New Insights Into Gauge Freedom and Constraints in Numerical Relativity

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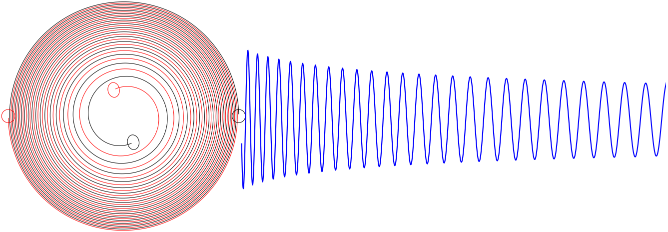


General Theme:

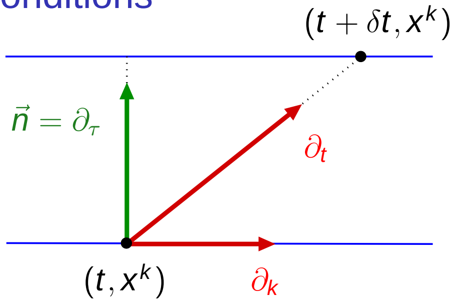
- Interesting and Unusual Ways to Specify Coordinates (Gauge).

Outline:

- Methods of Specifying Gauge (Coordinates).
- Generalized Harmonic (GH) Einstein Equations.
- Constraint Damping.
- Moving Black Holes.
- Dual Coordinate Frame Evolution.
- Choosing Coordinates by Feedback Control.
- Recent GH Binary Black Hole Results.

Traditional ADM Gauge Conditions

- Construct a foliation of spacetime by spatial slices.
- Choose a time function with t = const. on these slices.
- Choose spatial coordinates, x^k, on each slice.



- Decompose the 4-metric ψ_{ab} into its 3+1 parts: $ds^2 = \psi_{ab}dx^a dx^b = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$.
- The lapse N and shift N^i measure how coordinates are laid out on spacetime:

$$\vec{n} = \partial_\tau = \frac{\partial t}{\partial \tau}\partial_t + \frac{\partial x^k}{\partial \tau}\partial_k, \quad = \frac{1}{N}\partial_t - \frac{N^k}{N}\partial_k.$$

- Spacetime coordinates are determined in the traditional ADM method by specifying the lapse N and shift N^i.

ADM Evolution System

- When the gauge is determined by specifying the lapse N and shift N^k, the Einstein system becomes a set of evolution equations for the spatial metric g_{ij} and extrinsic curvature K_{ij}:

$$\begin{aligned} \partial_t g_{ij} - N^k \partial_k g_{ij} &= -2NK_{ij} + ..., \\ \partial_t K_{ij} - N^k \partial_k K_{ij} &= NR_{ij}(g) + ... \end{aligned}$$

- The Einstein equations also include constraints:

$$\begin{aligned} 0 &= \mathcal{M}_{\hat{t}} \equiv R - K_{ij}K^{ij} + K^2, \\ 0 &= \mathcal{M}_i \equiv \nabla^k K_{ki} - \nabla_i K. \end{aligned}$$

- This traditional form of the Einstein equations is not hyperbolic, and numerical solutions are found to suffer from generic constraint violating instabilities.

Generalized Harmonic Gauge Conditions

- An alternate way to specify the coordinates is through the gauge source function H^a:
- Let H^a denote the function obtained by the action of the covariant scalar wave operator on the coordinates x^a:

$$H^a \equiv \nabla^c\nabla_c x^a = \psi^{bc}(\partial_b\partial_c x^a - \Gamma^e_{bc}\partial_e x^a) = -\Gamma^a,$$

where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

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- Specifying coordinates by the *generalized harmonic* (GH) method can be accomplished by choosing a gauge-source function $H_a(x,\psi) = \psi_{ab}H^b$, and requiring that

$$H_a(x,\psi) = -\Gamma_a = -\psi_{ab}\psi^{cd}\Gamma^b_{cd}.$$

Important Properties of the GH Method

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

$$R_{ab} \;=\; -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + F_{ab}(\psi,\partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$. The vacuum Einstein equation, $R_{ab} = 0$, has the same principal part as the scalar wave equation when $H_a(x,\psi) = -\Gamma_a$ is imposed.

Generalized Harmonic Evolution System

- Frans Pretorius wrote a very nice second order finite difference AMR code to solve the generalized harmonic Einstein equations:

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

where $\mathcal{C}_a = H_a + \Gamma_a$.

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- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by the derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv \left[R_{ab} - \frac{1}{2}\psi_{ab}R\right]n^b = \left[\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\right]n^b.$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0 \left[n_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\, n^c\, \mathcal{C}_c \right],$$

where n^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

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- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

$$0 \quad = \quad \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[n_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \frac{1}{2}\gamma_0\, n_a\mathcal{C}^c\mathcal{C}_c.$$

  This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

First-Order Einstein Evolution System

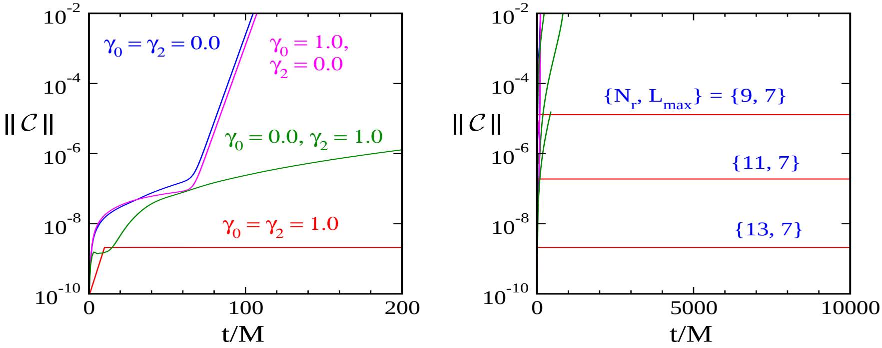
- Introduce new fields Π_{ab} and Φ_{iab} representing the time and space derivatives of the metric ψ_{ab}.
- Our code solves a first-order representation of the GH Einstein evolution system:

$$\partial_t \psi_{ab} = -N\Pi_{ab} + N^i \Phi_{iab},$$
$$\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + Ng^{ki}\partial_k \Phi_{iab} + \gamma_2 N^k \partial_k \psi_{ab} \simeq 0,$$
$$\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N\partial_i \Pi_{ab} - \gamma_2 N \partial_i \Psi_{ab} \simeq 0.$$

- Violations of the additional constraint, $\mathcal{C}_{iab} = \Phi_{iab} - \partial_i \psi_{ab}$, are suppressed on the timescale $1/\gamma_2$ by this evolution system.
- This evolution system is symmetric hyperbolic and linearly degenerate.

Numerical Tests of the First-Order GH System

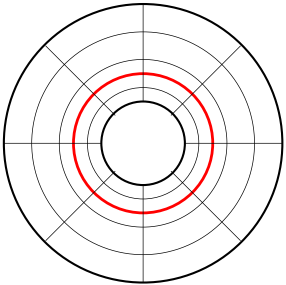
- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.



- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.



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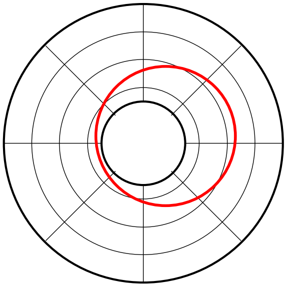

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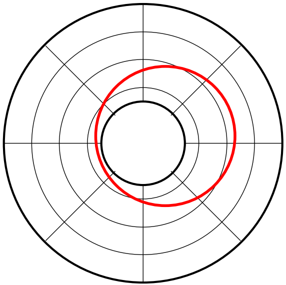

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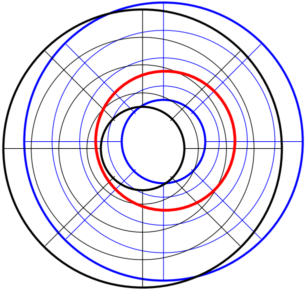

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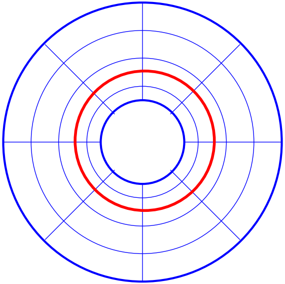

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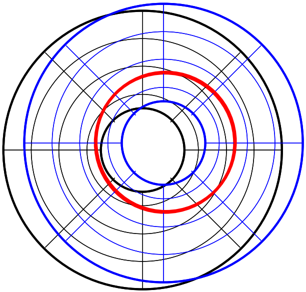

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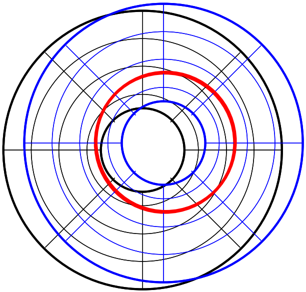

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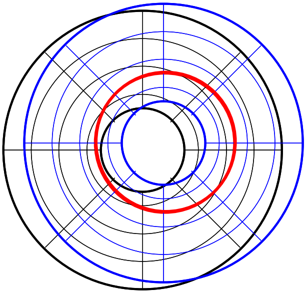

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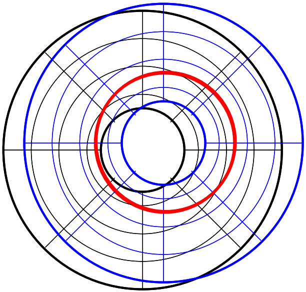

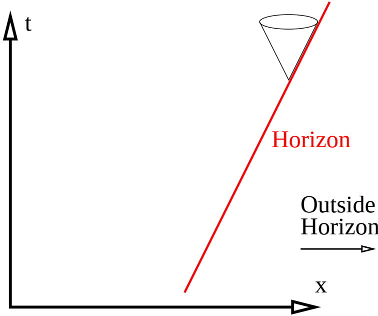

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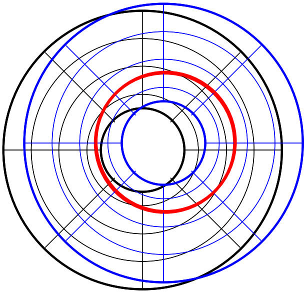

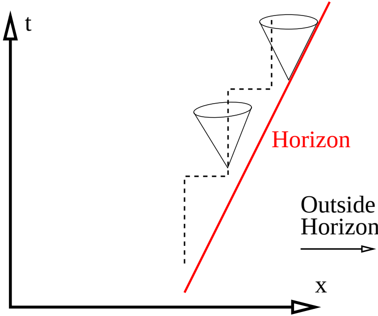

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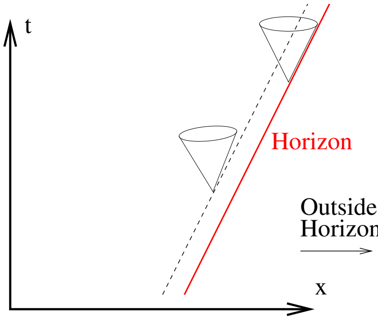
- Solution:

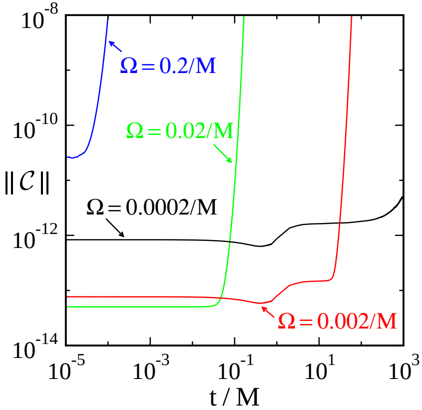
  Choose coordinates that smoothly track the location of the black hole.

  For a black hole binary this means using coordinates that rotate with respect to inertial frames at infinity.



Evolving Black Holes in Rotating Frames

- Coordinates that rotate with respect to the inertial frames at infinity are needed to track the horizons of orbiting black holes.
- Evolutions of Schwarzschild in rotating coordinates are unstable.



- Evolutions shown use a computational domain that extends to $r = 1000M$.
- Angular velocity needed to track the horizons of an equal mass binary at merger is about $\Omega \approx 0.2/M$.
- Problem caused by asymptotic behavior of metric in rotating coordinates: $\psi_{tt} \sim \rho^2\Omega^2$, $\psi_{ti} \sim \rho\Omega$, $\psi_{ij} \sim 1$.

Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar a} = \{\bar t, x^{\bar\imath}\}$, to define field components, $u^{\bar\alpha} = \{\psi_{\bar a\bar b}, \Pi_{\bar a\bar b}, \Phi_{\bar\imath\bar a\bar b}\}$, and the same coordinates to determine these components by solving Einstein's equation for $u^{\bar\alpha} = u^{\bar\alpha}(x^{\bar a})$:

$$\partial_{\bar t} u^{\bar\alpha} + A^{\bar k\,\bar\alpha}{}_{\bar\beta}\partial_{\bar k} u^{\bar\beta} = F^{\bar\alpha}.$$

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- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^{\bar a})$, to represent these components as functions, $u^{\bar\alpha} = u^{\bar\alpha}(x^a)$.

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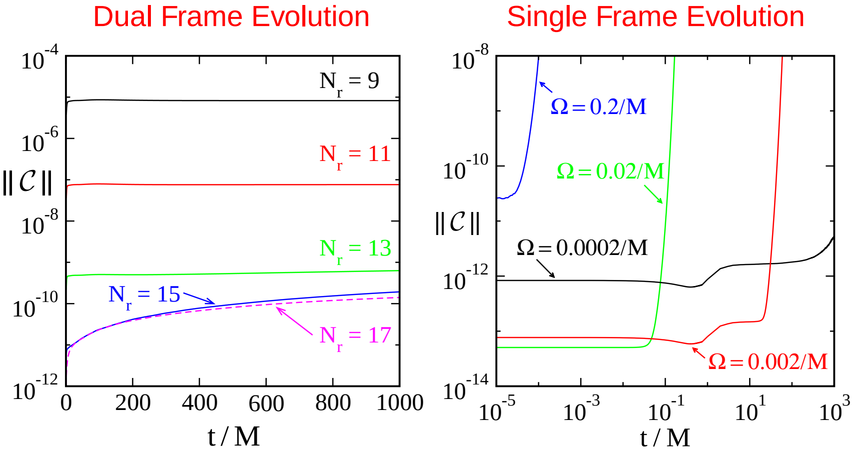
- These functions are determined by solving the transformed Einstein equation:

$$\partial_t u^{\bar{\alpha}} + \left[\frac{\partial x^i}{\partial \bar{t}} \delta^{\bar{\alpha}}{}_{\bar{\beta}} + \frac{\partial x^i}{\partial x^{\bar{k}}} A^{\bar{k}\,\bar{\alpha}}{}_{\bar{\beta}}\right] \partial_i u^{\bar{\beta}} = F^{\bar{\alpha}}.$$

Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:



- Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/M$ on a domain with outer radius $r = 1000M$.

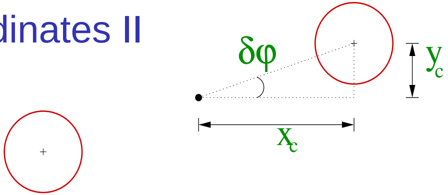
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- A coordinate transformation from inertial coordinates, $(\bar{x}, \bar{y}, \bar{z})$, to co-moving coordinates (x, y, z), consisting of a rotation followed by an expansion,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{a(\bar{t})} \begin{pmatrix} \cos\varphi(\bar{t}) & -\sin\varphi(\bar{t}) & 0 \\ \sin\varphi(\bar{t}) & \cos\varphi(\bar{t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix},$$

  with $t = \bar{t}$, is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions $a(\bar{t})$ and $\varphi(\bar{t})$.
- Since the motions of the holes are not known *a priori*, the functions $a(\bar{t})$ and $\varphi(\bar{t})$ must be chosen dynamically and adaptively as the system evolves.

Horizon Tracking Coordinates II



- Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.
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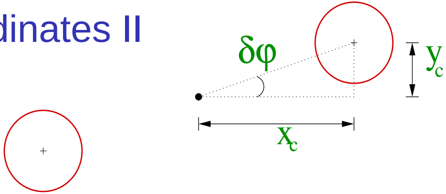

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- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.
- Changing the map parameters by the small amounts, δa and $\delta\varphi$, results in associated small changes in δQ^x and δQ^y:

$$\delta Q^x = -\delta a, \qquad \delta Q^y = -\delta\varphi.$$

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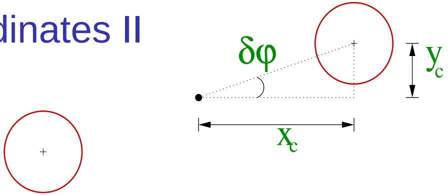

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- Measure the quantities $Q^y(t)$, $dQ^y(t)/dt$, $d^2Q^y(t)/dt^2$, and set

$$\frac{d^3\varphi}{dt^3} = \lambda^3 Q^y + 3\lambda^2 \frac{dQ^y}{dt} + 3\lambda \frac{d^2Q^y}{dt^2} = -\frac{d^3Q^y}{dt^3}.$$

The solutions to this “closed-loop” equation for Q^y have the form $Q^y(t) = (At^2 + Bt + C)e^{-\lambda t}$, so Q^y always decreases as $t \to \infty$.

Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times $t = t_i$.
- In the time interval $t_i < t < t_{i+1}$ we set:

$$\begin{aligned}\varphi(t) &= \varphi_i + (t - t_i)\frac{d\varphi_i}{dt} + \frac{(t-t_i)^2}{2}\frac{d^2\varphi_i}{dt^2} \\ &\quad + \frac{(t-t_i)^3}{2}\left(\lambda\frac{d^2Q_i^y}{dt^2} + \lambda^2\frac{dQ_i^y}{dt} + \lambda^3\frac{Q_i^y}{3}\right),\end{aligned}$$

where Q^x, Q^y, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop equation at $t = t_i$.

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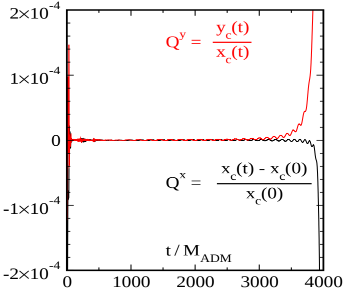
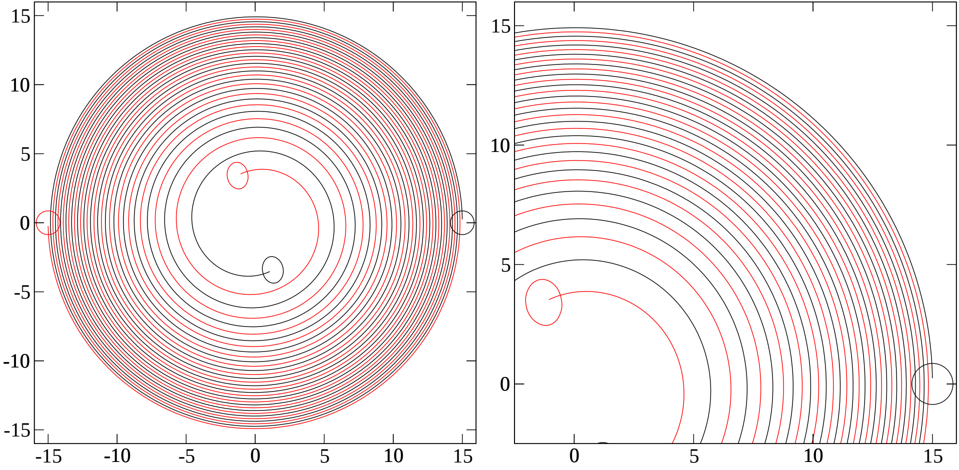


- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.

Evolving Binary Black Hole Spacetimes

- We can now evolve binary black hole spacetimes with excellent accuracy and computational efficiency through many orbits.

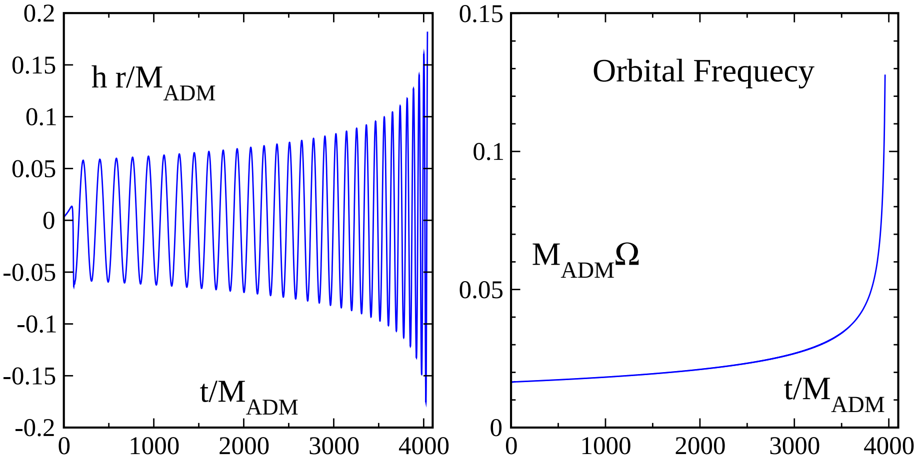


Head-on Merger Movie

Lapse-Ψ_4 Movie

Evolving Binary Black Hole Spacetimes II

- Gravitational waveform and frequency evolution for the equal mass non-spinning BBH evolution.



Summary

- Generalized Harmonic method produces manifestly hyperbolic representations of the Einstein equations for any choice of coordinates.
- Constraint damping makes the modified GH equations stable for numerical simulations.
- Dual coordinate frame evolution makes evolutions stable in coordinates that track the black hole motions.
- Feedback control systems can be used to construct co-moving coordinates that accurately track the black hole motions.