Solving Einstein's Equations: PDE Issues

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Mathematical and Numerical General Relativity Seminar
University of California at San Diego
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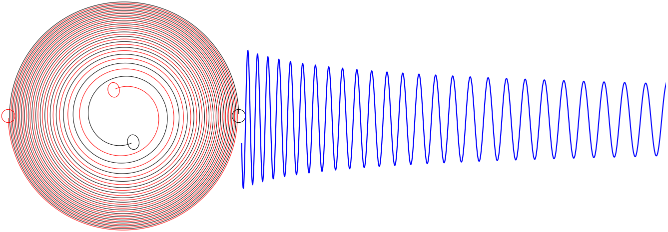

General Relativity Theory

- Einstein's theory of gravitation, general relativity theory, is a geometrical theory in which gravitational effects are described as geometrical structures on spacetime.
- The fundamental "gravitational" field is the spacetime metric ψ_{ab}, a symmetric ($\psi_{ab} = \psi_{ba}$) non-degenerate ($\psi_{ab} v^b = 0 \;\Rightarrow\; v^a = 0$) tensor field.

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- The metric ψ_{ab} defines an inner product, e.g. $\psi_{ab} v^a w^b$, which determines the physical angles between vectors for example.
- The spacetime metric determines the physical lengths of curves $x^a(\lambda)$ in spacetime, $L^2 = \pm \int \psi_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} d\lambda$.
- Coordinates can be chosen at any point in spacetime so that $ds^2 = \psi_{ab} dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2$ at that point.

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- Coordinates can be chosen at any point in spacetime so that $ds^2 = \psi_{ab} dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2$ at that point.
- The tensor ψ^{ab} is the inverse metric, i.e. $\psi^{ac}\psi_{cb} = \delta^a{}_b$.
- The metric and inverse metric are used to define the dual transformations between vector and co-vector fields, e.g. $v_a = \psi_{ab} v^b$ and $w^a = \psi^{ab} w_b$.

General Relativity Theory II

- The spacetime metric ψ_{ab} is determined by Einstein's equation:
  $$R_{ab} - \tfrac{1}{2}R\psi_{ab} = 8\pi T_{ab},$$
  where R_{ab} is the Ricci curvature tensor associated with ψ_{ab}, $R = \psi^{ab}R_{ab}$ is the scalar curvature, and T_{ab} is the stress-energy tensor of the matter present in spacetime.

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- For "vacuum" spacetimes (like binary black hole systems) $T_{ab} = 0$, so Einstein's equations can be reduced to $R_{ab} = 0$.
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- The Ricci curvature R_{ab} is determined by derivatives of the metric:
$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{bc} + \Gamma^c{}_{cd}\Gamma^d{}_{ab} - \Gamma^c{}_{ad}\Gamma^d{}_{bc},$$
where $\Gamma^c{}_{ab} = \frac{1}{2}\psi^{cd}(\partial_a\psi_{db} + \partial_b\psi_{da} - \partial_d\psi_{ab})$.
- Einstein's equations are second-order pde's that (should, hopefully) determine the spacetime metric, e.g. in vacuum
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- The important fundamental ideas needed to understand these questions are:
  - gauge freedom,
  - and constrints.
- Maxwell's equations are a simpler system in which these same fundamental issues play analogous roles.

Gauge and Hyperbolicity in Electromagnetism

- The usual representation of the vacuum Maxwell equations split into evolution equations and constraints:

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0,$$
$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0.$$

  These equations are often written in the more compact 4-dimensional form $\nabla^a F_{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$, where F_{ab} has components $\vec{E}$ and $\vec{B}$.

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- Maxwell's equations can be solved in part by introducing a vector potential $F_{ab} = \nabla_a A_b - \nabla_b A_a$. This reduces the system to the single equation: $\nabla^a \nabla_a A_b - \nabla_b \nabla^a A_a = 0.$

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$$\nabla^a \nabla_a A_b = \left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right) A_b = \nabla_b H.$$

Gauge and Hyperbolicity in General Relativity

- The spacetime Ricci curvature tensor can be written as:

$$R_{ab} \;=\; -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi, \partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi_{ad}\psi^{bc}\Gamma^d{}_{bc}$.

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- One way to impose the needed gauge conditions is to specify H^a, the source term for a wave equation for each coordinate x^a:

$$H^a = \nabla^c\nabla_c x^a \;=\; \psi^{bc}(\partial_b\partial_c x^a - \Gamma^e{}_{bc}\partial_e x^a) = -\Gamma^a,$$

  where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

Gauge Conditions in General Relativity

- Specifying coordinates by the *generalized harmonic* (GH) method is accomplished by choosing a gauge-source function $H^a(x, \psi)$, e.g. $H^a = \psi^{ab} H_b(x)$, and requiring that

$$H^a(x, \psi) = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b \psi_{dc} + \partial_c \psi_{db} - \partial_d \psi_{bc}).$$

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  $$R_{ab} \;=\; -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi,\partial\psi).$$

- The Generalized Harmonic Einstein equation is obtained by replacing $\Gamma_a = \psi_{ab}\Gamma^b$ with $-H_a(x,\psi) = -\psi_{ab}H^b(x,\psi)$:

$$R_{ab} - \nabla_{(a}\left[\Gamma_{b)} + H_{b)}\right] = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(\psi,\partial\psi).$$

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- The vacuum GH Einstein equation, $R_{ab} = 0$ with $\Gamma_a + H_a = 0$, is therefore manifestly hyperbolic, having the same principal part as the scalar wave equation:

$$0 = \nabla_a\nabla^a\Phi = \psi^{ab}\partial_a\partial_b\Phi + F(\partial\Phi).$$

The Constraint Problem

- Fixing the gauge in an appropriate way makes the Einstein equations hyperbolic, so the initial value problem becomes well-posed mathematically.
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- There is no guarantee, however, that constraints that are "small" initially will remain "small".
- Constraint violating instabilities were one of the major problems that made progress on solving the binary black hole problem so slow.
- Special representations of the Einstein equations are needed that control the growth of any constraint violations.

Constraint Damping in Electromagnetism

- Electromagnetism is described by the hyperbolic evolution equation $\nabla^a\nabla_a A_b = \nabla_b H$. Are there any constraints? Where have the usual $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ constraints gone?

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- Gauge condition becomes a constraint: $0 = \mathcal{C} \equiv \nabla^b A_b - H$.
- Maxwell's equations imply that this constraint is preserved:

$$\nabla^a\nabla_a\left(\nabla^b A_b - H\right) = \nabla^a\nabla_a \mathcal{C} = 0.$$

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- Modify evolution equations by adding multiples of the constraints:

$$\nabla^a\nabla_a A_b = \nabla_b H + \gamma_0 t_b \mathcal{C} = \nabla_b H + \gamma_0 t_b \left(\nabla^a A_a - H\right).$$

- These changes effect the constraint evolution equation,

$$\nabla^a\nabla_a \mathcal{C} - \gamma_0 t^b \nabla_b \mathcal{C} = 0,$$

  so constraint violations are damped when $\gamma_0 > 0$.

Constraints in the GH Evolution System

- The GH evolution system has the form,

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

  where $\mathcal{C}_a = H_a + \Gamma_a$ plays the role of a constraint. Without constraint damping, these equations are very unstable to constraint violating instabilities.

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- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv \Big[R_{ab} - \tfrac{1}{2}\psi_{ab}R\Big]t^b = \Big[\nabla_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\Big]t^b.$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 \;=\; R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[t_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\, t^c\,\mathcal{C}_c\right],$$

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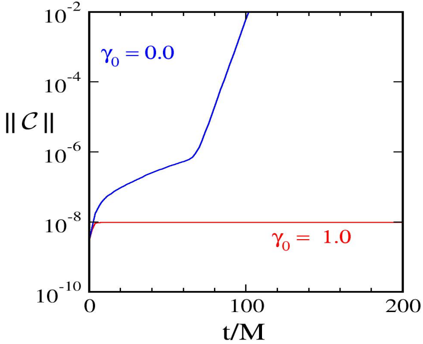
- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

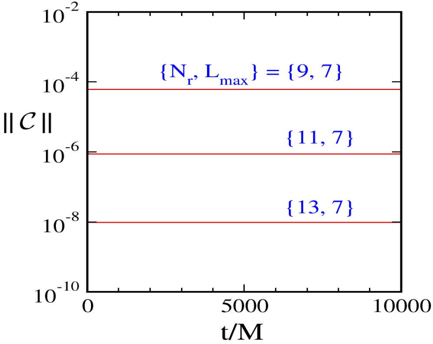
$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[t_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \tfrac{1}{2}\gamma_0\; t_a\mathcal{C}^c\mathcal{C}_c.$$

  This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

Numerical Tests of the GH Evolution System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = 1$.





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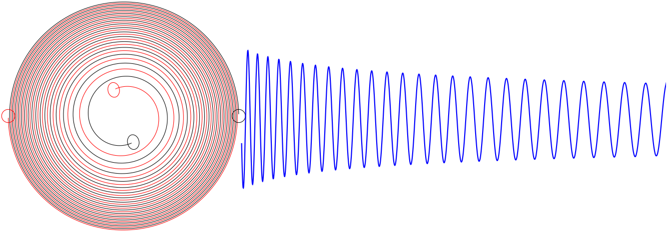

Summary of the GH Einstein System

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- Principal part of evolution system becomes manifestly hyperbolic:

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- Add constraint damping terms for stability:

$$0 \quad = \quad R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[t_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\, t^c\, \mathcal{C}_c\right],$$

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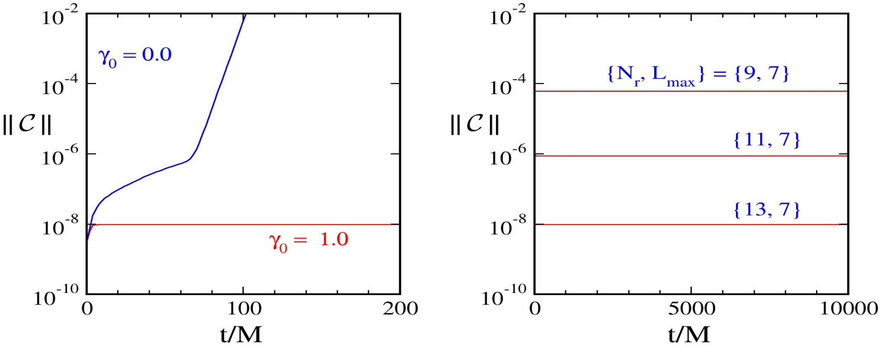

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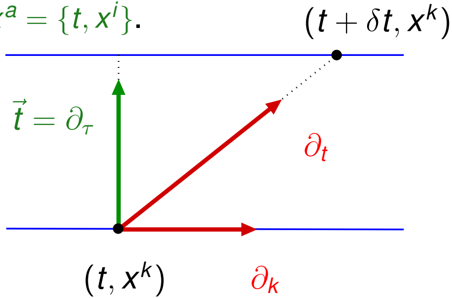
ADM 3+1 Approach to Fixing Coordinates

- Coordinates must be chosen to label points in spacetime before the Einstein equations can be solved. For some purposes it is convenient to split the spacetime coordinates x^a into separate time and space components: $x^a = \{t, x^i\}$.
- Construct spacetime foliation by spacelike slices.
- Choose time function with t = const. on these slices.
- Choose spatial coordinates, x^k, on each slice.



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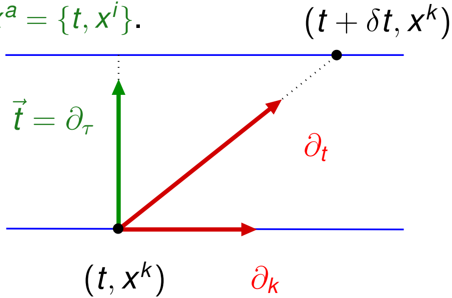
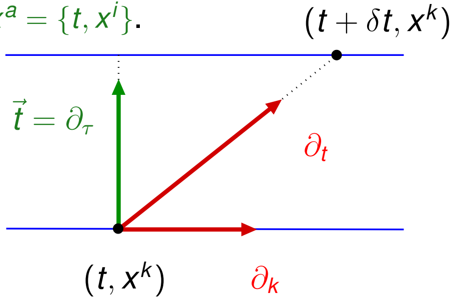


- Decompose the 4-metric ψ_{ab} into its 3+1 parts:

$$ds^2 = \psi_{ab}dx^a dx^b = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

ADM 3+1 Approach to Fixing Coordinates

- Coordinates must be chosen to label points in spacetime before the Einstein equations can be solved. For some purposes it is convenient to split the spacetime coordinates x^a into separate time and space components: $x^a = \{t, x^i\}$.
- Construct spacetime foliation by spacelike slices.
- Choose time function with t = const. on these slices.
- Choose spatial coordinates, x^k, on each slice.



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- The unit vector t^a normal to the $t =$constant slices depends only on the lapse N and shift N^i: $\vec{t} = \partial_\tau = \frac{\partial x^a}{\partial \tau}\partial_a = \frac{1}{N}\partial_t - \frac{N^k}{N}\partial_k$.

ADM Approach to the Einstein Evolution System

- Decompose the Einstein equations $R_{ab} = 0$ using the ADM 3+1 coordinate splitting. The resulting system includes evolution equations for the spatial metric g_{ij} and extrinsic curvature K_{ij}:

$$\begin{aligned}
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&\quad -\nabla_i \nabla_j N - 2NK_{ik}K^k{}_j + NK^k{}_k K_{ij}.
\end{aligned}$$

- The resulting system also includes constraints:

$$\begin{aligned}
0 &= R^{(3)} - K_{ij}K^{ij} + (K^k{}_k)^2, \\
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- System includes no evolution equations for lapse N or shift N^i. These quanties can be specified freely to fix the gauge.
- Resolving the issues of hyperbolicity (i.e. well posedness of the initial value problem) and constraint stability are much more complicated in this approach. The most successful version is the BSSN evolution system used by many (most) codes.

Dynamical GH Gauge Conditions

- The spacetime coordinates x^b are fixed in the generalized harmonic Einstein equations by specifying H^b:

$$\nabla^a \nabla_a x^b \equiv H^b.$$

- The generalized harmonic Einstein equations remain hyperbolic as long as the gauge source functions H^b are taken to be functions of the coordinates x^b and the spacetime metric ψ_{ab}.

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- This failure seems to occur because the coordinates themselves become very dynamical solutions of the wave equation $\nabla^a \nabla_a x^b = 0$ in these situations.
- Another simple choice – keeping H^b fixed in the co-moving frame of the black holes – works well during the long inspiral phase, but fails when the black holes begin to merge.

Dynamical GH Gauge Conditions II

- Some of the extraneous gauge dynamics could be removed by adding a damping term to the harmonic gauge condition:

$$\nabla^a\nabla_a x^b = H^b = \mu t^a \partial_a x^b = \mu t^b = -\mu N \psi^{tb}.$$

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- For the time coordinate t, this damped wave condition drives t to a time independent constant, which is not a good coordinate.
- A better choice sets $t^a H_a = -\mu \log \sqrt{g/N^2}$. The gauge condition in this case becomes

$$t^a \partial_a \log \sqrt{g/N^2} = -\mu \log \sqrt{g/N^2} + N^{-1} \partial_k N^k$$

  This coordinate condition keeps g/N^2 close to unity, even during binary black hole mergers (where it became of order 100 using simpler gauge conditions).

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- GH evolution system can be written as a symmetric-hyperbolic first-order system (Fischer and Marsden 1972, Alvi 2002):

$$\begin{aligned}
\partial_t\psi_{ab} - N^k\partial_k\psi_{ab} &= -N\,\Pi_{ab},\\
\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} &\simeq 0,\\
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where $\Phi_{kab} = \partial_k\psi_{ab}$.

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- This system has two immediate problems:
  - This system has new constraints, $\mathcal{C}_{kab} = \partial_k\psi_{ab} - \Phi_{kab}$, that tend to grow exponentially during numerical evolutions.
  - This system is not linearly degenerate, so it is possible (likely?) that shocks will develop (e.g. the components that determine shift evolution have the form $\partial_t N^i - N^k\partial_k N^i \simeq 0$).

A 'New' Generalized Harmonic Evolution System

- We can correct these problems by adding additional multiples of the constraints to the evolution system:

$$
\begin{aligned}
\partial_t\psi_{ab} - (1+\gamma_1)N^k\partial_k\psi_{ab} &= -N\Pi_{ab} - \gamma_1 N^k\Phi_{kab},\\
\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} - \gamma_1\gamma_2 N^k\partial_k\psi_{ab} &\simeq -\gamma_1\gamma_2 N^k\Phi_{kab},\\
\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} &\simeq -\gamma_2 N\Phi_{iab}.
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$$\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} \simeq -\gamma_2 N\Phi_{iab}.$$

- This ‘new’ generalized-harmonic evolution system has several nice properties:
  - This system is linearly degenerate for $\gamma_1 = -1$ (and so shocks should not form from smooth initial data).
  - The Φ_{iab} evolution equation can be written in the form, $\partial_t\mathcal{C}_{iab} - N^k\partial_k\mathcal{C}_{iab} \simeq -\gamma_2 N\mathcal{C}_{iab}$, so the new constraints are damped when $\gamma_2 > 0$.
  - This system is symmetric hyperbolic for all values of γ_1 and γ_2.

Constraint Evolution for the New GH System

- The evolution of the constraints, $c^A = \{\mathcal{C}_a, \mathcal{C}_{kab}, \mathcal{F}_a \approx t^c\partial_c\mathcal{C}_a, \mathcal{C}_{ka} \approx \partial_k\mathcal{C}_a, \mathcal{C}_{klab} = \partial_{[k}\mathcal{C}_{l]ab}\}$ are determined by the evolution of the fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$:
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$$\partial_t c^A + A^{k\,A}{}_B(u)\partial_k c^B = F^A{}_B(u, \partial u)\, c^B.$$
- This constraint evolution system is symmetric hyperbolic with principal part:
$$\begin{aligned} \partial_t\mathcal{C}_a &\simeq 0, \\ \partial_t\mathcal{F}_a - N^k\partial_k\mathcal{F}_a - Ng^{ij}\partial_i\mathcal{C}_{ja} &\simeq 0, \\ \partial_t\mathcal{C}_{ia} - N^k\partial_k\mathcal{C}_{ia} - N\partial_i\mathcal{F}_a &\simeq 0, \\ \partial_t\mathcal{C}_{iab} - (1+\gamma_1)N^k\partial_k\mathcal{C}_{iab} &\simeq 0, \\ \partial_t\mathcal{C}_{ijab} - N^k\partial_k\mathcal{C}_{ijab} &\simeq 0. \end{aligned}$$

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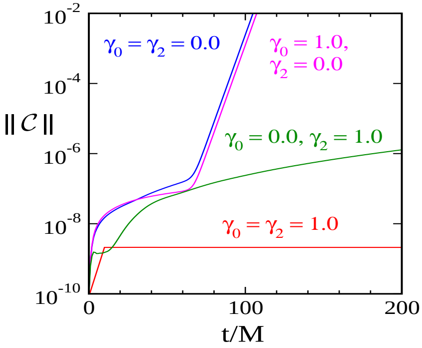
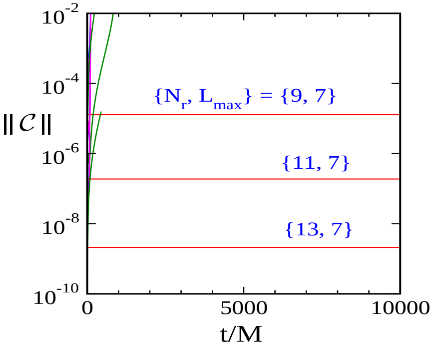
- An analysis of this system shows that all of the constraints are damped in the WKB limit when $\gamma_0 > 0$ and $\gamma_2 > 0$. So, this system has constraint suppression properties that are similar to those of the Pretorius (and Gundlach, et al.) system.

Numerical Tests of the New GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of our GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.





- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

Boundary Condition Basics

- We impose boundary conditions on first-order hyperbolic evolution systems, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\partial_k u^\beta = F^\alpha(u)$ in the following way (where in our case $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$):

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- We first find the eigenvectors of the characteristic matrix $n_k A^{k\,\alpha}{}_\beta$ at each boundary point:

$$e^{\hat\alpha}{}_\alpha\, n_k A^{k\,\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta,$$

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- Finally we impose a boundary condition on each incoming characteristic field (*i.e.* every field with $v_{(\hat\alpha)} < 0$), and impose no condition on any outgoing field (*i.e.* any field with $v_{(\hat\alpha)} \geq 0$).
- At internal boundaries (i.e. interfaces between computational subdomains) use outgoing characteristics of one subdomain to fix data for incoming characteristics of neighboring subdomain.

Evolutions of a Perturbed Schwarzschild Black Hole

- The simplest boundary conditions that correspond (roughly) to "no incoming waves" set $u^{\hat{\alpha}} = 0$ for each incoming field, or $d_t u^{\hat{\alpha}} \equiv e^{\hat{\alpha}}{}_{\beta}\partial_t u^{\beta} = 0$ for fields that include static "Coulomb" parts.

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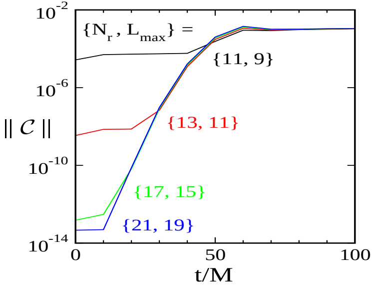
- Use boundary conditions that *Freeze* the remaining incoming characteristic fields: $d_t u^{\hat\alpha} = 0$.
- The resulting outgoing waves interact with the boundary of the computational domain and produce constraint violations.



Play Constraint Movie

Constraint Preserving Boundary Conditions

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$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$

- Set boundary conditions on the fields $\hat{u}^-$ by requiring

$$d_\perp \hat{u}^- = -\hat{F}(u, d_\parallel u).$$

Physical Boundary Conditions

- The Weyl curvature tensor C_{abcd} satisfies a system of evolution equations from the Bianchi identities: $\nabla_{[a}C_{bc]de} = 0$.
- The characteristic fields of this system corresponding to physical gravitational waves are the quantities:

$$\hat{w}^{\pm}_{ab} = (P_a{}^c P_b{}^d - \tfrac{1}{2}P_{ab}P^{cd})(t^e \mp n^e)(t^f \mp n^f)C_{cedf},$$

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- The incoming field w^-_{ab} can be expressed in terms of the characteristic fields of the primary evolution system:

$$\hat{w}^-_{ab} = d_\perp \hat{u}^-_{ab} + \hat{F}_{ab}(u, d_\| u).$$

- We impose boundary conditions on the physical graviational wave degrees of freedom then by setting:

$$d_\perp \hat{u}^-_{ab} = -\hat{F}_{ab}(u, d_\| u) + \hat{w}^-_{ab}|_{t=0}.$$

Imposing Neumann-like Boundary Conditions

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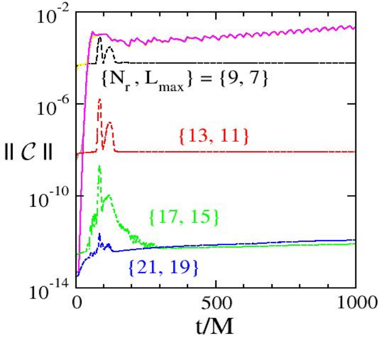
- We impose these Neumann-like boundary conditions by changing the appropriate components of the evolution equations at the boundary to:

$$d_t u^{\hat{\alpha}} = D_t u^{\hat{\alpha}} + v_{(\hat{\alpha})}\left(d_\perp u^{\hat{\alpha}} - d_\perp u^{\hat{\alpha}}|_{\mathrm{BC}}\right).$$

Tests of Constraint Preserving and Physical BC

- Evolve the perturbed black-hole spacetime using the resulting constraint preserving boundary conditions for the generalized harmonic evolution systems. Play Movies



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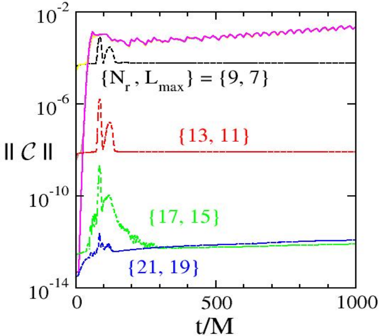

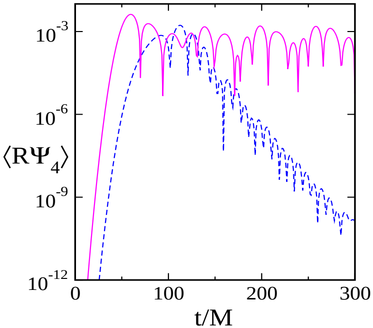

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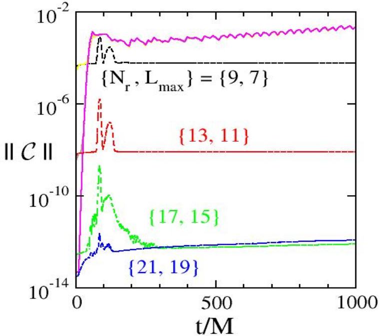

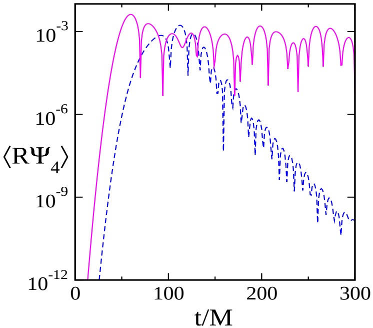

- Evolutions using these new constraint-preserving boundary conditions are still stable and convergent.
- The Weyl curvature component Ψ_4 shows clear quasi-normal mode oscillations in the outgoing gravitational wave flux when constraint-preserving boundary conditions are used.