A^2)}{(1 + X_A^2 + Y_A^2 + Z_A^2)^2} \left[ \frac{(1 + X_A^2)(1 + Y_A^2 + Z_A^2)}{(1 + Y_A^2)(1 + Z_A^2)} \, dx^2 + \frac{(1 + Y_A^2)(1 + X_A^2 + Z_A^2)}{(1 + X_A^2)(1 + Z_A^2)} \, dy^2 \right.$$

$$+ \frac{(1 + Z_A^2)(1 + X_A^2 + Y_A^2)}{(1 + X_A^2)(1 + Y_A^2)} \, dz^2 - \frac{2X_A Y_A}{1 + Z_A^2} \, dx \, dy - \frac{2X_A Z_A}{1 + Y_A^2} \, dx \, dz - \frac{2Y_A Z_A}{1 + X_A^2} \, dy \, dz \left. \right].$$

- Choose initial data with $\psi_{t=0} = \Re[Y_{k\ell m}(\chi, \theta, \varphi)]$, $\Pi_{t=0} = -\Re[i\omega \, Y_{k\ell m}(\chi, \theta, \varphi)]$ and $\Phi_{i \, t=0} = \Re[\nabla_i \, Y_{k\ell m}(\chi, \theta, \varphi)]$ where $\omega^2 = k(k + 2)/R_3^2$.

- The unique, exact, analytical solution to this problem is $\psi = \psi_E = \Re[e^{i\omega t} \, Y_{k\ell m}(\chi, \theta, \varphi)]$, $\Pi = -\partial_t \psi_E$, and $\Phi_i = \nabla_i \psi_E$. 

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Testing the Hyperbolic PDE Solver II

- Measure the accuracy of the numerical solution $\psi_N$ as a function of numerical resolution $N$ (grid points per dimension) in two ways:
  - First, with the solution error, $\Delta \psi = \psi_N - \psi_E$, and its norm:
    \[
    \mathcal{E}_\psi = \sqrt{\int \frac{\Delta \psi^2 \sqrt{g} d^3 x}{\int \psi^2 \sqrt{g} d^3 x}}.
    \]
  - Second, with the constraint error, $C_i = \Phi_i - \nabla_i \psi$, and its norm:
    \[
    \mathcal{E}_C = \sqrt{\int \frac{g^{ij} C_i C_j \sqrt{g} d^3 x}{\int g^{ij} (\Phi_i \Phi_j + \nabla_i \psi \nabla_j \psi) \sqrt{g} d^3 x}}.
    \]
Testing the Hyperbolic PDE Solver II

- Measure the accuracy of the numerical solution $\psi_N$ as a function of numerical resolution $N$ (grid points per dimension) in two ways:
  - First, with the solution error, $\Delta \psi = \psi_N - \psi_E$, and its norm:
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    \mathcal{E}_\psi = \sqrt{\frac{\int \Delta \psi^2 \sqrt{gd^3x}}{\int \psi^2 \sqrt{gd^3x}}}
    \]
  - Second, with the constraint error, $C_i = \Phi_i - \nabla_i \psi$, and its norm:
    \[
    \mathcal{E}_C = \sqrt{\frac{\int g^{ij}C_iC_j \sqrt{gd^3x}}{\int g^{ij}(\Phi_i\Phi_j + \nabla_i \psi \nabla_j \psi) \sqrt{gd^3x}}}
    \]
Solving Einstein’s Equation on Multi-Cube Manifolds

- Multi-cube methods were designed to solve first-order hyperbolic systems, 
  \( \partial_t u^\alpha + A^k{}_{\alpha \beta}(u) \tilde{\nabla}_k u^\beta = F^\alpha(u) \), where the dynamical fields \( u^\alpha \) are tensors that can be transformed across interface boundaries using the Jacobians \( J^{A\alpha i}_{B\beta k} \), etc.
Solving Einstein’s Equation on Multi-Cube Manifolds

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- The usual first-order representations of Einstein’s equation fail to meet these conditions in two important ways:
  - The usual choice of dynamical fields, \( u^\alpha = \{ \psi_{ab}, \Pi_{ab} = -t^c \partial_c \psi_{ab}, \Phi_{iab} = \partial_i \psi_{ab} \} \) are not tensor fields.
  - The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of \( \Pi_{ab} \), \( \Pi_{ab} = -t^c \partial_c \psi_{ab} \), and the one that comes from preserving the constraint \( C_{iab} = \Phi_{iab} - \partial_i \psi_{ab} \), \( t^c \partial_c C_{iab} = -\gamma_2 C_{iab} \).
Solving Einstein’s Equation on Multi-Cube Manifolds

- Multi-cube methods were designed to solve first-order hyperbolic systems, $\partial_t u^\alpha + A^k_{\alpha \beta}(u)\tilde{\nabla}_k u^\beta = F^\alpha(u)$, where the dynamical fields $u^\alpha$ are tensors that can be transformed across interface boundaries using the Jacobians $J_{B\beta k}^{A\alpha i}$, etc.

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  - The usual choice of dynamical fields, $u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c \partial_c \psi_{ab}, \Phi_{iab} = \partial_i \psi_{ab}\}$ are not tensor fields.
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- Our attempts to construct the transformations for non-tensor quantities like $\partial_i \psi_{ab}$ and $\Phi_{iab}$ across the non-smooth multi-cube interface boundaries failed to result in stable numerical evolutions.

- A spatially covariant first-order representation of the Einstein evolution system seems to be needed.
Covariant Representations of Einstein’s Equation

Let $\tilde{\psi}_{ab}$ denote a smooth reference metric on the manifold $R \times \Sigma$. For convenience we choose $ds^2 = \tilde{\psi}_{ab} dx^a dx^b = -dt^2 + \tilde{g}_{ij} dx^i dx^j$, where $\tilde{g}_{ij}$ is the smooth multi-cube reference three-metric on $\Sigma$. 

A fully covariant expression for the Ricci tensor can be obtained using the reference covariant derivative $\tilde{\nabla}^a$:

$$R_{ab} = -\frac{1}{2} \psi_{cd} \tilde{\nabla}^c \tilde{\nabla}^d \psi_{ab} + \nabla^a (\Delta^b) - \psi_{cd} \tilde{R}_{e}^{cd} (\Delta^a \psi^b)_{e} + \psi_{cd} \psi_{ef} (\tilde{\nabla}^e \psi^c \tilde{\nabla}^f \psi_{ab} - \Delta^ace \Delta^bd)$$

where $\Delta_{abc} = \psi_{ad} (\Gamma^d_{bc} - \tilde{\Gamma}^d_{bc})$, and $\Delta^a = \psi_{bc} \Delta_{abc}$.

A fully-covariant manifestly hyperbolic representation of the Einstein equations can be obtained by fixing the gauge with a covariant generalized harmonic condition: $\Delta^a = -H^a (\psi_{cd})$.

The vacuum Einstein equations then become:

$$\psi_{cd} \tilde{\nabla}^c \tilde{\nabla}^d \psi_{ab} = -2 \nabla^a (H^b) + 2 \psi_{cd} \psi_{ef} (\tilde{\nabla}^e \psi^c \tilde{\nabla}^f \psi_{ab} - \Delta^ace \Delta^bd) - 2 \psi_{cd} \tilde{R}_{e}^{cd} (\Delta^a \psi^b)_{e} + \gamma_0 [2 \delta^c_a (H^b) - \psi_{ab} t^c] (H^c + \Delta^c)$$
Covariant Representations of Einstein’s Equation

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$$R_{ab} = -\frac{1}{2} \psi^{cd} \tilde{\nabla}_c \tilde{\nabla}_d \psi_{ab} + \nabla (a \Delta_b) - \psi^{cd} \tilde{R}^e_{cd(a \psi b)e} + \psi^{cd} \psi^{ef} (\tilde{\nabla}_e \psi_{ca} \tilde{\nabla}_f \psi_{ab} - \Delta_{ace} \Delta_{bdf}),$$

where $\Delta_{abc} = \psi_{ad} \left( \Gamma^d_{bc} - \tilde{\Gamma}^d_{bc} \right)$, and $\Delta_a = \psi^{bc} \Delta_{abc}$.
Covariant Representations of Einstein’s Equation

Let $\tilde{\psi}_{ab}$ denote a smooth reference metric on the manifold $R \times \Sigma$. For convenience we choose $ds^2 = \tilde{\psi}_{ab} dx^a dx^b = -dt^2 + \tilde{g}_{ij} dx^i dx^j$, where $\tilde{g}_{ij}$ is the smooth multi-cube reference three-metric on $\Sigma$.

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R_{ab} = -\frac{1}{2} \psi^{cd} \tilde{\nabla}_c \tilde{\nabla}_d \psi_{ab} + \nabla (a \Delta_b) - \psi^{cd} \tilde{R}^e_{cd(a}\psi_{b)e} \\
\quad + \psi^{cd} \psi^{ef} (\tilde{\nabla}_e \psi_{ca} \tilde{\nabla}_f \psi_{ab} - \Delta_{ace} \Delta_{bdf}),
$$

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The vacuum Einstein equations then become:

$$
\psi^{cd} \tilde{\nabla}_c \tilde{\nabla}_d \psi_{ab} = -2\nabla (a H_b) + 2\psi^{cd} \psi^{ef} (\tilde{\nabla}_e \psi_{ca} \tilde{\nabla}_f \psi_{ab} - \Delta_{ace} \Delta_{bdf}) \\
\quad - 2\psi^{cd} \tilde{R}^e_{cd(a}\psi_{b)e} + \gamma_0 \left[ 2\delta^c_{(a} t_{b)} - \psi_{ab} t^c \right] (H_c + \Delta_c).
$$
A first-order representation of the Einstein equations can be obtained from this covariant generalized harmonic representation by choosing dynamical fields:

$$u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c\tilde{\nabla}_c\psi_{ab}, \Phi_{iab} = \tilde{\nabla}_i\psi_{ab}\},$$

which are tensors with respect to spatial coordinate transformations.
Covariant Representations of Einstein’s Equation II

- A first-order representation of the Einstein equations can be obtained from this covariant generalized harmonic representation by choosing dynamical fields:
  \[ u^\alpha = \{ \psi_{ab}, \Pi_{ab} = -t^c\tilde{\nabla}_c\psi_{ab}, \Phi_{iab} = \tilde{\nabla}_i\psi_{ab} \}, \]
  which are tensors with respect to spatial coordinate transformations.

- The first order equation that arises from the definition of \( \Pi_{ab} \),
  \[ t^c\tilde{\nabla}_c\psi_{ab} = -\Pi_{ab} \]
  is now covariant, as is the equation for \( t^c\tilde{\nabla}_c\Phi_{iab} \) that follows from the covariant constraint evolution equation,
  \[ t^c\tilde{\nabla}_cC_{iab} = -\gamma_2C_{iab}, \]
  where \( C_{iab} = \Phi_{iab} - \tilde{\nabla}_i\psi_{ab} \).
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that follows from the covariant constraint evolution equation,
\[ t^c\tilde{\nabla}_c C_{iab} = -\gamma_2 C_{iab}, \]
where \( C_{iab} = \Phi_{iab} - \tilde{\nabla}_i \psi_{ab} \).

The resulting first-order Einstein evolution system,
\[ \partial_t u^\alpha + A^k{}^\alpha{}^\beta(u)\tilde{\nabla}_k u^\beta = F^\alpha(u), \]
is symmetric-hyperbolic and covariant with respect to spatial coordinate transformations.

The characteristic speeds and fields of this covariant system have the same forms as the standard ones in terms of the dynamical fields \( \psi_{ab}, \Pi_{ab} \) and \( \Phi_{iab} \). These fields are now tensors, however, so the actual characteristic fields are somewhat different.
Testing the Einstein Solver: Non-Linear Gauge Wave

- This simple test evolves the non-linear gauge wave solution,
  \[ ds^2 = \psi_{Aab} dx^a dx^b = -(1 + F) dt^2 + (1 + F) dx^2 + dy^2 + dz^2, \]
  for the case \( F = 0.1 \sin[2\pi(2x - t)] \), on a manifold with spatial topology \( T^3 \).
Testing the Einstein Solver: Non-Linear Gauge Wave

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  for the case \( F = 0.1 \sin[2\pi(2x - t)] \), on a manifold with spatial topology \( T^3 \).

- Monitor how well the numerical solutions satisfy the Einstein system by evaluating the norm of the various constraints:
  \[ E_C = \sqrt{\int \sum |C|^2 \sqrt{g} d^3 x} \]
  \[ E_{\psi} = \sqrt{\int \sum |\Delta \psi_{ab}|^2 \sqrt{g} d^3 x} \]

\[ \begin{align*}
E_C & \quad \begin{cases}
N = 16 & \text{red}
N = 18 & \text{blue}
N = 20 & \text{green}
N = 22 & \text{black}
\end{cases}
\end{align*} \]
Testing the Einstein Solver: Non-Linear Gauge Wave

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  \[ ds^2 = \psi_{Aab} dx^a dx^b = -(1 + F) dt^2 + (1 + F) dx^2 + dy^2 + dz^2, \]
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- Monitor how well the numerical solutions satisfy the Einstein system by evaluating the norm of the various constraints:
  \[ E_C = \sqrt{\int \sum |C|^2 \sqrt{g} d^3x} / \sqrt{\int \sum |\partial_i u|^2 \sqrt{g} d^3x}. \]

- Monitor the accuracy of the numerical solution by evaluating the norm of its error, \( \Delta \psi_{ab} = \psi_{Nab} - \psi_{Aab} \):
  \[ E_\psi = \sqrt{\int \sum_{ab} |\Delta \psi_{ab}|^2 \sqrt{g} d^3x} / \sqrt{\int \sum_{ab} |\psi_{ab}|^2 \sqrt{g} d^3x}. \]
Testing the Einstein Solver: Static Universe on $S^3$

- Metric initial data is taken from the “Einstein Static Universe” geometry:

$$ds^2 = -dt^2 + R_3^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right],$$

This metric solves Einstein's equation with cosmological constant and complex scalar field source on a manifold with spatial topology $S^3$. Evolution of these initial data is the static universe geometry, if the cosmological constant is chosen to be $\Lambda = 1/R_3^2$, and the complex scalar field is $\phi = \phi_0 e^{i \mu t}$ with $\mu^2 |\phi_0|^2 = 1/(4\pi R_3^2)$. 

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Testing the Einstein Solver: Static Universe on $S^3$

- Metric initial data is taken from the “Einstein Static Universe” geometry:

\[
\begin{align*}
    ds^2 &= -dt^2 + R_3^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \\
    &= -dt^2 + \left( \frac{\pi R_3}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)(1 + Z_A^2)}{(1 + X_A^2 + Y_A^2 + Z_A^2)^2} \left[ \frac{(1 + X_A^2)(1 + Y_A^2 + Z_A^2)}{(1 + Y_A^2)(1 + Z_A^2)} \, dx^2 + \frac{(1 + Y_A^2)(1 + X_A^2 + Z_A^2)}{(1 + X_A^2)(1 + Z_A^2)} \, dy^2 \\
    &\quad + \frac{(1 + Z_A^2)(1 + X_A^2 + Y_A^2)}{(1 + X_A^2)(1 + Y_A^2)} \, dz^2 - \frac{2X_A Y_A}{1 + Z_A^2} \, dx \, dy - \frac{2X_A Z_A}{1 + Y_A^2} \, dx \, dz - \frac{2Y_A Z_A}{1 + X_A^2} \, dy \, dz \right].
\end{align*}
\]
Metric initial data is taken from the “Einstein Static Universe” geometry:

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\[ = -dt^2 + \left( \frac{\pi R_3}{2L} \right)^2 \frac{(1 + X_A^2)(1 + Y_A^2)(1 + Z_A^2)}{(1 + X_A^2 + Y_A^2 + Z_A^2)^2} \left[ \frac{(1 + X_A^2)(1 + Y_A^2 + Z_A^2)}{(1 + Y_A^2)(1 + Z_A^2)} \, dx^2 + \frac{(1 + Y_A^2)(1 + X_A^2 + Z_A^2)}{(1 + X_A^2)(1 + Z_A^2)} \, dy^2 \right. \]

\[ + \left. \frac{(1 + Z_A^2)(1 + X_A^2 + Y_A^2)}{(1 + X_A^2)(1 + Y_A^2)} \, dz^2 - \frac{2X_A Y_A}{1 + Z_A^2} \, dx \, dy - \frac{2X_A Z_A}{1 + Y_A^2} \, dx \, dz - \frac{2Y_A Z_A}{1 + X_A^2} \, dy \, dz \right]. \]

This metric solves Einstein’s equation with cosmological constant and complex scalar field source on a manifold with spatial topology \( S^3 \).
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- This metric solves Einstein’s equation with cosmological constant and complex scalar field source on a manifold with spatial topology $S^3$.

- Evolution of these initial data is the static universe geometry, if the cosmological constant is chosen to be $\Lambda = 1/R_3^2$, and the complex scalar field is $\varphi = \varphi_0 e^{i\mu t}$ with $\mu^2 |\varphi_0|^2 = 1/4\pi R_3^2$. 
Monitor how well the numerical solutions satisfy the Einstein system by evaluating the norm of the various constraints:

$$\mathcal{E}_C = \sqrt{\int \sum |C|^2 \sqrt{g} d^3 x} \cdot \sqrt{\int \sum |\partial_i u|^2 \sqrt{g} d^3 x}.$$
Testing the Einstein Solver: Static Universe on $S^3$ II

- Monitor how well the numerical solutions satisfy the Einstein system by evaluating the norm of the various constraints:
  \[ \mathcal{E}_C = \sqrt{\frac{\int \sum |C|^2 \sqrt{g}d^3x}{\int \sum |\partial_i u|^2 \sqrt{g}d^3x}}. \]

- Monitor the physical volume $V$ of the $S^3$ in comparison with the Einstein Static Solution value $V_0 = 2\pi^2 R_3^3$:
Testing the Einstein Solver: Static Universe on $S^3$ II

- Monitor how well the numerical solutions satisfy the Einstein system by evaluating the norm of the various constraints:

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- Monitor the accuracy of numerical metric solution by evaluating the norm of its error, $\Delta \psi_{ab} = \psi_{Nab} - \psi_{Aab}$:

\[ \mathcal{E}_\psi = \sqrt{\int \sum_{ab} |\Delta \psi_{ab}|^2 \sqrt{g} d^3x} \]
Testing the Einstein Solver: Static Universe on $S^3$ III

- The constraints are well satisfied for $t \lesssim 20$, so early time evolutions represent good (approximate) solutions to the Einstein-Klein-Gordon system.

\[ \omega^2 R^2 = 2 \mu^2 R^2 - 2 - 2 \sqrt{\mu^4 R^4 - \mu^2 R^2 + 1}. \]

For the mass and radius parameters used in these simulations $1/\tau_0 \equiv |\omega| \approx 1.10050$. 

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The constraints are well satisfied for $t \lesssim 20$, so early time evolutions represent good (approximate) solutions to the Einstein-Klein-Gordon system.

The static Einstein-Klein-Gordon solution has an unstable $k = 0$ mode with frequency $\omega$ given by

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For the mass and radius parameters used in these simulations $1/\tau_0 \equiv |\omega| \approx 1.10050$. 

![Graph 1](image1.png)

![Graph 2](image2.png)
Testing the Einstein Solver: Static Universe on $S^3$ IV

- Test the long term stability of the multi-cube method for the Einstein system by damping out the one unstable mode of the Einstein-Klein-Gordon static solution.

Define the spatial average $\bar{Q}$ of a quantity $Q$ by

$$\bar{Q} = \frac{\int Q \sqrt{\tilde{g}} \, d^3x}{\int \sqrt{\tilde{g}} \, d^3x}.$$ 

Modify the Einstein-Klein-Gordon evolution system by adding terms that damp out perturbations in the $k=0$ mode:

$$\partial_t \psi_{ab} = f_{ab} - \frac{1}{3} \left[ \bar{f} + \eta (\bar{\psi} - 3) \right] \tilde{g}_{ab} - \frac{1}{3} \left[ \bar{f}_{ttt} + \eta (\bar{\psi}_{ttt} + 1) \right] \tilde{t}^a \tilde{t}^b,$$

$$\partial_t \phi = f_{\phi} - \bar{f}_{\phi} + i \mu \phi_0 e^{i \mu t} - \eta (\bar{\phi} - \phi_0 e^{i \mu t}).$$
Testing the Einstein Solver: Static Universe on $S^3$ IV

- Test the long term stability of the multi-cube method for the Einstein system by damping out the one unstable mode of the Einstein-Klein-Gordon static solution.
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\begin{align*}
\partial_t \psi_{ab} &= f_{ab} - \frac{1}{3} \left[ \bar{f} + \eta(\bar{\psi} - 3) \right] g_{ab} - \left[ \bar{f}_{tt} + \eta(\bar{\psi}_{tt} + 1) \right] \tilde{t}_a \tilde{t}_b, \\
\partial_t \varphi &= f_\varphi - \bar{f}_\varphi + i \mu \varphi_0 e^{i\mu t} - \eta(\bar{\varphi} - \varphi_0 e^{i\mu t}).
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These methods have been tested by solving simple elliptic and hyperbolic equations on several compact manifolds.

A first-order symmetric-hyperbolic representation of the generalized harmonic Einstein evolution equations has been constructed that is covariant with respect to general spatial coordinate transformations.

The multi-cube methods are being tested now by finding simple solutions on compact manifolds to this covariant representation of Einstein's equation.
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