# A Spectral Approach to the Relativistic Inverse Stellar Structure Problem

Lee Lindblom

Theoretical Astrophysics, Caltech

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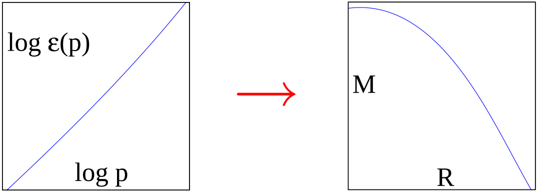
- What is the relativistic inverse stellar structure problem ($SSP^{-1}$)?
- Can spectral methods provide a more effective way to solve it?

# Relativistic Stellar Structure Problem (SSP)

- Given an equation of state, $\epsilon = \epsilon(p)$, solve Einstein's equations,

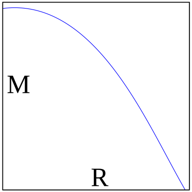
$$\begin{aligned}\frac{dm}{dr} &= 4\pi r^2 \epsilon, \\ \frac{dp}{dr} &= -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)},\end{aligned}$$

  to determine the structures of relativistic stars.
- Find the radius $p(R) = 0$ and mass $M = m(R)$ for each star.
- SSP can be thought of as a map from the equation of state $\epsilon = \epsilon(p)$ to the M-R curve $\{R(p_c), M(p_c)\}$.



# Relativistic Inverse Stellar Structure Problem ($SSP^{-1}$)

- When the equation of state is well understood – as in white dwarf stars – the standard stellar structure problem is useful.
- When the equation of state is poorly known – as in neutron stars – the inverse stellar structure problem may be more interesting.

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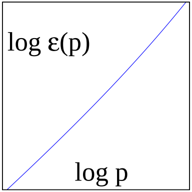

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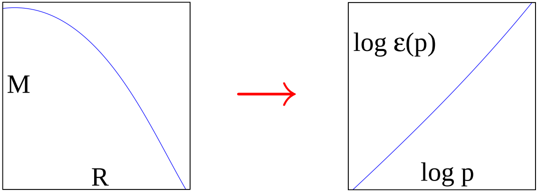
- When the equation of state is well understood – as in white dwarf stars – the standard stellar structure problem is useful.
- When the equation of state is poorly known – as in neutron stars – the inverse stellar structure problem may be more interesting.
- $SSP^{-1}$ finds the equation of state $\epsilon = \epsilon(p)$ from a given mass-radius curve.
- $SSP^{-1}$ can be thought of as a map from the M-R curve $\{R(p_c), M(p_c)\}$ to the equation of state $\epsilon = \epsilon(p)$.



# Standard Solution to SSP$^{-1}$

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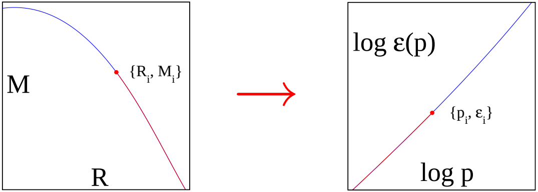
- Assume the complete M-R curve is known.



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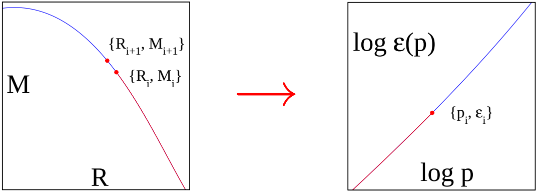
- Assume the equation of state is known for $\epsilon \le \epsilon_i = \epsilon(p_i)$.
- Assume the complete M-R curve is known.
- Choose a new point on the M-R curve, $\{R_{i+1}, M_{i+1}\}$, having slightly larger central density.



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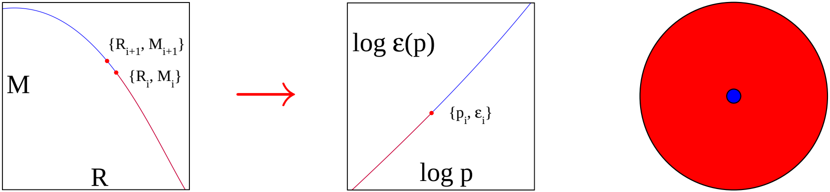
- Integrate Einstein's equations through the outer parts of the star, to determine the mass and radius, $\{r_{i+1}, m_{i+1}\}$, of the core.



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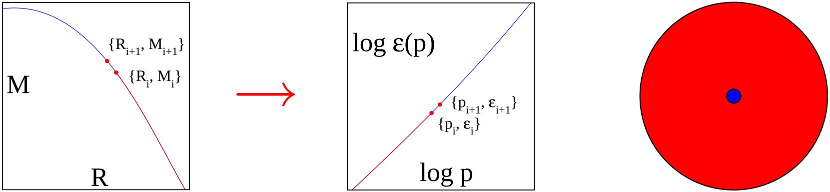
- Use a power series solution of Einstein's equations in the core to determine the central pressure and density, $\{p_{i+1}, \epsilon_{i+1}\}$.



# Can the Standard Solution to SSP$^{-1}$ be Improved?

- Standard solution to the relativistic SSP$^{-1}$ finds the equation of state, $\epsilon = \epsilon(p)$, represented as a table: $\{p_i, \epsilon_i\}$ for $i = 1, ..., N$.

- Standard solution has several weaknesses:
  - Solution converges (slowly) with the number of points, as $N^{-p}$.
  - Each equation of state point found, $\{p_i, \epsilon_i\}$, requires the knowledge of a separate M-R curve point, $\{R_i, M_i\}$.
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- Spectral numerical methods typically converge more rapidly, and represent functions more efficiently than finite difference methods.
- Can spectral methods provide better solutions to the SSP$^{-1}$?

# Outline for Solving SSP$^{-1}$ Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \lambda_\alpha)$, where the $\lambda_\alpha$ are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \lambda_\alpha) = \sum_\alpha \lambda_\alpha \Phi_\alpha(p)$, where the $\Phi_\alpha(p)$ are spectral basis functions.

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- Given a set of points from the "real" M-R curve, $\{R_i, M_i\}$, choose the parameters $\lambda_\alpha$ and $p_i$ that minimize the difference measure:

$$\Delta^2_{MR} = \frac{1}{N}\sum_{i=1}^{N}\left\{\left[\log\left(\frac{R(p_i, \lambda_\alpha)}{R_i}\right)\right]^2 + \left[\log\left(\frac{M(p_i, \lambda_\alpha)}{M_i}\right)\right]^2\right\}$$

# Outline for Solving $\mathrm{SSP}^{-1}$ Using Spectral Methods

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- Resulting $\lambda_\alpha$ for $\alpha = 1, ..., N$ determine an equation of state, $\epsilon = \epsilon(p, \lambda_\alpha)$, that provides an approximate solution of $\mathrm{SSP}^{-1}$.

# Faithful Spectral Expansions of the Equation of State

- Physical equations of state, $\epsilon = \epsilon(p)$, are positive monotonic increasing functions. These do not form a vector space.
- The representation, $\epsilon = \epsilon(p, \lambda_\alpha) = \sum_\alpha \lambda_\alpha \Phi_\alpha(p)$, is not faithful.
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- Faithful spectral expansions of the adiabatic index do exist:

$$\Gamma(p) = \frac{\epsilon c^2 + p}{pc^2} \frac{dp}{d\epsilon} = \exp\left[\sum_\alpha \gamma_\alpha \Phi_\alpha(p)\right].$$

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- Every equation of state is determined by the adiabatic index $\Gamma(p)$:

$$\begin{aligned} \mu(p) &= \exp\left[\int_{p_0}^{p} \frac{dp'}{p'\Gamma(p')}\right], \\ \epsilon(p) &= \frac{\epsilon_0}{\mu(p)} + \frac{1}{\mu(p)}\int_{p_0}^{p} \frac{\mu(p')}{c^2\Gamma(p')} dp'. \end{aligned}$$

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- Test its effectiveness by constructing expansions that minimize,

$$\Delta_\epsilon^2 \;=\; \frac{1}{N}\sum_{i=1}^{N}\left\{\left[\log\left(\frac{\epsilon(p_i,\gamma_\alpha)}{\epsilon_i}\right)\right]^2\right\}$$

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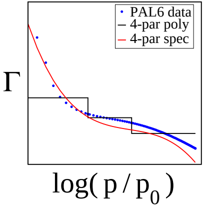


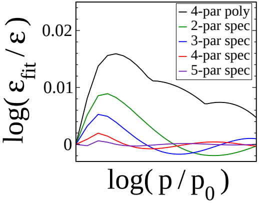

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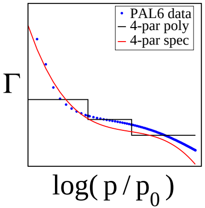


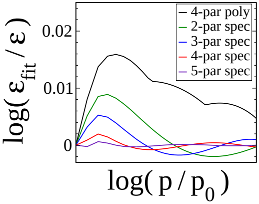


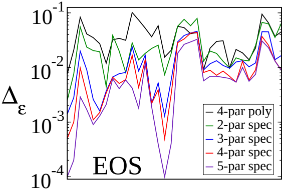

# Spectral Solution of SSP$^{-1}$

- Next step is to test this spectral approach by finding the approximate solution to SSP$^{-1}$ for realistic neutron star models.
- Choose points $\{R_i, M_i\}$ from realistic neutron star models, then fix the spectral expansion coefficients $\gamma_\alpha$ by minimizing,

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- Finally evaluate $\Delta_\epsilon$,

$$\Delta^2_\epsilon = \frac{1}{N}\sum_{i=1}^{N}\left\{\left[\log\left(\frac{\epsilon(p_i,\gamma_\alpha)}{\epsilon_i}\right)\right]^2\right\}$$

  to determine how well the spectral expansion $\epsilon = \epsilon(p,\gamma_\alpha)$, matches the original realistic neutron star equation of state $\epsilon = \epsilon(p)$.

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- The End.