Improved Gauge Drivers for the Generalized Harmonic (GH) Einstein Equations

Lee Lindblom and Béla Szilágyi
Caltech

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- Gauge conditions are specified in the GH Einstein system by the gauge source function $H^a \equiv \nabla^c \nabla_c x^a$.
- How do you choose H^a in a way that provides a reasonable coordinate system and keeps the GH Einstein system hyperbolic?

Gauge Conditions and Hyperbolicity

- The GH Einstein equations may be written (abstractly) as

$$\psi^{cd}\partial_c\partial_d\psi_{ab} \;=\; \partial_a H_b + \partial_b H_a + Q_{ab}(H,\psi,\partial\psi).$$

- These equations are manifestly hyperbolic when H^a is specified as a function of x^a and ψ_{ab}: $\quad H_a = H_a(x,\psi)$.

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- Unfortunately, most gauge conditions found useful in numerical relativity are conditions on ψ_{ab} and $\partial_c\psi_{ab}$.
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- Elevate H_a to the status of an independent dynamical field, by choosing an evolution equation for H_a whose solutions are the desired gauge conditions.

Solution: Gauge Driver Equations

- Pretorius proposed evolving H_a using an equation of the form:

$$\text{Gauge Driver :} \qquad \psi^{cd}\partial_c\partial_d H_a = Q_a(x, H, \partial H, \psi, \partial\psi),$$

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  - Also hyperbolic, but not obviously so.
  - Choose Q_a so that all solutions H_a evolve toward a target F_a:

$$t^c\partial_c H_a = -\mu(H_a - F_a) + \ldots$$

  - New gauge driver has fewer “interesting” solutions.

Damped-Wave Gauge Conditions

- Spatial coordinates satisfying $\nabla^c\nabla_c x^i = 2\mu_S t^c\partial_c x^i$ are called damped-wave coordinates.
- Choose target $F^i = 2\mu_S t^i = -2\mu_S N^{-1} N^i$.

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- Choose target $F^i = 2\mu_S t^i = -2\mu_S N^{-1}N^i$.
- The time-component $t^a H_a$ related to spacetime metric by constraints of GH Einstein system:

$$t^a H_a = t^a\partial_a \log\left(\frac{\sqrt{g}}{N}\right) + N^{-1}\partial_k N^k.$$

- Choose target $t^a F_a$ to suppress growth in $g = \det g_{ij}$:

$$t^a F_a = -2\mu_L \log\left(\frac{\sqrt{g}}{N}\right).$$

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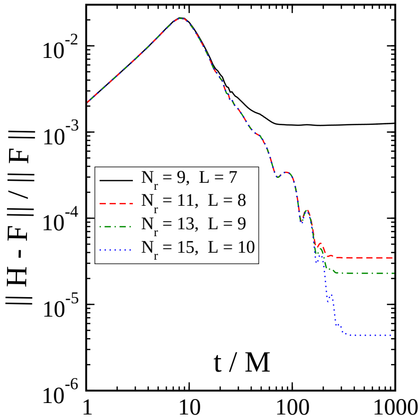
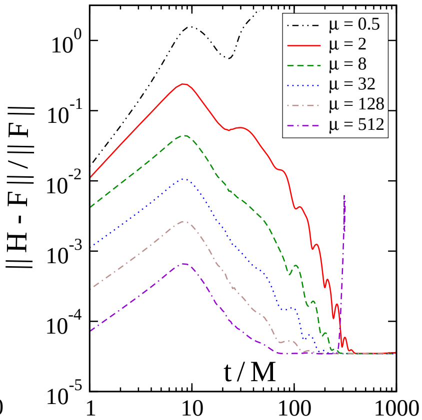
- Combined expression for damped-wave target F_a:

$$F_a = 2\mu_L \log\left(\frac{\sqrt{g}}{N}\right) t_a - 2\mu_S N^{-1} g_{ai} N^i.$$

Testing New Gauge-Driver System:

- Gauge Driver: $\partial_t H_a = -\mu N(H_a - F_a) + \ldots$
- Target Gauge: F_a representing damped-wave gauge.
- Initial Data: Schwarzschild with perturbed lapse and shift.





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- The new gauge driver $\partial_t H_a = -\mu N(H_a - F_a) + ...$ allows hyperbolic implementations of a wide variety of gauge conditions.
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- The new gauge driver $\partial_t H_a = -\mu N(H_a - F_a) + ...$ allows hyperbolic implementations of a wide variety of gauge conditions.
- Numerical tests show the new gauge driver is effective.
- Tests of numerous gauge conditions found the damped-wave gauge very stable and useful for black-hole evolutions.
- Binary black-hole systems have been evolved successfully through the last orbits plus merger using this new gauge driver and the damped-wave gauge condition.