Solving the Inverse Stellar Structure Problem

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Theoretical Astrophysics, Caltech

Institute of Theoretical Science Seminar
University of Oregon, 30 April 2013

- Can the equation of state of the matter in a neutron star be determined from astronomical observations?

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- Can the equation of state of the matter in a neutron star be determined from astronomical observations?
- This talk describes recent work on the mathematical, rather than observational or data analysis, aspects of this question.
  - What is the relativistic inverse stellar structure problem?
  - How can it be solved?
  - How well does the solution work in practice?
  - Can gravitational wave data be used to solve this problem?

Relativistic Stellar Structure Problem (SSP)

- Given an equation of state, $\epsilon = \epsilon(p)$, solve Einstein's equations,

$$\begin{aligned} \frac{dm}{dr} &= 4\pi r^2 \epsilon, \\ \frac{dp}{dr} &= -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)}, \end{aligned}$$

  to determine the structures of relativistic stars.

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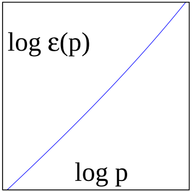
  to determine the structures of relativistic stars.
- Find the radius $p(R) = 0$ and mass $M = m(R)$ for each star.
- SSP can be thought of as a map from the equation of state $\epsilon = \epsilon(p)$ to the M-R curve $\{R(p_c), M(p_c)\}$.



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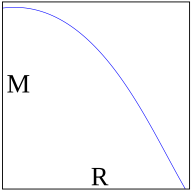

Relativistic Inverse Stellar Structure Problem (SSP^{-1})

- When the equation of state is well understood – as in white dwarf stars – the standard stellar structure problem is useful.
- When the equation of state is poorly known – as in neutron stars – the inverse stellar structure problem (SSP^{-1}) is more interesting.

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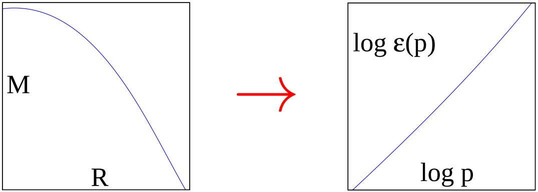
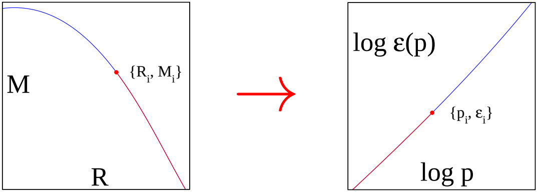
- When the equation of state is poorly known – as in neutron stars – the inverse stellar structure problem (SSP^{-1}) is more interesting.
- SSP^{-1} finds the equation of state $\epsilon = \epsilon(p)$ from a given mass-radius curve.
- SSP^{-1} can be thought of as the map from the M-R curve $\{R(p_c), M(p_c)\}$ to the equation of state $\epsilon = \epsilon(p)$.



Standard Solution to SSP^{-1}

- Assume the complete M-R curve is known, including the point $\{R_i, M_i\} = \{R(p_i), M(p_i)\}$.
- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.



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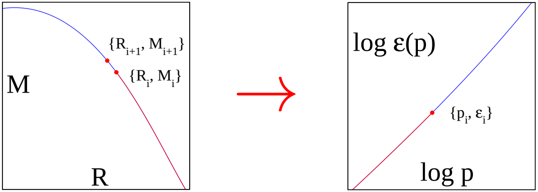

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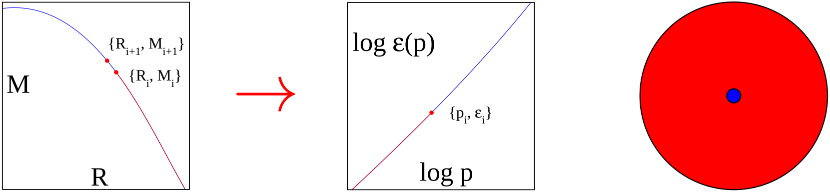


- Choose a new point on the M-R curve, $\{R_{i+1}, M_{i+1}\}$, having slightly larger central density.
- Integrate Einstein's equations,

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \qquad \frac{dp}{dr} = -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r - 2m)},$$

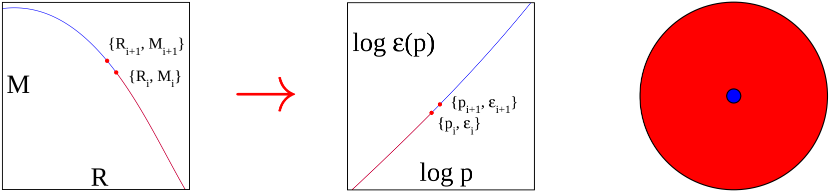
through the outer parts of the star, to determine the mass and radius, $\{r_{i+1}, m_{i+1}\}$, of the small core with large densities $\epsilon \geq \epsilon_i$.

Standard Solution to SSP^{-1} II

- For very small cores, $\{r_{i+1}, m_{i+1}\}$, the solution to the OV equations is described by the power series solution:

$$
\begin{aligned}
m_{i+1} &= \frac{4\pi}{3}\epsilon_{i+1} r_{i+1}^3 + \mathcal{O}(r_{i+1}^5),\\
p_i &= p_{i+1} - \frac{2\pi}{3}(\epsilon_{i+1} + p_{i+1})(\epsilon_{i+1} + 3p_{i+1}) r_{i+1}^2 + \mathcal{O}(r_{i+1}^4).
\end{aligned}
$$

- Invert these series to determine the central pressure and density, $\{p_{i+1}, \epsilon_{i+1}\}$, in terms of the known quantities, m_{i+1}, r_{i+1}, p_i, ϵ_i.



Can the Standard Solution to SSP^{-1} be Improved?

- Standard solution to the relativistic SSP^{-1} finds the equation of state, $\epsilon = \epsilon(p)$, represented as a table, $\{p_i, \epsilon_i\}$ for $i = 1, ..., N$, and an interpolation formula.

- Standard solution has several weaknesses:
  - Solution converges (slowly) with the number of points, as N^{-p}.
  - Each new equation of state point, $\{p_i, \epsilon_i\}$, requires the knowledge of a separate new M-R curve point, $\{R_i, M_i\}$.
  - Accurate M-R curve points $\{R_i, M_i\}$ for neutron stars are scarce.

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- Spectral numerical methods typically converge more rapidly, and represent functions more efficiently than finite difference methods.

- Can spectral methods provide better (*i.e.* more practical and more accurate) solutions to the SSP^{-1}?

- Can spectral methods provide interesting solutions to SSP^{-1} when only a few (*e.g.* two or three) M-R data points are available?

Outline for Solving SSP^{-1} Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \gamma_k)$, where the γ_k are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \gamma_k) = \sum_k \gamma_k \Phi_k(p)$, where the $\Phi_k(p)$ are spectral basis functions, *e.g.* $\Phi_k(p) = e^{ikp}$, or $\Phi_k(p) = P_k(p)$.

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- For a given equation of state, *i.e.* a particular choice of γ_k, solve the SSP to obtain a model M-R curve: $\{R(p_c, \gamma_k), M(p_c, \gamma_k)\}$.

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- Given a set of points from the "real" M-R curve, $\{R_i, M_i\}$, choose the parameters γ_k and p_c^i that minimize the difference measure:

$$\chi^2 = \frac{1}{N_{\rm stars}} \sum_{i=1}^{N_{\rm stars}} \left\{ \left[\log\left(\frac{R(p_c^i, \gamma_k)}{R_i} \right) \right]^2 + \left[\log\left(\frac{M(p_c^i, \gamma_k)}{M_i} \right) \right]^2 \right\}$$

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- Resulting γ_k for $k = 1, ..., N_{\gamma_k}$ determines an equation of state, $\epsilon = \epsilon(p, \gamma_k)$, that provides an approximate solution of SSP^{-1}.

Basic Questions

- Do spectral expansions provide an efficient way to represent realistic neutron-star equations of state?
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- These questions are best answered using a somewhat different form of the standard stellar structure problem (SSP).
- Digress (briefly) now to describe this alternate formulation that provides a more efficient and more accurate way to solve the SSP.

Alternative Representations of the SSP

- The standard Oppenheimer-Volkoff (OV) representation of the SSP equations determines $m(r)$ and $p(r)$, given an equation of state of the form $\epsilon = \epsilon(p)$.
- The outer boundary of the star is the point where $p(R) = 0$. This condition is difficult to solve numerically because the pressure goes to zero non-linearly there: $p \propto (R - r)^{\Gamma_0/(\Gamma_0 - 1)}$.

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- This problem can be simplified by introducing the relativistic enthalpy $h(p) = \int_0^p dp'/[\epsilon(p') + p']$, and re-writing the OV equations in terms of it:

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- The surface of the star is now the point where $h(R) = 0$. This condition is easier to solve numerically because the enthalpy goes to zero linearly there: $h(r) \propto (R - r)$.

Alternative Representations of the SSP II

- Simplify again by swapping the roles of h and r:

$$\frac{dm}{dh} = -\frac{4\pi\epsilon(h)r^3(r-2m)}{m+4\pi r^3 p(h)}, \qquad \frac{dr}{dh} = -\frac{r(r-2m)}{m+4\pi r^3 p(h)}.$$

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- This form of the equations is easier to solve numerically:
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  - The total mass M and radius R are determined simply by evaluating the solution at $h = 0$, $\{R, M\} = \{r(0), m(0)\}$.
- These alternative OV equations require that the equation of state, $\epsilon = \epsilon(p)$, be re-written as $\epsilon = \epsilon(h)$ and $p = p(h)$:
  - Start with the standard, $\epsilon = \epsilon(p)$.
  - Compute, $h(p) = \int_0^p dp'/[\epsilon(p') + p']$.
  - Invert to give $p = p(h)$.
  - Compose $\epsilon = \epsilon(p)$ with $p = p(h)$, to give $\epsilon = \epsilon(h) = \epsilon[p(h)]$.

Faithful Spectral Expansions of the Equation of State

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- Faithful here means *i)* that every choice of spectral parameters, α_k and β_k, corresponds to a possible physical equation of state, and *ii)* that every equation of state can be represented by such an expansion.

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- Faithful spectral expansions of the adiabatic index Γ do exist:

$$\Gamma(h) = \frac{\epsilon + p}{p}\frac{dp}{d\epsilon} = \exp\left[\sum_k \gamma_k \Phi_k(h)\right].$$

Faithful Spectral Expansions of the Equation of State II

- Every equation of state is determined by the adiabatic index $\Gamma(h)$:

$$\frac{d\epsilon}{dh} = \frac{(\epsilon + p)^2}{p\,\Gamma(h)}, \qquad \frac{dp}{dh} = \epsilon + p.$$

- The solutions to these equations can be reduced to quadratures:

$$\begin{aligned} p(h) &= p_0 \exp\left[\int_{h_0}^{h} \frac{e^{h'}dh'}{\mu(h')}\right], \\ \epsilon(h) &= p(h)\frac{e^h - \mu(h)}{\mu(h)}, \\ \mu(h) &= \frac{p_0 e^{h_0}}{\epsilon_0 + p_0} + \int_{h_0}^{h} \frac{\Gamma(h') - 1}{\Gamma(h')} e^{h'}dh'. \end{aligned}$$

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- Choosing, $\log\Gamma(h) = \sum_k \gamma_k \Phi_k(h)$, for any spectral basis functions, $\Phi_k(h)$, results in a faithful parametrized equation of state of the desired form: $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$.

Fitting Model Neutron-Star Equations of State

- How accurately and efficiently are realistic neutron-star equations of state represented by $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$, when $\Gamma(h)$ is given by
$$\Gamma(h) = \exp\left\{\sum_{k=0}^{N_{\gamma_k}-1} \gamma_k \left[\log\left(\frac{h}{h_0}\right)\right]^k\right\} ?$$

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- Let $\{p_i, \epsilon_i, h_i\}$, for $i = 1, ..., N_{\rm EOS}$ denote one of the standard tabulated realistic neutron-star equations of state.
- Find the spectral parameters γ_k that minimize the fitting error:

$$\left(\Delta^{EOS}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\rm EOS}} \sum_{i=1}^{N_{\rm EOS}} \left[\log\left(\frac{\epsilon(h_i, \gamma_k)}{\epsilon_i}\right)\right]^2.$$

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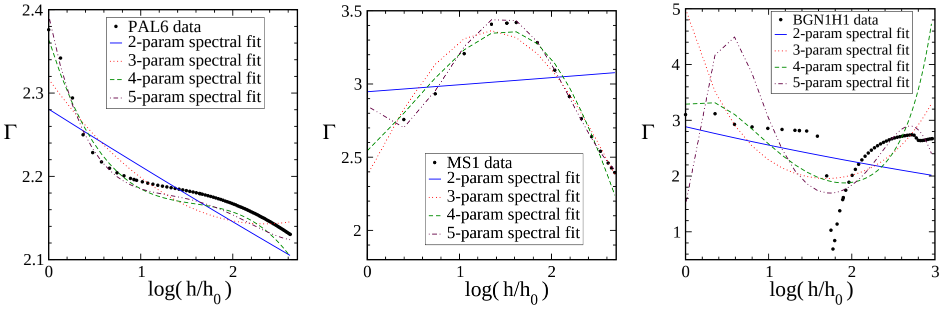
$$\left(\Delta^{EOS}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\text{EOS}}}\sum_{i=1}^{N_{\text{EOS}}}\left[\log\left(\frac{\epsilon(h_i, \gamma_k)}{\epsilon_i}\right)\right]^2.$$
- The average values of these fitting errors, $\Delta^{EOS}_{N_{\gamma_k}}$, for 34 realistic neutron-star equations of state are:
$$\Delta^{EOS}_2 = 0.032, \qquad \Delta^{EOS}_3 = 0.017,$$
$$\Delta^{EOS}_4 = 0.012, \qquad \Delta^{EOS}_5 = 0.0089.$$

Spectral Fits of Model Neutron-Star Equations of State



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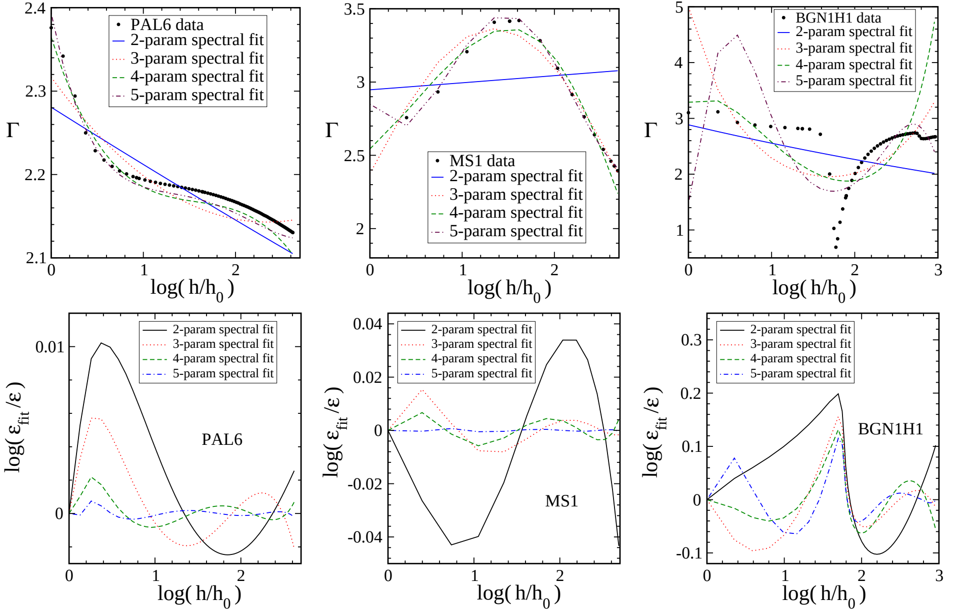

Spectral Solution of SSP^{-1}

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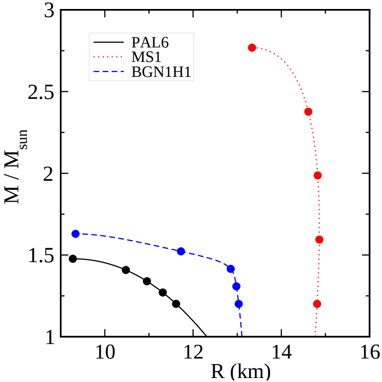
- Work done with Caltech undergraduate Nathaniel Indik.
- Choose mock data points $\{R_i, M_i\}$ for neutron-star models computed with 34 realistic equations of state.



Spectral Solution of SSP^{-1} II

- Fix the spectral expansion coefficients γ_k by minimizing,

$$\chi^2 = \frac{1}{N_{\text{stars}}} \sum_{i=1}^{N_{\text{stars}}} \left\{ \left[\log \left(\frac{M(h_c^i, \gamma_k)}{M_i} \right) \right]^2 + \left[\log \left(\frac{R(h_c^i, \gamma_k)}{R_i} \right) \right]^2 \right\}.$$

  with respect to variations in γ_k, and variations in the central values of the enthalpy for each star, h_c^i.

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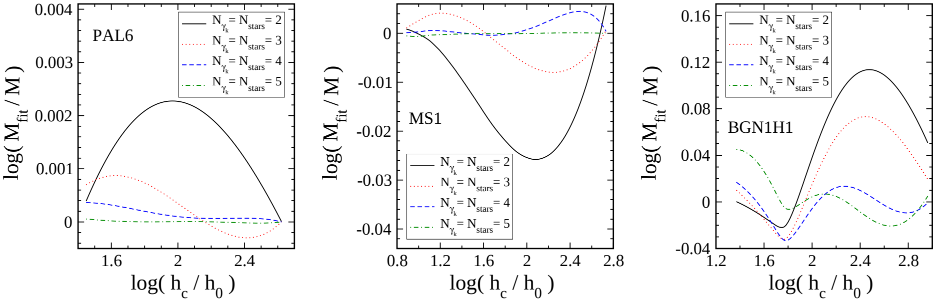
  with respect to variations in γ_k, and variations in the central values of the enthalpy for each star, h_c^i.
- Compare the resulting M-R curve $\{R(h_c, \gamma_k), M(h_c, \gamma_k)\}$ with the exact curve from the known equation of state $\{R(h_c), M(h_c)\}$.



Spectral Solutions to SSP^{-1} III

- Next evaluate the equation of state fitting errors, $\Delta^{MR}_{N_{\gamma_k}}$,

$$\left(\Delta^{MR}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\mathrm{EOS}}}\sum_{i=1}^{N_{\mathrm{EOS}}}\left[\log\left(\frac{\epsilon(h_i,\gamma_k)}{\epsilon_i}\right)\right]^2$$

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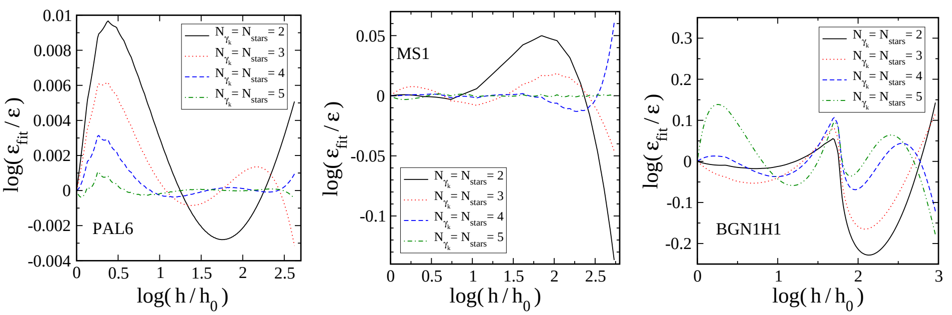
- The average values of $\Delta^{MR}_{N_{\gamma_k}}$ (with $N_{\gamma_k} = N_{\rm stars}$) determined in this way for 34 realistic model equations of state are:

$$\begin{array}{ll} \Delta^{MR}_2 = 0.040, & \Delta^{EOS}_2 = 0.032, \\ \Delta^{MR}_3 = 0.029, & \Delta^{EOS}_3 = 0.017, \\ \Delta^{MR}_4 = 0.023, & \Delta^{EOS}_4 = 0.012, \\ \Delta^{MR}_5 = 0.017, & \Delta^{EOS}_5 = 0.0089. \end{array}$$

- The accuracy of these solutions to the SSP^{-1} is quite impressive, even though the number of M-R data used is very small.

Spectral Solutions to SSP^{-1} IV

- Compare the spectral equation of state, $\epsilon(h, \gamma_k)$, determined by fitting the M-R data with the exact equation of state $\epsilon(h)$:



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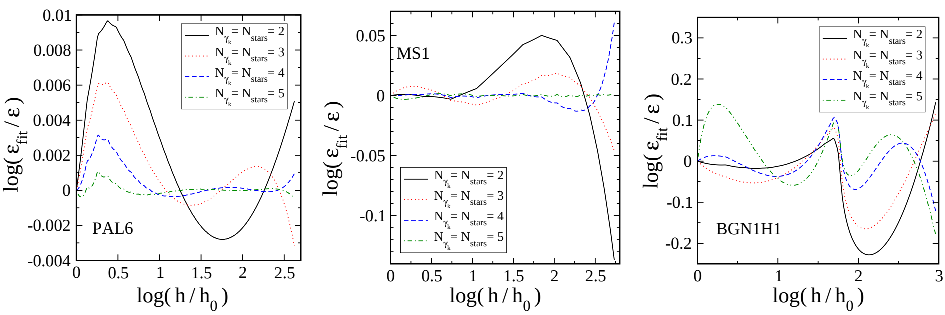

- The convergence of $\Delta^{MR}_{N_{\gamma_k}}$ is not as good as $\Delta^{EOS}_{N_{\gamma_k}}$. Perhaps our χ^2 minimization finds local not global minima?

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- Compare average accuracy of the resulting equations of state $\Delta_N^{M\Lambda}$ with those obtained using mass-radius data, Δ_N^{MR}. (Warning: Preliminary results, hot-off-the-press.)

$$\Delta_2^{M\Lambda} = 0.040, \ \Delta_3^{M\Lambda} = 0.029, \ \Delta_4^{M\Lambda} = 0.023, \ \Delta_5^{M\Lambda} = 0.018.$$
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- Gravitational wave measurements of neutron star masses M_i and tidal deformabilities Λ_i can be used to determine the neutron star equation of state with about the same level of accuracy as that obtainable with more traditional mass M_i radius R_i data.